$K_4$ of curves and syntomic regulators

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The goal of this talk is to describe joint work with Amnon Besser regarding the syntomic regulator on a particular subspace of $K_4^{(3)}(C)$ (conjecturally all of $K_4^{(3)}(C)$) for a smooth, geometrically irreducible curve $C$ over a number field $k$. 
\( C \) smooth geometrically irreducible curve over a number field \( k \).

Fix \( k \subseteq \mathbb{C} \)

Let \( F = k(C) \), \( \mathbb{C}(C) = \mathbb{C}(C_{\mathbb{C}}) \)

Define

\[
K_2(F) \to H^1_{dR}(\mathbb{C}(C), \mathbb{R}(1)) = \lim_{\substack{\cdot \to 0 \\ \cdot \subseteq C_{\mathbb{C}} \setminus \text{finite}}} H^1_{dR}(U; \mathbb{R}(1))
\]

\[
\{ f, g \} \mapsto \log |f| \, d\arg g - \log |g| \, d\arg f
\]

This works as

\[
\log |z| \, d\arg(1 - z) - \log |1 - z| \, d\arg z = dP_2(z),
\]

\( P_2(z) \) a \( C^\infty \)-function on \( \mathbb{C} \setminus \{0, 1\} \)

For \( \omega \) in \( H^0(\mathbb{C}, \Omega^1) \), the composition

\[
K_2(C) \longrightarrow H^1_{dR}(\mathbb{C}(C), \mathbb{R}(1)) \stackrel{\int_{C_{\mathbb{C}}} \cdot \wedge \omega}{\longrightarrow} \mathbb{C}
\]

is given by

\[
\{ f, g \} \mapsto \int_{C_{\mathbb{C}}} (\log |f| \, d\arg g - \log |g| \, d\arg f) \wedge \omega,
\]

which extends over the map \( K_2(C) \to K_2(F) \).
Let $C$ be a complete, smooth, geometrically irreducible curve over a number field $k$
$F = k(C)$.

$\tilde{\mathcal{M}}_{(3)}(F)$ is the cohomological complex

$$
\tilde{M}_3(F) \xrightarrow{\text{d}} \tilde{M}_2(F) \otimes_{\mathbb{Q}} F_{\mathbb{Q}}^* \xrightarrow{\text{d}} \bigwedge^3 F_{\mathbb{Q}}^*
$$

in degrees 1, 2, 3

$d[f]_3 = [f]_2 \otimes f$

$d([f]_2 \otimes g) = (1 - f) \wedge f \wedge g$

[For any field $F$ of characteristic zero, $\tilde{M}_j(F)$ is a $\mathbb{Q}$–vector space

generated by symbols $[z]_j$ with $z$ in $F^*$, and unknown relations

(which include $[z]_n + (-1)^n [z^{-1}]_n = 0$) and so $[1]_2 = 0$.]

$\tilde{M}_{(3)}(C)$: the total complex associated to the double complex

$$
\begin{array}{cccc}
\tilde{M}_3(F) & \xrightarrow{\text{d}} & \tilde{M}_2(F) \otimes_{\mathbb{Q}} F_{\mathbb{Q}}^* & \xrightarrow{\text{d}} & \bigwedge^3 F_{\mathbb{Q}}^* \\
\downarrow \partial_1 & & \downarrow \partial_1 & & \\
0 & \longrightarrow & \bigsqcup_x \tilde{M}_2(k(x)) & \xrightarrow{\text{d}} & \bigsqcup_x \bigwedge^2 k(x)^*_{\mathbb{Q}}
\end{array}
$$

[coproduct over all (closed) points $x$ in $C$]

with the top row being $\tilde{M}_{(3)}(F)$

$d[z]_2 = (1 - z) \land z$

$\partial_{1,x}([f]_2 \otimes g) = \text{ord}_x(g) \cdot [f(x)]_2$ (with $[0]_2 = [\infty]_2 = 0$)

