K_4 of curves and syntomic regulators

R. de Jeu

University of Durham

email: rob.de-jeu@durham.ac.uk

website: http://www.maths.dur.ac.uk/~dma0rdj

The goal of this talk is to describe joint work with Amnon Besser regarding the syntomic regulator on a particular subspace of $K_4^{(3)}(C)$ (conjecturally all of $K_4^{(3)}(C)$) for a smooth, geometrically irreducible curve C over a number field k.

C smooth geometrically irreducible curve over a number field k.
Fix $k \subset \mathbb{C}$
Let $F = k(C)$, $\mathbb{C}(C) = \mathbb{C}(C_{\mathbb{C}})$

Define

$$K_2(F) \to H^1_{\mathrm{dR}}(\mathbb{C}(C), \mathbb{R}(1)) = \varinjlim_{\substack{U \subset C_{\mathbb{C}} \\ C_{\mathbb{C}} \setminus U \,\text{finite}}} H^1_{\mathrm{dR}}(U; \mathbb{R}(1))$$

$$\{f, g\} \mapsto \log|f| \mathrm{d}i \arg g - \log|g| \mathrm{d}i \arg f$$

This works as

$$\log|z| \mathrm{d}i \arg(1-z) - \log|1-z| \mathrm{d}i \arg z = \mathrm{d}P_2(z),$$

$P_2(z)$ a C^∞–function on $\mathbb{C} \setminus \{0, 1\}$
For ω in $H^0(C_{\mathbb{C}}, \Omega^1)$, the composition

$$K_2(C) \longrightarrow H^1_{\mathrm{dR}}(\mathbb{C}(C), \mathbb{R}(1)) \xrightarrow{\int_{C_{\mathbb{C}}} \cdot \wedge\ \omega} \mathbb{C}$$

is given by

$$\{f, g\} \mapsto \int_{C_{\mathbb{C}}} (\log|f| \mathrm{d}i \arg g - \log|g| \mathrm{d}i \arg f) \wedge \omega,$$

which extends over the map $K_2(C) \to K_2(F)$.

Let C be a complete, smooth, geometrically irreducible curve over a number field k
$F = k(C)$.

$\widetilde{\mathcal{M}}_{(3)}(F)$ is the cohomological complex

$$\widetilde{M}_3(F) \xrightarrow{\mathrm{d}} \widetilde{M}_2(F) \otimes_{\mathbb{Q}} F^*_{\mathbb{Q}} \xrightarrow{\mathrm{d}} \bigwedge^3 F^*_{\mathbb{Q}}$$

in degrees 1, 2, 3
$\mathrm{d}[f]_3 = [f]_2 \otimes f$
$\mathrm{d}([f]_2 \otimes g) = (1-f) \wedge f \wedge g$

[For any field F of characteristic zero, $\widetilde{M}_j(F)$ is a $\mathbb{Q}$–vector space generated by symbols $[z]_j$ with z in F^*, and unknown relations (which include $[z]_n + (-1)^n[z^{-1}]_n = 0$) and so $[1]_2 = 0$.]

$\widetilde{\mathcal{M}}_{(3)}(C)$: the total complex associated to the double complex

$$\begin{array}{ccccc}
\widetilde{M}_3(F) & \xrightarrow{\mathrm{d}} & \widetilde{M}_2(F) \otimes_{\mathbb{Q}} F^*_{\mathbb{Q}} & \xrightarrow{\mathrm{d}} & \bigwedge^3 F^*_{\mathbb{Q}} \\
\downarrow & & \downarrow \partial_1 & & \downarrow \partial_2 \\
0 & \longrightarrow & \coprod_x \widetilde{M}_2(k(x)) & \xrightarrow{\mathrm{d}} & \coprod_x \bigwedge^2 k(x)^*_{\mathbb{Q}}
\end{array}$$

[coproduct over all (closed) points x in C]
with the top row being $\widetilde{\mathcal{M}}_{(3)}(F)$
$\mathrm{d}[z]_2 = (1-z) \wedge z$
$\partial_{1,x}([f]_2 \otimes g) = \mathrm{ord}_x(g) \cdot [f(x)]_2$ (with $[0]_2 = [\infty]_2 = 0$)
$\partial_{2,x}$ determined by (π a uniformizer at x, u_j units at x):
$\pi \wedge u_1 \wedge u_2 \mapsto u_1(x) \wedge u_2(x)$
$u_1 \wedge u_2 \wedge u_3 \mapsto 0$

