Algebraic K-theory of fields and curves

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Motivation

k: a number field.
$\mathcal{O}_k$: the ring of algebraic integers of k.

r_1: the number of embeddings $k \to \mathbb{R}$
$2r_2$: the number of nonreal embeddings $k \to \mathbb{C}$
$[k:\mathbb{Q}] = r_1 + 2r_2$
$\mathcal{O}_k^*$ has rank $r = r_1 + r_2 - 1$
Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of k into $\mathbb{C}$ up to complex conjugation.
If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^*/\text{torsion}$, let

$$R = \frac{2^{r_2}}{[k:\mathbb{Q}]} \left| \det \begin{pmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{pmatrix} \right|$$

$$\zeta_k(s) = \sum_{\substack{(0)\neq I \subset \mathcal{O}_k \\ I \text{ an ideal of } \mathcal{O}_k}} (\#\mathcal{O}_k/I)^{-s}$$

$$= \prod_{\substack{0\neq \mathcal{P} \subset \mathcal{O}_k \\ \mathcal{P} \text{ prime ideal}}} \frac{1}{1-(\#\mathcal{O}_k/\mathcal{P})^{-s}}$$

$$\mathrm{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R |\mathrm{Cl}(\mathcal{O}_k)|}{w\sqrt{\Delta_k}}$$

Δ_k = the absolute value of the discriminant of k.
$w = |\mathcal{O}_{k,\mathrm{tor}}^*|$ = #roots of unity in k.

$$K_0(\mathcal{O}_k) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O}_k)$$
$$K_1(\mathcal{O}_k) \cong \mathcal{O}_k^*$$
$$|\mathrm{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\mathrm{tor}}|$$
$$w = |K_1(\mathcal{O}_k)_{\mathrm{tor}}|$$

If F is a field, then

$$K_0(F) \cong \mathbb{Z}$$
$$K_1(F) \cong F^* = F \setminus \{0\}$$
$$K_2(F) \cong F^* \otimes_{\mathbb{Z}} F^* / < x \otimes (1-x), x \in F^* \setminus \{1\} > .$$

The class of $a \otimes b$ in $K_2(F)$ is denoted $\{a, b\}$, so $K_2(F)$ is generated by symbols $\{a, b\}$ with a, b in F^*, and rules

$$\{a_1 a_2, b\} = \{a_1, b\} + \{a_2, b\}$$
$$\{a, b_1 b_2\} = \{a, b_1\} + \{a, b_2\}$$
$$\{x, 1-x\} = 0.$$

It follows that $\{a, b\} + \{b, a\} = \{x, -x\} = 0$.

Example

$$K_2(\mathbb{Q}) \cong \{\pm 1\} \oplus \bigoplus_{\substack{p \text{ prime} \\ p>2}} (\mathbb{Z}/p\mathbb{Z})^* .$$

Borel's theorem

k: number field (hence $K_{2n-1}(\mathcal{O}_k) \cong K_{2n-1}(k)$ if $n \geq 2$)
$K_n(\mathcal{O}_k)$ is finitely generated for all $n \geq 0$.
$m_n =$ the rank of $K_n(\mathcal{O}_k)$.

Theorem (Borel) $K_{2n}(\mathcal{O}_k)$ is a finite group if $n \geq 1$. For $n \geq 2$, $K_{2n-1}(\mathcal{O}_k)$ has rank $m_{2n-1} = \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd,} \\ r_2 & \text{if } n \text{ is even.} \end{cases}$
Furthermore, there exists a natural regulator map

$$K_{2n-1}(\mathcal{O}_k)/\text{torsion} \to \mathbb{R}^{m_{2n-1}}.$$

The image is a lattice with volume V_n of a fundamental domain

$$V_n \sim_{\mathbb{Q}^*} \frac{\zeta_k(n)}{\pi^{n([k:\mathbb{Q}]-m_{2n-1})}\sqrt{\Delta_k}}$$

where Δ_k is the absolute value of the discriminant of k.

[$a \sim_{\mathbb{Q}^*} b$ means $a = qb$ for some q in $\mathbb{Q}^*$.]

Example $\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:
$K_{2n-1}(\mathbb{Z})$ is finite for n even;
$K_{2n-1}(\mathbb{Z})$ has rank 1 for n odd, and $V_n \sim_{\mathbb{Q}^*} \zeta(n)$.

n	2	3	4	5	6	7	...
m_{2n-1}	0	1	0	1	0	1	...
$\zeta(n)$	$\pi^2/6$	irrat.	$\pi^4/90$	???	$\pi^6/945$	???	...

Getting a hold on higher K-groups.

> "Algebraic K-theory is a functor that associates to your favourite exact category Abelian groups K_n $(n \geq 0)$, about which you know nothing."

