On the K-theory of curves over number fields

Rob de Jeu
r.m.h.de.jeu@vu.nl
http://www.few.vu.nl/~jeu

Department of Mathematics
Vrije Universiteit Amsterdam

31st October 2024, Mainz, Germany

Borel's theorem

Quillen defined Abelian groups $K_n(R)$ $(n \geq 0)$ for rings R, as well as for algebraic varieties.

Let k be a number field, with r_1 real and $2r_2$ non-real embeddings, $d = r_1 + 2r_2$, and ring of algebraic integers $\mathcal{O}$, and Δ_k the absolute value of the discriminant of k

Then

- $K_0(\mathcal{O}) \simeq \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$
- $K_1(\mathcal{O}) \simeq \mathcal{O}^*$ has rank $r_1 + r_2 - 1$

Classically

$$\mathrm{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R \, |\mathrm{Cl}(\mathcal{O})|}{w\sqrt{\Delta_k}}$$

with R the regulator of $\mathcal{O}^*$, which has lots of K-theory in it.

Theorems of Quillen and Borel

Theorem (Quillen)

$K_n(\mathcal{O})$ is finitely generated for all $n \geq 0$.

Theorem (Borel)

(1) $K_{2n}(\mathcal{O})$ is a finite group if $n \geq 1$.
(2) For $n \geq 2$, $K_{2n-1}(\mathcal{O})$ has rank $m_{2n-1} = r_1 + r_2$ if n is odd, and rank $m_{2n-1} = r_2$ if n is even.
(3) There exists a natural regulator map

$$K_{2n-1}(\mathcal{O}) \to \mathbb{R}^{m_{2n-1}} \qquad (n \geq 2).$$

Its image is a lattice with volume of a fundamental domain

$$V_n = q_n \frac{\zeta_k(n)}{\pi^{n(d-m_{2n-1})}\sqrt{\Delta_k}}$$

with q_n in $\mathbb{Q}^*$.

Example: the K-theory of $\mathbb{Z}$

$\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:

$K_{2n-1}(\mathbb{Z})$ is finite for n even;

$K_{2n-1}(\mathbb{Z})$ has rank 1 for n odd, and $V_n = q_n \zeta(n)$ with q_n in $\mathbb{Q}^*$.

n	2	3	4	5	6	7	...
m_{2n-1}	0	1	0	1	0	1	...
$\zeta(n)$	$\pi^2/6$	irrational	$\pi^4/90$	???	$\pi^6/945$	???	...

Generalisation: curves (continued)

If k is a field then $K_2(k)$ is an Abelian group written additively, with

$$\begin{array}{ll} \text{generators} & \{a,b\} \text{ for } a,b \text{ in } k^* \\ \text{relations} & \{a_1a_2,b\} = \{a_1,b\}+\{a_2,b\} \\ & \{a,b_1b_2\} = \{a,b_1\}+\{a,b_2\} \\ & \{a,1-a\} = 0 \text{ if } a \neq 0,1 \end{array}$$

Then also $\{a,b\} = -\{b,a\}$ and $\{c,-c\} = 0$ for a, b, c in k^*.

Let C be a regular curve over a field F. There is an exact localisation sequence

$$\cdots \to \oplus_{P\in C} K_2(F(P)) \to K_2(C) \to K_2(F(C)) \xrightarrow{T} \oplus_{P\in C} F(P)^* \to \dots$$

where T_P is the tame symbol for P:

$$T_P : \{f,g\} \mapsto (-1)^{\mathrm{ord}_P(f)\,\mathrm{ord}_P(g)} \frac{f^{\mathrm{ord}_P(g)}}{g^{\mathrm{ord}_P(f)}}|_P$$

with $\mathrm{ord}_P(f)$ the order of vanishing of f at P.

Generalisation: curves (continued)

Now assume C is a curve over $\mathbb{C}$.
For two non-zero meromorphic functions f and g on C, $\log|f| \mathrm{d} \arg g - \log|g| \mathrm{d} \arg f$ is a closed 1-form on some Zariski open U. Because

$$\log|z| \mathrm{d} \arg(1-z) - \log|1-z| \mathrm{d} \arg z = \mathrm{d} P_2(z),$$

where $P_2(z)$ is a C^∞-function on $\mathbb{C} \setminus \{0,1\}$, we get a map

$$\begin{aligned} \mathrm{reg} : K_2(F) &\to \frac{\{\text{closed 1-forms on some } U\}}{\{\text{exact 1-forms on some } U\}} \\ \{f,g\} &\mapsto \log|f| \mathrm{d} \arg g - \log|g| \mathrm{d} \arg f . \end{aligned}$$

