On K_2 of curves

Rob de Jeu
r.m.h.de.jeu@vu.nl

(Partly joint work with Bogdan Banu and Hang Liu)

Department of Mathematics
VU University Amsterdam

Contents

K_2 of a field

For a field F let $F^\flat = F \setminus \{0, 1\}$. Then

$$K_2(F) = F^* \otimes_{\mathbb{Z}} F^* / \langle x \otimes (1-x) | x \in F^\flat \rangle$$

is an Abelian group written additively with

$$\begin{aligned} \text{generators} \qquad \{a, b\} &= \text{ the class of } a \otimes b \\ \text{relations} \qquad \{a_1 a_2, b\} &= \{a_1, b\} + \{a_2, b\} \\ \{a, b_1 b_2\} &= \{a, b_1\} + \{a, b_2\} \\ \{a, 1-a\} &= 0 \text{ if } a \text{ is in } F^\flat \end{aligned}$$

Then also $\{a, b\} = -\{b, a\}$ and $\{c, -c\} = 0$ for a, b, c in F^*.

K-theory of a curve

Let C be an irreducible regular curve over a field k and $F = k(C)$. We have the exact localization sequence

$$\cdots \to K_3(F) \to \coprod_{P\in C^{(1)}} K_2(k(P)) \to K_2(C) \to K_2(F)$$
$$\xrightarrow{T} \coprod_{P\in C^{(1)}} k(P)^* \to K_1(C) \to F^*$$
$$\xrightarrow{\mathrm{div}} \coprod_{P\in C^{(1)}} \mathbb{Z} \to K_0(C) \to \mathbb{Z} \to 0$$

with the tame symbol $T : K_2(F) \to \coprod_{P\in C^{(1)}} k(P)^*$ given by

$$T_P(\{f,g\}) = (-1)^{\mathrm{ord}_P(f)\mathrm{ord}_P(g)} \frac{f^{\mathrm{ord}_P(g)}}{g^{\mathrm{ord}_P(f)}}|_P$$

We try to approximate $K_2(C)$ as $K_2^T(C) \stackrel{\text{def}}{=} \ker(T)$.
(If k is a number field then all $K_{2n}(k(P))$ ($n \geq 1$) are infinite torsion groups.)

$K_2(\mathbb{Q})$

$K_2(\mathbb{Q}) \overset{T}{\twoheadrightarrow} \oplus_{p \text{ prime}} \mathbb{F}_p^*$ with $\ker(T) = \langle\{-1,-1\}\rangle \cong \mathbb{Z}/2\mathbb{Z}$

Here T is the tame symbol, defined by

$$T_p(\{a,b\}) = (-1)^{v_p(a)v_p(b)} \frac{a^{v_p(b)}}{b^{v_p(a)}} \text{ modulo } p$$

More precisely, for q prime or -1, let
$F_q = \langle\{a,b\} \text{ with } a,b \in \{-1,2,3,5,7,11,\ldots,q\}\rangle \subseteq K_2(\mathbb{Q})$

Then

$$F_q/F_{q'} \overset{\sim}{\to} \mathbb{F}_q^* \text{ via } T_q \qquad (q \geq 2)$$

with q' the subprime of q (= one prime smaller) $(2' = -1)$

Proof Let $q \geq 2$
- surjectivity: $\{i,q\} \mapsto \bar{i} \in \mathbb{F}_q^*$ $(i = 1,\ldots,q-1)$

• injectivity: the kernel of $F_q \stackrel{T_q}{\to} \mathbb{F}_q^*$ is $F_{q'}$: $F_{q'} \subseteq \ker(T_q)$: clear;
if $q = 2$ then $F_2 = F_{-1}$ as $\{2,2\} = \{2,-1\} = \{-1,2\} = 0$
if $q > 2$ then $F_q/F_{q'}$ is generated by the classes of $\{a,q\} - \{b,q\}$
with $a, b \in M_q \stackrel{\text{def}}{=} \{-1, 1, 2, 3, 4, 5, \dots, q-1\}$