$\partial_{2,x}$ determined by ($\pi$ a uniformizer at $x$, $u_j$ units at $x$):

$\pi \land u_1 \land u_2 \mapsto u_1(x) \land u_2(x)$

$u_1 \land u_2 \land u_3 \mapsto 0$
Similar complexes $\mathcal{M}_3(F)$

\[
\begin{array}{ccc}
M_3(F) & \xrightarrow{d} & M_2(F) \otimes_{\mathbb{Q}} F_{\mathbb{Q}}^* \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\partial_1} & \coprod_x \widetilde{M}_2(k(x)) \\
\downarrow & & \downarrow \\
& \xrightarrow{\partial_2} & \coprod_x \wedge^2 k(x)^*_\mathbb{Q}
\end{array}
\]

$[\text{coproduct over all (closed) points } x \text{ in } C]$

$\mathcal{M}_3(F)$ in top row

vertical maps induced from quotient map $\mathcal{M}_3(F) \rightarrow \widetilde{\mathcal{M}}_3(F)$

$$d[f]_3 = [f]_2 \otimes f$$ $$d([f]_2 \otimes g) = (1 - f) \otimes f \wedge g$$

$$\widetilde{M}_n(F) = M_n(F)/<[f]_n + (-1)^n [f^{-1}]_n>$$
There are maps:

\[
K_4^{(3)}(C) \hookrightarrow K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F_Q^* \longrightarrow K_4^{(3)}(F)
\]

\[
H^2(\mathcal{M}_{(3)}(C)) \longrightarrow H^2(\mathcal{M}_{(3)}(F))
\]

\[
\cong \quad \cong
\]

\[
H^2(\widetilde{\mathcal{M}}_{(3)}(C)) \longrightarrow H^2(\widetilde{\mathcal{M}}_{(3)}(F))
\]

Fix embedding \( k \subset \mathbb{C} \), and \( \omega \) in \( H^0(C \otimes_k \mathbb{C}, \Omega^1_{/\mathbb{C}}) \). Then the regulator for Deligne cohomology

\[
K_4^{(3)}(C) \rightarrow H^2_D(C_{\mathbb{C}}, \mathbb{R}(3)) \cong H^1_{dR}(C_{\mathbb{C}}; \mathbb{R}(2))
\]

followed by

\[
\int \cdot \wedge \omega : H^1_{dR}(C_{\mathbb{C}}; \mathbb{R}(2)) \rightarrow \mathbb{C}
\]

extends over

\[
K_4^{(3)}(C) \rightarrow K_4^{(3)}(F).
\]

The resulting map

\[
H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \longrightarrow \mathbb{C}
\]

is induced by

\[
\widetilde{M}_2(F) \otimes F_Q^* \rightarrow \mathbb{C}
\]

\[
[g]_2 \otimes f \mapsto \pm \frac{8}{3} \int_{C\mathbb{C}} P_2 \circ g \, d \log |f| \wedge \omega
\]
**Coleman integration**

\[ \mathbb{C}_p = \hat{\mathbb{Q}}_p \]

\(| \cdot |: p\text{-adic valuation with } |p| = p^{-1} \]

\(\mathcal{O}: \text{ring of integers of } \mathbb{C}_p \)

\(\mathbb{F}_p: \text{residue field} \)

\(X/\mathcal{O}: \text{smooth curve over } \mathcal{O} \)

(=smooth projective surjective scheme of relative dimension 1)

For \( x \) in \( X(\mathbb{F}_p) \), put

\( U_x = \text{residue disc of } x = \{ \text{all pts in } X(\mathbb{C}_p) \text{ reducing to } x \} \), a copy of the maximal ideal of \( \mathcal{O} \).