This fits into a commutative diagram

$$\begin{tikzcd}
 & K_2(C) \arrow[r] \arrow[d, dotted, "\mathrm{reg}"'] & K_2(\mathbb{C}(C)) \arrow[r] \arrow[d, "\mathrm{reg}"'] & \oplus_{P\in C}\mathbb{C}^* \arrow[d, "\log|\cdot|"'] \\
0 \arrow[r] & H^1_{\mathrm{dR}}(C;\mathbb{R}) \arrow[r] & H^1_{\mathrm{dR}}(\mathbb{C}(C);\mathbb{R}) \arrow[r, "i\cdot\mathrm{res}"'] & \oplus_{P\in C}\mathbb{R}
\end{tikzcd}$$

with $H^1_{\mathrm{dR}}(\mathbb{C}(C);\mathbb{R}) = \varinjlim_{U} H^1_{\mathrm{dR}}(U;\mathbb{R})$.

Bloch's result

Theorem (Bloch)

Let E be an elliptic curve defined over $\mathbb{Q}$ with complex multiplication. Then there exists α in $K_2(E)$ with

$$L'(E,0) = q\frac{1}{2\pi}\int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha)\wedge\omega$$

for some q in $\mathbb{Q}^*$, or, using the functional equation for the L-function:

$$\frac{1}{2\pi}L(E,2) = q'\int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha)\wedge\omega.$$

ω is the non-zero holomorphic form on $E_{\mathbb{C}}$ with $\int_{E(\mathbb{R})}\omega = 1$.

Beilinson proposed conjectures that include vast generalisations of this result, and Borel's theorem.

'Integrality'; choice of model

Fix a regular, flat, proper model $\mathcal{C}/\mathcal{O}_F$ of C/F, with $\mathcal{O}_F$ the ring of algebraic integers in a number field F. Then we define

$$K_2^T(C)_{\mathrm{int}} = \ker\left(K_2(F(C)) \stackrel{T_{\mathcal{C}}}{\to} \oplus_{\mathcal{D}} \mathbb{F}(\mathcal{D})^\times\right),$$

where $\mathcal{D}$ runs through all irreducible curves on $\mathcal{C}$, and $\mathbb{F}(\mathcal{D})$ is the residue field at $\mathcal{D}$. The component of $T_{\mathcal{C}}$ for $\mathcal{D}$ is given by the tame symbol corresponding to $\mathcal{D}$,

$$\{a, b\} \mapsto (-1)^{v_{\mathcal{D}}(a) v_{\mathcal{D}}(b)} \frac{a^{v_{\mathcal{D}}(b)}}{b^{v_{\mathcal{D}}(a)}}(\mathcal{D}),$$

where $v_{\mathcal{D}}$ is the valuation on F corresponding to $\mathcal{D}$.

$K_2^T(C)_{\mathrm{int}}$ is the subgroup of $K_2^T(C)$ consisting of integral elements.

Theorem (Liu-dJ, 2015)

$K_2^T(C)_{\mathrm{int}}$ is independent of $\mathcal{C}$, and is equal to the image of $K_2(\mathcal{C})$ in $K_2(k(C))$ under localisation.

Beilinson regulator for K_2 of curves

For starters,

- $C/\mathbb{C}$ a regular, proper curve,
- $\alpha = \sum_j \{f_j, g_j\}$ in $K_2^T(C)$
- γ in $H_1(C(\mathbb{C}), \mathbb{Z})$
- their regulator pairing is (well-)defined by

$$\langle \gamma, \alpha \rangle = \frac{1}{2\pi} \int_\gamma \sum_j \eta(f_j, g_j)$$

with $\eta(f, g) = \log|f| \mathrm{d} \arg(g) - \log|g| \mathrm{d} \arg(f)$ for non-zero functions f and g on C; we use a representative of γ that avoids all zeroes and poles of the functions involved.