If $a_1, a_2 \in M_q$ then $\{a_1, q\} + \{a_2, q\} \stackrel{F_{q'}}{\equiv} \{a_3, q\}$ for $a_3 \in M_q$:
division with remainder gives $a_1 a_2 - a_3 = qA$ with
$a_3 = 1, 2, \dots, q-1 \in M_q$ and $A = -1, 0, 1, \dots, q-2$, and
if $A = 0$: $a_1 a_2 = a_3$ so clear;
if $A \neq 0$: $0 = \{\frac{a_1 a_2}{qA}, \frac{a_3}{-qA}\} \stackrel{F_{q'}}{\equiv} \{a_3, q\} - \{a_1, q\} - \{a_2, q\}$.
So $F_q/F_{q'} = \{\{a,q\} - \{b,q\} \text{ with } a, b \in M_q\}$
Finally, if $T_q(\{a,q\}) = T_q(\{b,q\})$ for $a, b \in M_q$ then
$a - b = 0, \pm q$ and $\{a,q\} \equiv \{b,q\}$ modulo F_q' as before

$K_2(k(x))$

If k is a field and x a variable then one can treat $K_2(k(x))$ similarly using division with remainder in $k[x]$:

if $d = \deg(f(x)) \geq d' = \deg(g(x))$ for $f(x), g(x) \neq 0$, then

$$\begin{aligned} \mathrm{Pol}_{\leq 0} \times \mathrm{Pol}_{\leq d-d'} \times \mathrm{Pol}_{\leq d'-1} &\to \mathrm{Pol}_{\leq d} \\ (c, q, r) &\mapsto cf - qg - r \end{aligned}$$

is a k-linear map with non-trivial kernel:
$1 + (d - d' + 1) + d' > d + 1$

If $(c, q, r) \neq 0$ is in this kernel then $c \neq 0$ because of degrees, so the kernel has basis $\{(1, q/c, r/c)\}$.

A useful lemma

The Euclidean algorithm in $\mathbb{Z}$ converges faster if we allow negative remainders: $a = qb + r$ with $|r| \leq |b|/2$. So(?) in $k[x]$ it "converges faster" if we allow $a(x) = q(x)b(x) + r(x)$ with "$|\deg(r(x))| \leq |\deg(b(x))|/2$". This inspired the

Lemma (VW-lemma)

Let K/k be a finite extension of fields, $V, W \subseteq K$ k-subspaces with $\dim_k(V) + \dim_k(W) > \dim_k(K) = [K : k]$. Then $K^* = \{vw^{-1} \text{ with } v \in V^*, w \in W^*\}$ where $V^* = V \setminus \{0\}$ and $W^* = W \setminus \{0\}$.

Example $\mathbb{Q}[\sqrt[3]{2}]^* = \left\{a_1 + a_2\sqrt[3]{2} + a_3\sqrt[3]{2}^2\right\}^* = \left\{\frac{b_1+b_2\sqrt[3]{2}^2}{c_1+c_2\sqrt[3]{2}}\right\}$, where all $a_i, b_j, c_l \in \mathbb{Q}$, and $b_1 + b_2\sqrt[3]{2}^2, c_1 + c_2\sqrt[3]{2} \neq 0$

Proof of the VW-lemma If $\beta \in K^*$ then $V \cap \beta W \neq \{0\}$.

The tame symbol on a curve

Let C/k be a smooth, proper, geometrically irreducible curve over a number field k, $F = k(C)$, and $g = \mathrm{genus}(C)$.
According to Beilinson

$$K_2(F) \xrightarrow{T} \oplus_{P \in C^{(1)}} k(P)^* \xrightarrow{\mathrm{Nm}} k^*$$

with $\mathrm{Nm} = \prod_P \mathrm{Nm}_{k(P)/k}$, should become exact after tensoring with $\mathbb{Q}$,

Example $E : y^2 = x^3 + x - 1$, $P = \{x = 0\} \cap E \setminus \{O\}$, so $\mathbb{Q}(P) = \mathbb{Q}(i)$. For q a prime number with $q \equiv 1$ modulo 4 write $q = \alpha\overline{\alpha}$ in $\mathbb{Z}[i]$. Then for some $n > 0$, $\left(\frac{\overline{\alpha}}{\alpha}\right)^n_{|P} \in \mathrm{im}(T)$?

Beilinson(+Deligne+Bass) also expect: $K_2^T(C)/K_2(k)$ is a finitely generated group of rank $[k:\mathbb{Q}]\cdot g+\delta$
where δ depends on the primes of bad reduction of C

So it would be good to control $\mathrm{coker}(T)$ and $\ker(T)=K_2^T(C)$

Assume k is an arbitrary field, and C/k is smooth, proper, geometrically irreducible with genus g.