\( Y \subseteq X_{\overline{\mathbb{F}}_p} \text{ nonempty open affine subscheme, smooth over } \overline{\mathbb{F}}_p, \) so

\( X(\overline{\mathbb{F}}_p) = Y(\overline{\mathbb{F}}_p) \bigsqcup \{ e_1, \ldots, e_n \}. \)

\( U_r = \text{rigid space obtained by removing discs of radius } r < 1 \)

from \( X(\mathbb{C}_p) \) for all \( e_i; \ e_i \text{ locally given by } \overline{h} = 0, \) so leave out \( |h| \leq r. \)

\( U \ \text{“} \lim_{r \uparrow 1} U_r \text{ is independent of the choices, a basic wide open} \)

in the sense of Coleman (i.e. for \( \cdots \) on \( U \) work in \( \lim_{r \uparrow 1} \cdots (U_r) \))
Make a choice of logarithm log : \( \mathbb{C}_p^* \to \mathbb{C}_p \) such that

1. \( \log(ab) = \log a + \log b \)
2. \( \log(1 + z) = \) usual powerseries expansion for \( |z| < 1 \)
   (I.e., fix a choice of \( \log p \).)

For \( x \in Y(\mathbb{F}_p) \), put
\[
A(U_x) = \left\{ \sum_{n=0}^{\infty} a_n z^n \text{ conv. for } |z| < 1 \right\}
\]
\[
A_{\log}(U_x) = A(U_x)
\]
\[
\Omega_{\log}(U_x) = A_{\log}(U_x)dz
\]
\( [z = z_x \text{ is a local parameter on } U_x] \)

For \( x \notin Y(\mathbb{F}_p) \) (i.e., \( x = e_1, \ldots, e_n \), the ends), put
\[
A(U_x) = \left\{ \sum_{n=-\infty}^{\infty} a_n z^n \text{ conv. for } r < |z| < 1, \text{ some } r < 1 \right\}
\]
\[
A_{\log}(U_x) = A(U_x)[\log z]
\]
\[
\Omega_{\log}(U_x) = A_{\log}(U_x)dz
\]

Put
\[
A_{loc}(U) = \prod_{x \in X(\mathbb{F}_p)} A_{\log}(U_x)
\]
(locally analytic functions, with choice of logs around the \( e_i \))
\[
\Omega_{loc}(U) = \prod_{x \in X(\mathbb{F}_p)} \Omega_{\log}(U_x)
\]
(locally analytic forms, with choice of log around the \( e_i \))

\[
0 \xrightarrow{\prod_{x \in X(\mathbb{F}_p)} \mathbb{C}_p} A_{loc}(U) \xrightarrow{d} \Omega_{loc}(U) \to 0
\]
is exact by reduction formulae for \( \int z^n \log^k z dz \) as \( d \log z = \frac{dz}{z} \).
If $X_{\overline{\mathbb{F}_p}}$ is defined over $\mathbb{F}_q$, so $X' \times_{\overline{\mathbb{F}_q}} \overline{\mathbb{F}_p}$ for some $X'/\mathbb{F}_q$, let $\phi$ be the geometric Frobenius of $X'/\mathbb{F}_q$.

**Coleman:**

1. There exists a lift $\varphi$ of $\phi$ to $U$ (coming from a rigid analytic map $\phi : U_r \to U_s$ for some $s < r < 1$).
2. There exists a subspace $A_{\text{Col}}(U)$ of $A_{\text{loc}}(U)$, containing the rigid analytic functions $A(U)$ on $U$, such that with $\Omega_{\text{Col}}(U) = A_{\text{Col}}(U) \cdot \Omega^1(U)$

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathbb{C}_p & \longrightarrow & A_{\text{Col}}(U) & \longrightarrow & \Omega_{\text{Col}}(U) & \longrightarrow & 0
\end{array}
\]

is exact.

Denote $\int : \Omega_{\text{Col}}(U) \to A_{\text{Col}}(U) \mod \mathbb{C}_p$ by $\omega \mapsto \int \omega$ or $\omega \mapsto F_\omega$. Then

(a) $dF_\omega = \omega$

(b) $\int \varphi^*(\omega) = \varphi^*(\int \omega)$

[A_{\text{Col}} is independent of $\varphi$.]