Let k be a number field.
$B_2(k)$: a free Abelian group on $[x]_2$, x in k, $x \neq 0, 1$.
Define

$$\begin{aligned} d_2 : B_2(k) &\to \bigwedge^2 k^* \\ [x]_2 &\mapsto (1-x) \wedge x \end{aligned}$$

Define $D(z)$ on $\mathbb{C}^* \setminus \{1\}$ by

$$dD(z) = \log|z| d\arg(1-z) - \log|1-z| d\arg(z)$$

and a linear map

$$\begin{aligned} \tilde{D} : B_2(k) &\to \mathbb{R}^{[k:\mathbb{Q}]} \\ [x]_2 &\mapsto (D(\sigma(x)))_\sigma \end{aligned}$$

where σ runs through the embeddings of k into $\mathbb{C}$.
Then

$$\frac{\mathrm{Ker}(d_2)}{\mathrm{Ker}(d_2) \cap \mathrm{Ker}(\tilde{D})} \Big/ \text{torsion} \cong K_3(k)/\text{torsion}$$

and Borel's regulator map is essentially given by $\tilde{D}$.

Zagier: inductive procedure.

This involves polylogarithms:

$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ for $n \geq 1$ on $|z| < 1$.
$Li_1(z) = -\mathrm{Log}(1-z)$
Using $\mathrm{d}Li_{n+1}(z) = Li_n(z)\mathrm{d}\log z$ for $n \geq 1$, $Li_n(z)$ extends to a multivalued function on $\mathbb{C}^* \setminus \{1\}$.
Single valued versions for $n \geq 2$:

$$P_n(z) = \pi_{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \log^k |z| Li_{n-k}(z)$$

$$P_{n,\mathrm{Zag}}(z) = \pi_{n-1} \sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \log^k |z| Li_{n-k}(z)$$

[π_m: real part for m even, imaginary part for m odd;
B_k: Bernoulli number.]

$$P_2(z) = P_{2,\mathrm{Zag}}(z) = D(z)$$

$B_n(k)$: a free Abelian group on $[x]_n$, x in k, $x \neq 0, 1$ $(n \geq 3)$
Define a linear map

$$\begin{aligned}\tilde{P}_{n,\mathrm{Zag}} : B_n(k) &\to \mathbb{R}^{[k:\mathbb{Q}]} \\ [x]_n &\mapsto (P_{n,\mathrm{Zag}}(\sigma(x)))_\sigma\end{aligned}$$

where σ runs through the embeddings of k into $\mathbb{C}$.

For $n \geq 3$ define inductively

$$C_{n-1}(k) = \frac{B_{n-1}(k)}{\mathrm{Ker}(d_{n-1}) \cap \mathrm{Ker}(\tilde{P}_{n-1,\mathrm{Zag}})}$$

and

$$\begin{aligned}d_n : B_n(k) &\to C_{n-1}(k) \otimes k^* \\ [x]_n &\mapsto [x]_{n-1} \otimes x\end{aligned}$$

Then there exists an injection

$$\frac{\mathrm{Ker}(d_n)}{\mathrm{Ker}(d_n) \cap \mathrm{Ker}(\tilde{P}_{n,\mathrm{Zag}})} \Bigg/ \text{torsion} \to K_{2n-1}(k)/\text{torsion}$$

and on its image Borel's regulator map is essentially given by $\tilde{P}_{n,\mathrm{Zag}}$ (RdJ). [This map is an isomorphism if $n = 3$ (Goncharov) or k is cyclotomic.]

Curves.

Let k be a number field and C/k a (smooth, proper, geometrically irreducible) curve of genus g.

Conjecture (Beilinson) Let $n \geq 1$.
(1) $K_{2n}(C) \otimes \mathbb{Q}$ is of dimension $r = [k:\mathbb{Q}] \cdot g$ if $n \geq 2$, and $K_2(C) \otimes \mathbb{Q}$ contains a subspace of dimension r consisting of "integral" elements;
(2) there is an isomorphism

$$\mathrm{reg} : K_{2n}(C)_{\mathrm{int}} \otimes \mathbb{R} \to H^1_{\mathrm{dR}}(C(\mathbb{C}); \mathbb{R})^{\pm} \cong \mathbb{R}^r$$

where $(-1)^n = \pm 1$, and if $\alpha_1, \ldots, \alpha_r$ form a $\mathbb{Q}$-basis of $K_{2n}(C)_{\mathrm{int}}$ and $\gamma_1, \ldots, \gamma_r$ form a $\mathbb{Z}$-basis of $H_1(C(\mathbb{C}); \mathbb{Z})^{\pm}$, then

$$R_{2n}(C) = \left| \det \left(\int_{\gamma_i} \mathrm{reg}(\alpha_j) \right) \right|$$

is related to $L(C, n+1)$.
[$\pm$ relates to the action of complex conjugation on $C(\mathbb{C})$, a disjoint union of $[k:\mathbb{Q}]$ Riemann surfaces, each of genus g.]