Assume we have a rational point $O\in C(k)$ and let

$$RR_n = L(n(O)) = H^0(C,\mathcal{O}(n(O)))$$
$$RR_n^* = RR_n\setminus\{0\}$$
$$S = \{O\}\cup\{P\in C^{(1)} \text{ such that } f(P)=0 \text{ for some } f\in RR_{2g}^*\}$$
$$S' = \{O\}\cup\{P\in C^{(1)} \text{ such that } f(P)=0 \text{ for some } f\in RR_{3g}^*\}$$

The cokernel of T

Proposition

The restriction of the tame symbol

$$K_2(F) \stackrel{T_S}{\to} \oplus_{P \in C^{(1)} \setminus S} k(P)^*$$

is surjective, so $\mathrm{coker}(T)$ *is generated by* $\oplus_{P \in S} k(P)^*$.

Idea of proof Induction on $\deg(P)$. Initial step:
if $P \notin S$, $\deg(P) \leq 2g$, and $\beta \in k(P)^*$, then $RR_{2g} \to k(P)$, $f \mapsto f(P)$ is injective, so there exist f_i in RR_{2g}^* with $f_1(P)/f_2(P) = \beta$ (VW-lemma!), and $T_S(\{f_1/f_2, f_P\}) = \beta_{|P}$ for a suitable f_P.

Example If E/k is an elliptic curve defined by a Weierstrass equation $y^2 + a_1 xy + a_3 = x^3 + a_2 x^2 + a_4 x + a_6$, then $RR_2 = k \oplus kx$ and $S = \{O\} \cup \{\{x = c\} \text{ for } c \in k\}$

The kernel of T

Controlling the kernel of T is more involved (just as for $\mathbb{Q}$).

Division with remainder in $k[x]$ generalizes to the non-triviality of kernels of the form

$$
\begin{aligned}
L(D_1) \times L(D_2) \times L(D_3) &\to L(D_4) \\
(f_1, f_2, f_3) &\mapsto g_1 f_1 + g_2 f_2 + g_3 f_3
\end{aligned}
$$

for suitable divisors D_i on C, where $L(D) = H^0(C, \mathcal{O}(D))$.

Let

$$
C^{(1)}_{\leq d} = \{P \in C^{(1)} \text{ with } \deg(P) \leq d\}
$$

and similarly for $C^{(1)}_d$.

The kernel and cokernel of T

Proposition

Let $F_d = \langle \{f_1, f_2\} \text{ with } |(f_i)| \subseteq C^{(1)}_{\leq d} \rangle$. *If* $d \geq 3g+1$, *then* $F_d / F_{d-1} \cong \oplus_{O \neq P \in C^{(1)}_d} k(P)^*$, *and the inclusions give a quasi-isomorphism*

$$\begin{array}{ccc} F_{3g} & \xrightarrow{T} & \oplus_{P \in \{O\} \cup C^{(1)}_{\leq 3g}} k(P)^* \\ & \downarrow & \\ K_2(F) & \xrightarrow{T} & \oplus_{P \in C^{(1)}} k(P)^* \end{array}$$

Proposition

If $g \geq 1$, *and* $L_{3g} = \langle \{f_1, f_2\} \text{ with } f_i \in RR^*_{3g} \rangle$, *then the inclusions give a quasi-isomorphism*

$$\begin{array}{ccc} L_{3g} & \xrightarrow{T} & \oplus_{P \in S'} k(P)^* \\ & \downarrow & \\ K_2(F) & \xrightarrow{T} & \oplus_{P \in C^{(1)}} k(P)^* \end{array}$$

A computational example

So for an ellipic curve E given by a Weierstrass equation,
$K_2^T(E) \subseteq \langle \{l, m\} \text{ with } l, m \text{ of the form } ax + by + c \ (a, b, c \in k) \rangle$

If $k = \mathbb{Q}$ we only need such $ax + by + c$ with a, b and c in $\mathbb{Z}$ and $\gcd(a, b, c) = 1$, or $a = b = 0$, $c = -1$ or a prime number.

Example $E : y^2 + xy + y = x^3 + x^2$
with conductor $286 = 2 \cdot 11 \cdot 13$ and $E(\mathbb{Q}) = \{O\}$

E has split multiplicative reduction at 2, 11 and 13 and conjecturally $K_2^T(E)/K_2(\mathbb{Q})$ is finitely generated of rank 4.