Similar complexes $\mathcal{M}_{(3)}(F)$

$$M_3(F) \xrightarrow{\mathrm{d}} M_2(F) \otimes_{\mathbb{Q}} F_{\mathbb{Q}}^* \xrightarrow{\mathrm{d}} F_{\mathbb{Q}}^* \otimes \bigwedge^2 F_{\mathbb{Q}}^*$$

$\mathrm{d}[f]_3 = [f]_2 \otimes f$
$\mathrm{d}([f]_2 \otimes g) = (1-f) \otimes f \wedge g$

$\widetilde{M}_n(F) = M_n(F)/{<}[f]_n + (-1)^n[f^{-1}]_n{>}$

$\mathcal{M}_{(3)}(C)$ is the total complex associated to the double complex

$$\begin{array}{ccccc}
M_3(F) & \xrightarrow{\mathrm{d}} & M_2(F) \otimes_{\mathbb{Q}} F_{\mathbb{Q}}^* & \xrightarrow{\mathrm{d}} & F_{\mathbb{Q}}^* \otimes \bigwedge^2 F_{\mathbb{Q}}^* \\
\downarrow & & \partial_1 \downarrow & & \partial_2 \downarrow \\
0 & \longrightarrow & \coprod_x \widetilde{M}_2(k(x)) & \xrightarrow{\mathrm{d}} & \coprod_x \bigwedge^2 k(x)_{\mathbb{Q}}^*
\end{array}$$

[coproduct over all (closed) points x in C]

$\mathcal{M}_{(3)}(F)$ in top row
vertical maps induced from quotient map $\mathcal{M}_{(3)}(F) \to \widetilde{\mathcal{M}}_{(3)}(F)$

There are maps:

$$\begin{array}{ccccc}
K_4^{(3)}(C) & \longleftarrow & K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}} & \longrightarrow & K_4^{(3)}(F) \\
 & & \uparrow & & \uparrow \\
 & & H^2(\mathcal{M}_{(3)}(C)) & \longrightarrow & H^2(\mathcal{M}_{(3)}(F)) \\
 & & \downarrow \cong & & \downarrow \cong \\
 & & H^2(\widetilde{\mathcal{M}}_{(3)}(C)) & \longrightarrow & H^2(\widetilde{\mathcal{M}}_{(3)}(F))
\end{array}$$

Fix embedding $k \subset \mathbb{C}$, and ω in $H^0(C \otimes_k \mathbb{C}, \Omega^1_{/\mathbb{C}})$. Then the regulator for Deligne cohomology

$$K_4^{(3)}(C) \to H^2_{\mathcal{D}}(C_{\mathbb{C}}, \mathbb{R}(3)) \cong H^1_{\mathrm{dR}}(C_{\mathbb{C}}; \mathbb{R}(2))$$

followed by

$$\int \cdot \wedge\ \omega : H^1_{\mathrm{dR}}(C_{\mathbb{C}}; \mathbb{R}(2)) \to \mathbb{C}$$

extends over

$$K_4^{(3)}(C) \to K_4^{(3)}(F).$$

The resulting map

$$H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \longrightarrow \mathbb{C}$$

is induced by

$$\begin{aligned}
\widetilde{M}_2(F) \otimes F^*_{\mathbb{Q}} &\to \mathbb{C} \\
[g]_2 \otimes f &\mapsto \pm \frac{8}{3} \int_{C_{\mathbb{C}}} P_2 \circ g \ \mathrm{d}\log|f| \wedge \omega
\end{aligned}$$

Coleman integration

$\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$
$|\cdot|$: p–adic valution with $|p| = p^{-1}$
$\mathcal{O}$: ring of integers of $\mathbb{C}_p$
$\overline{\mathbb{F}}_p$: residue field