A few results

(1) there are enough elements for elliptic curves over $\mathbb{Q}$ with complex multiplication, and the expected relation with the L-function holds (Deninger);
(2) there are enough elements for modular curves and the relation with the L-function holds (Beilinson);
(3) same holds for elliptic curves over $\mathbb{Q}$ because they are modular;
(4) for every $g \geq 2$, there are g-dimensional families of hyperelliptic curves C over $\mathbb{Q}$ of genus g with g "integral" elements in $K_2(C)/\text{torsion}$, and in general non-vanishing regulator map (RdJ);
(5) for every real quadratic field k there are infinitely many elliptic curves over k, non-isomorphic over $\overline{\mathbb{Q}}$, with irrational j-invariant, and with two "integral" elements in K_2, and non-vanishing regulator (RdJ).

Explicit example of (4):
For the curves $C = C_{a,b}$ defined by

$$y^2 = -4x^6 + (2x^3 + (ax+1)(bx+1))^2$$

with integers $0 < b < a$ (of genus 2 if $a \gg 0$):

(1) there are 2 "integral" elements in $K_2(C)/\text{torsion}$;
(2) the Jacobian of $C_{a,b}$ does not split over $\overline{\mathbb{Q}}$ if, e.g., a and b satisfy $a \equiv 1$ and $b \equiv 2$ modulo 3;
(3)

$$\lim_{a \to \infty} \frac{R_2(C_{a,b})}{(\log a)^2} = 48\pi^2 \ .$$

More for curves and fields

Let F be an infinite field. Then one has an Adams decomposition

$$K_n(F) \otimes_{\mathbb{Z}} \mathbb{Q} = K_n^{(2)}(F) \oplus \cdots \oplus K_n^{(n)}(F)$$

for $n \geq 2$.

Write $F^*_{\mathbb{Q}}$ for $F^* \otimes_{\mathbb{Z}} \mathbb{Q}$.
Let $B_n(F)$ be a free $\mathbb{Q}$-vector space on $[x]_n$, x in F, $x \neq 0, 1$, modulo some inductively defined relations.

Complex $\Gamma(F, n)$ in cohomological degrees $1, \ldots, n$ for $n \geq 2$:

$$B_n(F) \to B_{n-1}(F) \otimes F^*_{\mathbb{Q}} \to \ldots \to B_2(F) \otimes \bigwedge^{n-2} F^*_{\mathbb{Q}} \to \bigwedge^{n} F^*_{\mathbb{Q}}$$

$$\mathrm{d}[x]_l \otimes y_1 \wedge \ldots \wedge y_{n-l} = [x]_{l-1} \otimes x \wedge y_1 \wedge \ldots \wedge y_{n-l} \quad (l \geq 3)$$
$$\mathrm{d}[x]_2 \otimes y_1 \wedge \ldots \wedge y_{n-2} = (1-x) \wedge x \wedge y_1 \wedge \ldots \wedge y_{n-2}$$

Conjecture (Goncharov)
$H^p(\Gamma(n, F)) \cong K^{(n)}_{2n-p}(F)$ for all $n \geq 2$ and all p.

Let C a complete nonsingular curve over a field k. Then one has an Adams decomposition

$$K_n(C) \otimes_{\mathbb{Z}} \mathbb{Q} = K_n^{(2)}(C) \oplus \cdots \oplus K_n^{(n+1)}(C)$$

for $n \geq 2$.

$F = k(C)$: the field of rational functions on C.
$\Gamma(n, C)$: total complex associated to double complex, in cohomological degrees $1, \ldots, n+1$,

$$\begin{array}{ccccc} B_n(F) & \longrightarrow & B_{n-1}(F) \otimes F_{\mathbb{Q}}^* & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \coprod B_{n-1}(k(x)) & \longrightarrow & \cdots \end{array}$$

with the coproduct over the closed points x in C. The vertical maps are "based on" $[f]_m \otimes g \mapsto \mathrm{ord}_x(g) \cdot [f(x)]_m$ with $[0]_m = [\infty]_m = 0$.

Conjecture (Goncharov)
$H^p(\Gamma(n, C)) \cong K_{2n-p}^{(n)}(C)$ for all $n \geq 2$ and all p.

Some results
(1) There exists maps $K_4^{(3)}(F) \to H^2(\Gamma(3, F))$ (RdJ) and $K_4^{(3)}(F) \to H^2(\Gamma(F, 3))$ (Goncharov);
(2) if k is a number field, we get a complete description of the image of the regulator map of $K_4(C)$;
[only $K_4^{(3)}(C)$ contributes]
(3) we also get a complete description of the image of the regulator map of $K_6(C)$, although the results corresponding to (1) for $K_6^{(4)}(F)$ are less precise.