Let $P$ and $Q$ be in $U$, $\omega$ in $\Omega_{\text{Col}}(U)$, and $F_\omega$ in $A_{\text{Col}}(U)$ with $dF_\omega = \omega$. Put $\int^Q_P \omega = F_\omega(Q) - F_\omega(P)$.

More generally, if $D = \sum_i a_i P_i$ with all $P_i$ in $U$ and $\sum_i a_i = 0$, put $\int_D \omega = \sum_i a_i F(P_i)$. 

8
Example

\[ X = \mathbb{P}^1_{\mathbb{C}_p} \]
\[ Y = \mathbb{P}^1_{\mathbb{F}_p} \setminus \{1, \infty\} \]
\[ U \text{ “=” } \mathbb{P}^1_{\mathbb{C}_p} \setminus \bigcup_{U_1 \cup \cdots \cup U_\infty} \]
\[ \phi(z) = z^p \text{ (working over } \mathbb{F}_p) \]
\[ \varphi(z) = (z - 1)^p + 1 \]

Put \( L_{i_{n+1}}(z) = \int_{0}^{z} L_{i_n}(z) \, d\log z \) starting with \( L_{i_0}(z) = \frac{z}{1-z} \).

\[ L_{i_n}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \text{ for } |z| < 1. \]

[In fact, \( L_{i_n}(z) \) extends naturally to \( \mathbb{C}_p \setminus \{1\} \).]

\[ L_2(z) = L_{i_2}(z) + \log z \cdot \log(1 - z) \]
\[ L_{2, \text{mod}}(z) = L_{i_2}(z) + \frac{1}{2} \log z \cdot \log(1 - z) \]

\( L_2 \) does not satisfy a nice functional equation for \( z \) versus \( z^{-1} \), but

\[ L_{2, \text{mod}}(z) + L_{2, \text{mod}}(z^{-1}) = 0. \]

Use Coleman integration for regulators in the following context:

\( K \) complete discrete valuation subfield of \( \mathbb{C}_p \)
\( C \) smooth, complete, geometrically irreducible curve over number field \( k \subset K \) with good reduction at the corresponding valuation ideal.
\( \mathcal{O}_k \) valuation ring in \( k \)
\( \mathcal{C} \) smooth proper model of \( C \) over \( \mathcal{O}_k \)
\( \omega \) in \( H^0(C, \Omega^1_{/K}) \)
Theorem (Besser) With \( \text{reg}_{\text{syn}} \) the syntomic regulator, the composition

\[
K_2^{(2)}(C) \cong K_2^{(2)}(C) \overset{\text{reg}_{\text{syn}}}{\longrightarrow} H^2_{\text{syn}}(C, (2)) \cong H^1_{\text{dR}}(C/K) \overset{\text{Tr}(\cdot \cup \omega)}{\longrightarrow} K
\]

factorizes through

\[
K_2^{(2)}(C) \to K_2^{(2)}(F'),
\]

and then is induced by

\[
K_2^{(2)}(F) \to K
\]

\[
\{f, g\} \mapsto \int_{(f)} \log(g)\omega.
\]

\( \int \log(g)\omega(x) \) is given by the constant term at \( x \): if \( \int \log(g)\omega = \sum_j f_j(z_x) \log^j z_x \), the constant term is \( f_0(0) \).

For \( \int \log(g)\omega \), the result is independent of the choice of a uniformizing parameter \( z_x \) at \( x \): fix \( \int \log(g)\omega \), fix \( F_\omega \) such that \( F_\omega(x) = 0 \), and define \( \int F_\omega \log(g) \) via \( \int \log(g)\omega = \log(g)F_\omega - \int F_\omega \log(g) \). Then \( \log(g)F_\omega \) has constant term zero for any \( z_x \), and \( F_\omega \log(g) \) is holomorphic at \( x \), so its integral can be evaluated at \( x \) to give the constant term.
Theorem (Besser+RdJ)