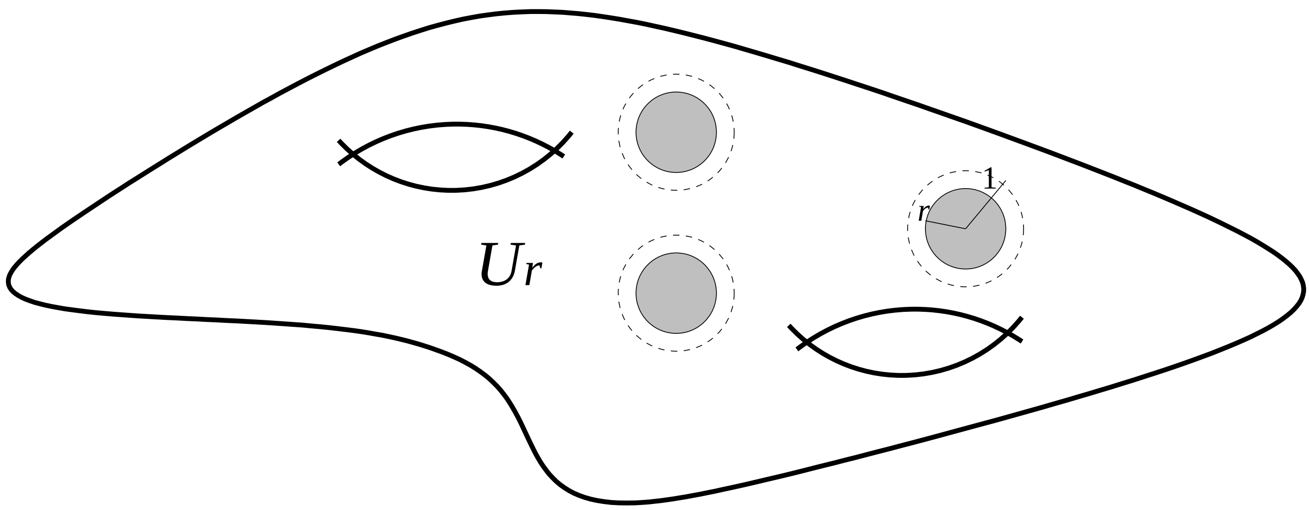
$X/\mathcal{O}$: smooth curve over $\mathcal{O}$
(=smooth projective surjective scheme of relative dimension 1)

For x in $X(\overline{\mathbb{F}}_p)$, put
U_x = residue disc of x = $\{$all pts in $X(\mathbb{C}_p)$ reducing to $x\}$, a copy of the maximal ideal of $\mathcal{O}$.

$Y \subseteq X_{\overline{\mathbb{F}}_p}$ nonempty open affine subscheme, smooth over $\overline{\mathbb{F}}_p$, so $X(\overline{\mathbb{F}}_p) = Y(\overline{\mathbb{F}}_p) \coprod \{e_1, \ldots, e_n\}$.

U_r = rigid space obtained by removing discs of radius $r < 1$ from $X(\mathbb{C}_p)$ for all e_i: e_i locally given by $\overline{h} = 0$, so leave out $|h| \leq r$.

U "=" $\varprojlim_{r \uparrow 1} U_r$ is independent of the choices, a basic wide open in the sense of Coleman (i.e. for $\cdots$ on U work in $\varinjlim_{r \uparrow 1} \cdots (U_r)$)



Make a choice of logarithm $\log : \mathbb{C}_p^* \to \mathbb{C}_p$ such that

(1) $\log ab = \log a + \log b$

(2) $\log(1+z) =$ usual powerseries expansion for $|z|$ small.

(I.e., fix a choice of $\log p$.)

For $x \in Y(\overline{\mathbb{F}}_p)$, put

$A(U_x) = \{\sum_{n=0}^{\infty} a_n z^n \text{ conv. for } |z| < 1\}$

$A_{\log}(U_x) = A(U_x)$

$\Omega_{\log}(U_x) = A_{\log}(U_x)\mathrm{d}z$

[$z = z_x$ is a local parameter on U_x.]

For $x \notin Y(\overline{\mathbb{F}}_p)$ (i.e., $x = e_1, \ldots, e_n$, the *ends*), put

$A(U_x) = \{\sum_{n=-\infty}^{\infty} a_n z^n \text{ conv. for } r < |z| < 1, \text{ some } r < 1\}$

$A_{\log}(U_x) = A(U_x)[\log z]$

$\Omega_{\log}(U_x) = A_{\log}(U_x)\mathrm{d}z$

Put

$A_{\mathrm{loc}}(U) = \prod_{x \in X(\overline{\mathbb{F}}_p)} A_{\log}(U_x)$

(locally analytic functions, with choice of logs around the e_i)

$\Omega_{\mathrm{loc}}(U) = \prod_{x \in X(\overline{\mathbb{F}}_p)} \Omega_{\log}(U_x)$

(locally analytic forms, with choice of log around the e_i)

$$0 \longrightarrow \prod_{x \in X(\overline{\mathbb{F}}_p)} \mathbb{C}_p \longrightarrow A_{\mathrm{loc}}(U) \xrightarrow{\mathrm{d}} \Omega_{\mathrm{loc}}(U) \longrightarrow 0$$

is exact by reduction formulae for $\int z^n \log^k z \mathrm{d}z$ as $\mathrm{d}\log z = \frac{\mathrm{d}z}{z}$.