By imposing a boundary condition at the primes of split multiplicative reduction ("integrality conditions") we expect to get a subgroup of rank 1 ($\delta = 3$ here)

There is a regulator $R \in \mathbb{R}$ for this subgroup and Beilinson expects $R/L'(E, 0) \in \mathbb{Q}^*$

Using the "lines"

$l_1 = x$	$l_2 = x - 1$	$l_3 = x + 1$
$l_4 = y$	$l_5 = y - 1$	$l_6 = y + 1$
$l_7 = y + x$	$l_8 = y - x$	$l_9 = y - x - 1$
$l_{10} = y - x + 1$	$l_{11} = y + x - 1$	$l_{12} = y + x + 1$
$l_{13} = y - x - 2$	$l_{14} = y - x + 2$	$l_{15} = y + x - 2$
$l_{16} = y + x + 2$	$l_{17} = y - 2x$	$l_{18} = y + 2x$
$l_{19} = y - 2x - 1$	$l_{20} = y - 2x + 1$	$l_{21} = y + 2x - 1$
$l_{22} = y + 2x + 1$	$l_{23} = 2$	

all elements we get in $K_2^T(E)$ are integral at 11 and 13.

Using also 11, 13, and suitable lines one can hit all three "integrality obstructions", and $K_2^T(E)$ has rank at least 4.

To approximate $\langle\{l_i, l_j\}\rangle \subseteq K_2(\mathbb{Q}(E))$ we use a free Abelian group Gen on generators $\{l_i, l_j\}$ $(i < j)$ with relations the kernel Rel of

$$\mathrm{Gen} \to K_2(\mathbb{Q}(x,y)) \xrightarrow{\widetilde{T}} \oplus_C \mathbb{Q}(D)^*$$

(where D runs through the irreducible curves in $\mathbb{A}^2_{\mathbb{Q}}$) since $\ker(\widetilde{T}) = K_2(\mathbb{Q})$ is torsion. This gives

$$\mathrm{Gen}/\mathrm{Rel} \to K_2(F)/K_2(\mathbb{Q}) \xrightarrow{T} \oplus_{P\in E^{(1)}} \mathbb{Q}(P)^*$$

T+integrality conditions give a kernel in $\mathrm{Gen}/\mathrm{Rel}$ of rank 31 instead of 1 but numerically all regulators are integral multiples of $\frac{1}{12}L'(E,0)$.

We ignored torsion in the $\mathbb{Q}(P)^*$ that occur so may have to multiply everything by 12 to be in $K_2^T(E)$.

For example, the conjectures suggest that, modulo torsion,

- $2\{x,y\}+\{x+1,y\}+2\{x,x+y+1\}+\{x+1,x+y+1\} \overset{?}{=} 0$
- $-2\{x,y+1\}-2\{x,y+x\}+\{x+1,x-1\}-\{x+1,y+1\}-\{x+1,y+x\}+\{x-1,y+1\}+\{x-1,y+x\} \overset{?}{=} 0$

Herbert Gangl observed: many relations can be explained using Steinberg relations $\{u,1-u\}=0$ coming from linear triplet relations $A_1A_2A_3+B_1B_2B_3=C_1C_2C_3$ on E with each A_i, B_i and C_i a "line": $u=\frac{A_1A_2A_3}{C_1C_2C_3}$, $1-u=\frac{B_1B_2B_3}{C_1C_2C_3}$.

E.g., $x^2(x+1)-y(y+x+1)=1$ proves the first relation above (up to 2-torsion) since $\{x^2(x+1),-y(y+x+1)\}=0$.

There are about 765 really different linear triplet relations. Replacing Rel with the resulting Steinberg relations then gives a kernel of rank 3 for T+integrality conditions.