\[ H^2(\mathcal{M}_3(C)) \rightarrow K_4^{(3)}(C) \cong K_4^{(3)}(C) \rightarrow \]

\[ \rightarrow H^2_{syn}(C, (3)) \cong H^1_{dR}(C/K) \xrightarrow{\text{Tr}(\cdot \cup \omega)} K \]

is given by

\[ \sum_i [g_i]_2 \otimes f_i \mapsto 2 \sum_i \int_{(f_i)} L_2(g_i) \omega, \]

provided that none of the \( f_i, g_i \) and \( 1 - g_i \) have a zero or pole along the special fibre of \( C \).

[The Coleman integral is carried out on a wide open space \( U \) on which \( f_i, g_i \) and \( 1 - g_i \) are holomorphic.]

\( (\int L_2(g) \omega)(x) \) is given by the constant term.

For \( \int L_2(g) \omega \) that is again independent of the choice of local parameter \( z_x \).
Back to a single residue disc, an end $e$. For three functions $F, G, H$ in $A(U_e) + K \cdot \log z$, fix for each two functions $R$ and $S$ out of $F, G, H$ a choice of $\int R dS$ (i.e., a function in $A_{\log}(U_e)$ whose differential is $RdS$) and of $\int S dR$ in such a way that

$$\int R dS + \int S dR = RS.$$ 

Denote all this data (including the auxiliary data $\int FdG$ etc.) by $(F, G; H)$.

**Proposition (Besser+RdJ)**

There exists a unique function $(F, G; H) \rightarrow \langle F, G; H \rangle$, called the triple index on the end $e$, from data as above to $K$, satisfying

1. Multilinearity - the triple index is linear in each of the three variables (with linear choices of $\int R dS$ etc.).
2. Symmetry - we have $\langle F, G; H \rangle = \langle G, F; H \rangle$ (the $\int FdG$ etc. must also be swapped).
3. Triple identity - We have, again with the obvious additional choices,

   $\langle F, G; H \rangle + \langle F, H; G \rangle + \langle G, H; F \rangle = 0.$

4. Some compatibility for changing the auxiliary data.
5. Reduction to double index - if $G$ is in $A(U_e)$, then

   $$\langle F, G; H \rangle = \left\langle F, \int GdH \right\rangle,$$

   where $\int GdH$ is from the auxiliary data and is in $A(U_e) + K \cdot \log z$ because by assumption $GdH$ is in $A(U_e) \cdot dz$. 

12
The double index

\[ \langle \cdot, \cdot \rangle : (A(U_e) + K \cdot \log z) \times (A(U_e) + K \cdot \log z) \rightarrow K \]

is the unique alternating \( K \)-linear map such that

\[ \langle F, G \rangle = \text{Res} FdG \]

if \( F \) is in \( A(U_e) \) and \( G \) is in \( A(U_e) + K \cdot \log z \).

Furthermore, if \( F, G \) and \( H \) are Coleman functions on a basic wide open \( U \), such that \( dF \) and \( dG \) are holomorphic on \( U \), \( dH \) is holomorphic on each end \( U_{e_j} \), and we use Coleman integrals for \( \int FdG \) etc., then the global index

\[ \langle F, G; H \rangle_{gl} = \sum_{e \in \text{Ends}(U)} \langle F, G; H \rangle_e \]

depends only on \( F, G \) and \( H \) and not on the auxiliary data.

**Theorem (Besser+RdJ)**

If we allow \( \omega \) to be a form of the second kind on \( C_K \) in the previous Theorem, then the composition is given by

\[ \sum_i [g_i]_2 \otimes f_i \mapsto \left\langle \log(f_i), \log(g_i); \int F_\omega d\log(1 - g_i) \right\rangle_{gl}, \]

where \( F_\omega \) is any Coleman integral of \( \omega \) and the sum of triple indices is done with respect to a wide open space \( U \) on which all \( f_i, g_i, 1 - g_i \) as well as \( \omega \) are holomorphic.