If $X_{\overline{\mathbb{F}}_p}$ is defined over $\mathbb{F}_q$, so $X' \times_{\mathbb{F}_q} \overline{\mathbb{F}}_p$ for some $X'/\mathbb{F}_q$, let ϕ be the geometric Frobenius of $X'/\mathbb{F}_q$.

Coleman:

(1) There exists a lift φ of ϕ to U (coming from a rigid analytic map $\phi : U_r \to U_s$ for some $s < r < 1$).

(2) There exists a subspace $A_{\mathrm{Col}}(U)$ of $A_{\mathrm{loc}}(U)$, containing the rigid analytic functions $A(U)$ on U, such that with $\Omega_{\mathrm{Col}}(U) = A_{\mathrm{Col}}(U) \cdot \Omega^1(U)$

$$0 \longrightarrow \mathbb{C}_p \longrightarrow A_{\mathrm{Col}}(U) \xrightarrow{\mathrm{d}} \Omega_{\mathrm{Col}}(U) \longrightarrow 0$$

is exact.

Denote $\int : \Omega_{\mathrm{Col}}(U) \to A_{\mathrm{Col}}(U) \bmod \mathbb{C}_p$ by $\omega \mapsto \int \omega$ or $\omega \mapsto F_\omega$. Then

(a) $\mathrm{d}F_\omega = \omega$

(b) $\int \varphi^*(\omega) = \varphi^*(\int \omega)$

[A_{Col} is independent of φ.]

Let P and Q be in U, ω in $\Omega_{\mathrm{Col}}(U)$, and F_ω in $A_{\mathrm{Col}}(U)$ with $\mathrm{d}F_\omega = \omega$. Put $\int_P^Q \omega = F_\omega(Q) - F_\omega(P)$.

More generally, if $D = \sum_i a_i P_i$ with all P_i in U and $\sum_i a_i = 0$, put $\int_D \omega = \sum_i a_i F(P_i)$.

Example

$X = \mathbb{P}^1_{\mathbb{C}_p}$
$Y = \mathbb{P}^1_{\overline{\mathbb{F}}_p} \setminus \{1, \infty\}$
U "=" $\mathbb{P}^1_{\mathbb{C}_p} \setminus U_1 \coprod U_\infty$
$\phi(z) = z^p$ (working over $\mathbb{F}_p$)
$\varphi(z) = (z-1)^p + 1$
Put $Li_{n+1}(z) = \int_0^z Li_n(z) \mathrm{d} \log z$ starting with $Li_0(z) = \frac{z}{1-z}$.

$Li_n(z) = \sum_{k=1}^\infty \frac{z^k}{k^n}$ for $|z| < 1$.

[In fact, $Li_n(z)$ extends naturally to $\mathbb{C}_p \setminus \{1\}$.]

$$L_2(z) = Li_2(z) + \log z \cdot \log(1-z)$$

$$L_{2,\mathrm{mod}}(z) = Li_2(z) + \frac{1}{2} \log z \cdot \log(1-z)$$

L_2 does not satisfy a nice functional equation for z versus z^{-1}, but

$$L_{2,\mathrm{mod}}(z) + L_{2,\mathrm{mod}}(z^{-1}) = 0.$$

Use Coleman integration for regulators in the following context:

K complete discrete valuation subfield of $\mathbb{C}_p$
C smooth, complete, geometrically irreducible curve over number field $k \subset K$ with good reduction at the corresponding valuation ideal.
$\mathcal{O}_k$ valuation ring in k
$\mathcal{C}$ smooth proper model of C over $\mathcal{O}_k$
ω in $H^0(C, \Omega^1_{/K})$

Theorem (Besser) With $\mathrm{reg}_{\mathrm{syn}}$ the syntomic regulator, the composition

$$K_2^{(2)}(C) \cong K_2^{(2)}(\mathcal{C}) \xrightarrow{\mathrm{reg}_{\mathrm{syn}}} H^2_{\mathrm{syn}}(\mathcal{C},(2)) \cong H^1_{\mathrm{dR}}(C/K) \xrightarrow{\mathrm{Tr}(\cdot\cup\omega)} K$$

factorizes through

$$K_2^{(2)}(C) \to K_2^{(2)}(F),$$

and then is induced by

$$\begin{aligned} K_2^{(2)}(F) &\to K \\ \{f,g\} &\mapsto \int_{(f)} \log(g)\omega. \end{aligned}$$

$\int \log(g)\omega(x)$ is given by the constant term at x: if $\int \log(g)\omega = \sum_j f_j(z_x)\log^j z_x$, the constant term is $f_0(0)$.