An optimistic conjecture

One of the two still unexplained relations is

$$10\{x,y\}+5\{x+1,y\} \overset{?}{=} 3\{x,y\}+6\{x,y+1\}-9\{x,y+x+1\}$$
$$+\{x,y-2x\}-6\{x+1,y\}-3\{x+1,y-1\}+3\{x+1,y+1\}$$
$$-3\{x-1,y\}+3\{x-1,y-1\}-3\{x-1,y+1\}$$
$$-6\{y-1,y+x+1\}+3\{y-1,y-2x\}-3\{x+y+1,y-2x\}$$

The calculations suggests the

Optimistic conjecture

Let k be any field, E/k an elliptic curve, $O \in E(k)$, $F = k(E)$, and $LF^* = \langle RR_3^* \rangle \subseteq F^*$ $(RR_3 = H^0(E, \mathcal{O}(3(O))))$

(so the image of $LF^* \otimes_{\mathbb{Z}} LF^*$ in $K_2(F)$ is L_3).

Then (perhaps up to torsion or so)

$$L_3 \overset{?}{=} \frac{LF^* \otimes_{\mathbb{Z}} LF^*}{\langle u \otimes (1-u) \text{ with } u, 1-u \text{ from a linear triplet relation}\rangle}$$

Some theoretical evidence

Proposition If E/k is elliptic with k algebraically closed (so $LF^* = F^*$ and $L_3 = K_2(F)$) then we only need relations $u \otimes (1-u)$ with $\deg(u) \leq 3$, and those come from linear triplet relations:

- if $\deg(u) = 0$ then $u + (1-u) = 1$;
- if $\deg(u) = 2$ then from some $l + m = n$;
- if $\deg(u) = 3$ then from some $l_1 l_2 n_2 + m_1 m_2 n_1 = n_1 n_2 N$.

Here all l_i, etc., are in RR_3^* for O the neutral element.

In the last case we have $u = \frac{l_1 l_2}{n_1 N}$ and $1 - u = \frac{m_1 m_2}{n_2 N}$.

Some experimental evidence (with Bogdan Banu)

Back to a curve of genus g.

$$\frac{\mathbb{Z}[RR^*_{3g} \times RR^*_{3g}]}{R_{\leq w}} \twoheadrightarrow L_{3g}$$

by mapping $[l, m]$ to $\{l, m\}$, where $R_{\leq w}$ consists of those $[A/C, B/C]$ worked out into generators $[l, m]$ for which $A + B = C$ with each term of the form $l_1 l_2 \cdots l_t$ for some $t \leq w$.

Over a finite field, everything is finite (but the calculations may be big so have to be done in clever ways). Then experimentally the size of the group on the left is the same as the one on the right for 25 elliptic curves (5 each over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$ and $\mathbb{F}_{11}$) with $w = 3$, and for 5 curves of genus 2 over $\mathbb{F}_2$ with $w = 2$. O is the point at infinity in all cases.

Variation Using normalized lines (and allowing a constant in each of A, B and C) gives an isomorphic group on the left and faster calculations.

“Integral elements” in K_2^T of curves over number fields

Let C/k be a smooth, proper, geometrically irreducible curve of genus g over the number field k with ring of integers $\mathcal{O}$.

Proposition

Let $\mathcal{C}/\mathcal{O}$ be a regular proper model of C/k. Then the subgroup

$$K_2^T(C)_{\text{int}} \stackrel{\text{def}}{=} \ker(K_2(F) \xrightarrow{T} \oplus_{\mathcal{D}} \mathbb{F}(\mathcal{D})^*) \subseteq K_2^T(C)$$

where $\mathcal{D}$ runs through all irreducible curves of $\mathcal{C}$, and $\mathbb{F}(\mathcal{D})$ is the residue field at $\mathcal{D}$, does not depend on $\mathcal{C}$.

Regulator pairing

Let X be the analytic manifold associated to $C \times_{\mathbb{Q}} \mathbb{C}$; it is a disjoint union of $[k : \mathbb{Q}]$ Riemann surfaces of genus g, indexed by the embeddings $\sigma : k \to \mathbb{C}$. Complex conjugation acts on X through the action on $\mathbb{C}$.