For $\int \log(g)\omega$, the result is independent of the choice of a uniformizing parameter z_x at x: fix $\int \log(g)\omega$, fix F_ω such that $F_\omega(x)=0$, and define $\int F_\omega \mathrm{d}\log(g)$ via $\int \log(g)\omega = \log(g)F_\omega - \int F_\omega \mathrm{d}\log(g)$. Then $\log(g)F_\omega$ has constant term zero for any z_x, and $F_\omega \mathrm{d}\log(g)$ is holomorphic at x, so its integral can be evaluated at x to give the constant term.

Theorem (Besser+RdJ)

$$H^2(\mathcal{M}_{(3)}(C)) \longrightarrow K_4^{(3)}(C) \cong K_4^{(3)}(\mathcal{C}) \longrightarrow$$

$$\longrightarrow H^2_{\mathrm{syn}}(\mathcal{C},(3)) \cong H^1_{\mathrm{dR}}(C/K) \xrightarrow{\mathrm{Tr}(\cdot\cup\omega)} K$$

is given by

$$\sum_i [g_i]_2 \otimes f_i \mapsto 2\sum_i \int_{(f_i)} L_2(g_i)\omega,$$

provided that none of the f_i, g_i and $1-g_i$ have a zero or pole along the special fibre of $\mathcal{C}$.

[The Coleman integral is carried out on a wide open space U on which f_i, g_i and $1-g_i$ are holomorphic.]
$\left(\int L_2(g)\omega\right)(x)$ is given by the constant term.
For $\int L_2(g)\omega$ that is again independent of the choice of local parameter z_x.

Back to a single residue disc, an end e. For three functions F, G, H in $A(U_e) + K \cdot \log z$, fix for each two functions R and S out of F, G, H a choice of $\int RdS$ (i.e., a function in $A_{\log}(U_e)$ whose differential is RdS) and of $\int SdR$ in such a way that

$$\int RdS + \int SdR = RS \ .$$

Denote all this data (including the *auxiliary data* $\int F\mathrm{d}G$ etc.) by $(F, G; H)$.

Proposition (Besser+RdJ)

There exists a unique function $(F, G; H) \to \langle F, G; H \rangle$, called the triple index on the end e, from data as above to K, satisfying

(1) Multilinearity - the triple index is linear in each of the three variables (with linear choices of $\int R\mathrm{d}S$ etc.).

(2) Symmetry - we have $\langle F, G; H \rangle = \langle G, F; H \rangle$ (the $\int F\mathrm{d}G$ etc. must also be swapped).

(3) Triple identity - We have, again with the obvious additional choices,

$$\langle F, G; H \rangle + \langle F, H; G \rangle + \langle G, H; F \rangle = 0.$$

(4) Some compatibility for changing the auxiliary data.

(5) Reduction to double index - if G is in $A(U_e)$, then

$$\langle F, G; H \rangle = \left\langle F, \int GdH \right\rangle,$$

where $\int GdH$ is from the auxiliary data and is in $A(U_e) + K \cdot \log z$ because by assumption GdH is in $A(U_e) \cdot \mathrm{d}z$.

The *double index*

$$\langle \cdot, \cdot \rangle : (A(U_e) + K \cdot \log z) \times (A(U_e) + K \cdot \log z) \to K$$

is the unique alternating K–linear map such that

$$\langle F, G \rangle = \mathrm{Res} F \mathrm{d} G$$

if F is in $A(U_e)$ and G is in $A(U_e) + K \cdot \log z$.

Furthermore, if F, G and H are Coleman functions on a basic wide open U, such that $\mathrm{d}F$ and $\mathrm{d}G$ are holomorphic on U, $\mathrm{d}H$ is holomorphic on each end U_{e_j}, and we use Coleman integrals for $\int F \mathrm{d}G$ etc., then the *global index*

$$\langle F, G; H \rangle_{\mathrm{gl}} = \sum_{e \in \mathrm{Ends(U)}} \langle F, G; H \rangle_e$$

depends only on F, G and H and not on the auxiliary data.

Theorem (Besser+RdJ)
If we allow ω to be a form of the second kind on C_K in the previous Theorem, then the composition is given by

$$\sum_i [g_i]_2 \otimes f_i \mapsto \left\langle \log(f_i), \log(g_i); \int F_\omega \mathrm{d} \log(1 - g_i) \right\rangle_{\mathrm{gl}} ,$$

where F_ω is any Coleman integral of ω and the sum of triple indices is done with respect to a wide open space U on which all f_i, g_i, $1 - g_i$ as well as ω are holomorphic.