There is a regulator pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : H_1(X;\mathbb{Z})^- \times K_2^T(C)/\text{torsion} &\to \mathbb{R} \\ (\gamma, \alpha) &\mapsto \frac{1}{2\pi}\int_\gamma \eta(\alpha) \end{aligned}$$

where for $\alpha = \{a, b\}$,
$\eta(\alpha) = (\log|a^\sigma|d\arg(b^\sigma) - \log|b^\sigma|d\arg(a^\sigma))_\sigma$.
Let $r = g \cdot [k : \mathbb{Q}]$. If $\gamma_1, \dots, \gamma_r$ form a basis of $H_1(X;\mathbb{Z})^-$ and $M_1, \dots, M_r$ are in $K_2^T(C)/\text{torsion}$, define the regulator by

$$R = |\det(\langle \gamma_i, M_j \rangle)|.$$

Beilinson's conjecture on K_2 of curves

Conjecture

1. $K_2^T(C)_{\text{int}}/\text{torsion}$ *is free of rank r and the regulator pairing restricted to it is non-degenerate.*
2. *Let R denote the regulator with respect to a basis of* $H_1(X;\mathbb{Z})^-$ *and of* $K_2^T(C)_{\text{int}}/\text{torsion}$. *Assuming* $L(C,s)$ *is defined around 0, we have* $L^{(r)}(C,0) = qR$ *with* $q \in \mathbb{Q}^*$

If $g > 0$ then only 2 is known/tested for some specially constructed $M_1, \ldots, M_r$ in special cases, e.g.:

- (Bloch) E a CM elliptic curve over $\mathbb{Q}$.
- (Beilinson) C a modular curve, k Abelian over $\mathbb{Q}$ + refinement by using decomposition by Hecke algebra.
- elliptic curves over $\mathbb{Q}$ with base field extended to $k/\mathbb{Q}$ Abelian.
- certain hyperelliptic curves over $\mathbb{Q}$ (q computed numerically)

Non-Abelian base fields: computations (Bogdan Banu)

Example $k = \mathbb{Q}(t)$ with $t^3 - t^2 - 1 = 0$ (discriminant -31)
C defined by $y^2 + xy = x^3 - t$, norm of the conductor is 80621999 (a prime number).
Pick 55 elements of RR_3^*:
$X-1, X+t-1, X-t, X+t, X-t+1, X+1, Y-1, Y+t-1, Y-t, Y, Y+t, Y+-t+1, Y+1, -Y+X-1, (t-1)Y+X-1, -tY+X-1, tY+X-1, (-t+1)Y+X-1, Y+X-1, -Y+X+t-1, (t-1)Y+X+t-1, -tY+X+t-1, tY+X+t-1, (-t+1)Y+X+t-1, Y+X+t-1, -Y+X-t, (t-1)Y+X-t, -tY+X-t, tY+X-t, (-t+1)Y+X-t, Y+X-t, -Y+X, (t-1)Y+X, -tY+X, tY+X, (-t+1)Y+X, Y+X, -Y+X+t, (t-1)Y+X+t, -tY+X+t, tY+X+t, (-t+1)Y+X+t, Y+X+t, -Y+X-t+1, (t-1)Y+X-t+1, -tY+X-t+1, tY+X-t+1, (-t+1)Y+X-t+1, Y+X-t+1, -Y+X+1, (t-1)Y+X+1, -tY+X+1, tY+X+1, t^2-t, -1$

We obtain 1537 generators of the form $[l, m]$ (modulo stupid relations), and 554 elements in the kernel of T, all integral. For all subsets of 3 of these elements, we compute the regulator R and obtain 1000316 different non-zero $q^{-1} = R/L^{(3)}(C, 0)$:

5073057476467/40, 95072797701/10, 29463253479/2, 1645554280, 60792814793/10, 74307570342/5, ...

Assuming all $R/L^{(3)}(C, 0)$ are in $\frac{1}{40}\mathbb{Z}$, they generate $\frac{1}{40}\mathbb{Z}$, and some 3 elements in $K_2^T(C)_{\text{int}}$ give a regulator for which $q^{-1} = \frac{1}{40}$.

Other examples over this field or other non-Abelian cubical fields of small absolute discriminant (about 50 curves in total) give such results with typically $q^{-1} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{48}$, once $\frac{9}{8}$ and once 8. Mild lie: we can only compute on the 0-component of the model so may have to multiply by some natural number to kill the remaining tame symbol on the other components. For the example with $\frac{1}{40}$ this problem does not occur.

Bloch's construction of elements in $K_2^T(C)$

Assume that $O, P, Q \in C(k)$

$$\begin{aligned} \mathrm{div}(f) &= d(P) - d(O), \\ \mathrm{div}(g) &= e(Q) - e(O), \\ f(Q) &= 1, \\ g(P) &= 1. \end{aligned}$$

Then $\{f, g\} \in K_2^T(C)$ by the product formula

$$\prod_{x \in C^{(1)}} \mathrm{Nm}_{k(x)/k} T_x(\alpha) = 1$$

for α in $K_2(F)$.

Some constructions for higher genus

For $g > 0$, if $b_1, \ldots, b_g$ in $\mathbb{Q}^*$ are distinct, λ is in $\mathbb{Q}^*$,

$$p(x) = 2x^{g+1} + q(x), \quad q(x) = \lambda \prod_{j=1}^{g} (b_j x + 1), \quad r(x) = 4x^{g+1} + q(x),$$

then $y^2 = -4x^{2g+2} + p(x)^2 = q(x)r(x)$ defines a hyperelliptic curve over $\mathbb{Q}$ of genus g provided r has no multiple roots, with one point O at infinity. Because

$$(y - p(x)) = (2g + 2)((0, \lambda)) - (2g + 2)(O)$$
$$(b_l x + 1) = 2((-1/b_l, 0)) - 2(O)$$

we have elements in $K_2^T(C)$ for $l = 1, \ldots, g$:

$$\alpha_l \stackrel{\text{def}}{=} 2\left\{ \frac{y - p(x)}{-p(-1/b_l)}, b_l x + 1 \right\} = 2\left\{ \frac{y - p(x)}{2x^{g+1}}, b_l x + 1 \right\}$$

Proposition

If the b_i and λ are in $\mathbb{Z}$ then the α_l are in $K_2^T(C)_{\text{int}}$.

The regulators and $L(C,2)$ can be computed numerically so we can verify (2) in Beilinson's conjecture (relation between R and $L(C,2)$) numerically.

Example ($g=2$) For $b_1 = 1$, $b_2 = 8$, $\lambda = 1$ we get

$$\pm\frac{L^{(2)}(C,0)}{2!} \stackrel{?}{=} \frac{N}{(2\pi)^4} L(C,2) \stackrel{?}{=} 2^{-4}3^{-1}7^{-1}R$$

with $N = 2 \cdot 7 \cdot 59$.

Example ($g=4$) For $b_i = i$ ($i = 1, \ldots, 4$) and $\lambda = 1$ we get

$$\pm\frac{L^{(4)}(C,0)}{4!} \stackrel{?}{=} \frac{N}{(2\pi)^8} L(C,2) \stackrel{?}{=} 2^{-6}5^{-2}R$$

with $N = 2^9 \cdot 3^2 \cdot 17 \cdot 113$.

Another construction (with Hang Liu): an example

Let $L_1, \cdots, L_n$ be n distinct lines given by equations $a_i x + b_i y + c_i = 0$ which are not parallel to each other. Consider the curve defined by

$$\prod_{i=1}^{n} L_i = 1$$

and C the normalisation of its projective closure. It has n distinct points P_i corresponding to the points $[-b_i, a_i, 0]$, and $\operatorname{div}(\frac{L_k}{L_i}) = n(P_k) - n(P_i)$.
Denote $a_i b_k - a_k b_i$ by $[i, k]$

$$\left\{ \frac{[i,m]}{[k,m]} \frac{L_k}{L_i}, \frac{[i,k]}{[m,k]} \frac{L_m}{L_i} \right\} \in K_2^T(C), \quad i, k, m \text{ distinct}$$

Another construction (with Hang Liu): general case

Let $L_{i,j}, 1 \leqslant i \leqslant N, N \geqslant 2, 1 \leqslant j \leqslant N_i$ be a family of distinct lines such that they are parallel to each other only for the same i and defined by $a_i x + b_i y + c_{i,j} = 0$.
Consider the curve defined by the equation

$$f(x,y) = \lambda \prod_{i=1}^{N} \prod_{j=1}^{N_i} L_{i,j} - 1 = 0$$

and C the normalisation of its projective closure.
We have two types of elements of $K_2^T(C)$

$$T_{i,j;k,l;m,n} = \left\{ \frac{[i,m]}{[k,m]} \frac{L_{k,l}}{L_{i,j}}, \frac{[i,k]}{[m,k]} \frac{L_{m,n}}{L_{i,j}} \right\} = \{h_1, h_2\}, \quad i,k,m \text{ distinct}$$

$$R_{i,j,k;l,m,n} = \left\{ \frac{L_{i,j}}{L_{i,k}}, \frac{L_{l,m}}{L_{l,n}} \right\}, \quad i \neq l, j \neq k, m \neq n,$$

Proposition

Let V be the subgroup of $K_2^T(C)$ generated by all the elements $R_{i,j,k;l,m,n}$ and $T_{i,j;k,l;m,n}$. Then V is generated by

$$\begin{aligned} &R_{1,1,j;2,1,m}, && 1 < j \leqslant N_1,\ 1 < m \leqslant N_2; \\ &T_{1,1;k,l;m,n}, && 2 \leqslant k < m \leqslant N,\ 1 \leqslant l \leqslant N_k,\ 1 \leqslant n \leqslant N_m; \\ &T_{1,j;2,1;m,n}, && 3 \leqslant m \leqslant N,\ 2 \leqslant j \leqslant N_1,\ 1 \leqslant n \leqslant N_m. \end{aligned}$$

In characteristic 0, for almost all λ, the projective completion of the curve defined by $f(x,y)=0$ has as singularities only the simple singular points $[b_i, -a_i, 0], 1 \leqslant i \leqslant n$ with multiplicity N_i. By the degree-genus formula the genus g of C is

$$\binom{\sum_{i=1}^N N_i - 1}{2} - \sum_{i=1}^{N} \binom{N_i}{2} = \sum_{1 \leqslant i < j \leqslant N} N_i N_j - \sum_{1 \leqslant i \leqslant N} N_i + 1.$$

This is also the number of generators in the proposition above.

Integrality of the elements

For $N = 2$ or 3, C is the projective regular model of an affine curve

$$f(x,y) = \lambda \prod_{i=1}^{N_1}(x+a_i)\prod_{j=1}^{N_2}(y+b_j)\prod_{k=1}^{N_3}(y-x+c_k) - 1 = 0.$$

Theorem

Let a_i, b_j, c_k, λ be algebraic integers with $\lambda \neq 0$. Then the elements $R_{i,j,k;l,m,n}$ and $T_{i,j;k,l;m,n}$ are in $K_2^T(C)_{\text{int}}$.

Example

On the elliptic curve corresponding to $\lambda x(x+a)y(y+b) = 1$ for non-zero integers λ, a and b with $\lambda a^2 b^2 \neq 16$, the element $\{\frac{x}{x+a}, \frac{y}{y+b}\}$ is in $K_2^T(C)_{\text{int}}$.

When is C hyperelliptic?

Assume $N_1 \geqslant N_2 \geqslant N_3 \geq 0$ and the characteristic is 0. If the affine curve defined by $f(x, y) = 0$ is nonsingular, then C is hyperelliptic if and only if $N_2 = 2, N_3 = 0$ or $N_2 = N_3 = 1$.
When $N_2 = 2, N_3 = 0$, C has affine model defined by

$$y(y + 2x^{g+1} + \lambda \prod_{i=1}^{g}(a_i x + 1)) + x^{2g+2} = 0.$$

(Already studied by T. Dokchitser, R. de Jeu and D. Zagier.)
When $N_2 = N_3 = 1$, C has affine model defined by

$$y(y + 2x^{g+2} + \lambda \prod_{i=1}^{g}(a_i x + 1)) + x^{2g+4} = 0.$$

Results on limits of regulators

C a regular proper model of

$$f(x,y) = \lambda \prod_{i=1}^{N} \prod_{j=1}^{N_i} L_{i,j} - 1 = 0.$$

Theorem

Let C be as above with fixed $a_i, b_i, c_{i,j}$ in $\mathbb{R}$, such that no three of the lines meet in an affine point. If we take $t = 1/\lambda$ in $\mathbb{R}$ and let $t \to 0$, then the generators in the proposition give regulator $R = R(t)$ satisfying

$$\lim_{t\to 0} \frac{R(t)}{|\log|t||^g} = 1.$$

C a regular proper model of

$$f(x,y) = \lambda \prod_{i=1}^{N_1}(x+a_i) \prod_{j=1}^{N_2}(y+b_j) \prod_{k=1}^{N_3}(y-x+c_k) - 1 = 0.$$

Corollary

Let C be as above and assume no three of the lines meet in an affine point. If the a_i, b_j, c_k are fixed integers, and $|\lambda| \gg 0$, $\lambda \in \mathbb{Z}$, then in general the regulator of the elements we constructed does not vanish, hence for general C in the family $K_2^T(C)_{\mathrm{int}}$ has rank at least g.