Beilinson regulator for K_2 of curves (continued)

As main course,

• C a regular, proper, geometrically irreducible curve over a number field k of degree m, of genus g; let $n = mg$

• X the Riemann surface consisting of all $\mathbb{C}$-valued points of C, a disjoint union of the complex points of m curves C^σ, indexed by the embeddings σ of k into $\mathbb{C}$. Complex conjugation acts on X through its action on $\mathbb{C}$, and $H_1(X,\mathbb{Z})^- \simeq \mathbb{Z}^n$

• Define a pairing

$$\begin{aligned} H_1(X,\mathbb{Z}) \times K_2^T(C) &\to \mathbb{R} \\ (\gamma,\alpha) &\mapsto \langle \gamma,\alpha\rangle_X = \sum_\sigma \langle \gamma_\sigma, \alpha^\sigma\rangle \end{aligned}$$

if $\gamma = (\gamma_\sigma)_\sigma$ in $H_1(X,\mathbb{Z}) = \oplus_\sigma H_1(C^\sigma(\mathbb{C}),\mathbb{Z})$, α^σ the pullback of α to C^σ.

Beilinson's conjecture for K_2 of curves (continued)

Assume $L(C, s)$ can be analytically continued to the complex plane and satisfies a functional equation for s versus $2 - s$ as in the Hasse-Weil conjecture.
Then $L(C, s)$ should have a zero of order n at $s = 0$, and we let $L^*(C, 0) = (n!)^{-1} L^{(n)}(C, 0)$ be the first non-vanishing coefficient in its Taylor expansion in s at 0.

Conjecture

• *Let $\gamma_1, \dots, \gamma_n$ and $\alpha_1, \dots, \alpha_n$ form $\mathbb{Z}$-bases of $H_1(X, \mathbb{Z})^-$ and $K_2^T(C)_{\mathrm{int}}$ modulo torsion respectively Let the Beilinson regulator of the α_j be $R = |\det(\langle \gamma_i, \alpha_j \rangle_X)_{i,j}|$. Then*

$$L^*(C, 0) = Q \cdot R$$

for some Q in $\mathbb{Q}^\times$

We borrowed finite generation of $K_2^T(C)_{\mathrm{int}}$ from Bass's conjecture

A new integrality criterion

Proposition

Let C be a regular, projective, geometrically irreducible curve over a number field F, with regular, flat, proper model $\mathcal{C}$ over the ring of algebraic integers $\mathcal{O}_F$. Suppose f, g in $F(C)^\times$ and $N \geq 1$ satisfy $(f) = N(P) - N(O)$, $(g) = N(Q) - N(O)$ for some distinct F-rational points O, P and Q on C, and $f(Q) = g(P) = 1$. Then $\alpha = \{f, g\}$ is in $K_2^T(C)$.

Let $\mathfrak{P}$ be a maximal ideal of $\mathcal{O}_F$, with fibre $\mathcal{F} = \mathcal{C}_{\mathfrak{P}}$, and let $\mathcal{D}$ be an irreducible component of $\mathcal{F}$. Then

- *$T_{\mathcal{D}}(\alpha) = 1$ if O and P, or O and Q, hit the same irreducible component of $\mathcal{F}$*
- *$T_{\mathcal{D}}(\alpha) = T_{\mathcal{D}}(\{f, \varepsilon\}) = T_{\mathcal{D}}(\{g, \varepsilon\})$ if P and Q hit the irreducible component $\mathcal{B}$ and O hits the irreducible component $\mathcal{A} \neq \mathcal{B}$; $T_{\mathcal{D}}(\alpha) = 1$ if $\mathcal{D}$ is not in the connected component of $\cup_{\mathcal{D}' \neq \mathcal{B}} \mathcal{D}'$ that contains $\mathcal{A}$. Here $\varepsilon = (-g/f)(O)$, an Nth root of unity.*

A new integrality criterion (continued)

With $e_N(\cdot,\cdot)$ the Weil pairing on the N-torsion in the Jacobian of C over an algebraic closure of F, we have (Mazo, 2005)

$$\varepsilon = (-1)^{N+1} e_N((P)-(O),(Q)-(O)) \,.$$

Example

On the elliptic curve over $\mathbb{Q}$ defined by $y^2 = x^3 + 1$, with $P = (2,3)$, $Q = -P = (2,-3)$, $N = 6$,

$$f = \frac{1}{108}\frac{(y-2x+1)^3}{y+1} \qquad g = \frac{1}{108}\frac{(-y-2x+1)^3}{-y+1},$$

$\varepsilon = -1$. The reduction at $p = 3$ is of type III. It has two irreducible components $\mathcal{A}$, hit by O, and $\mathcal{B}$, hit by P and Q. Then $T_{\mathcal{A}}(\{f,g\}) = -1$ and $T_{\mathcal{B}}(\{f,g\}) = 1$.

Joint work with Brunault, Liu, Villegas

Let E be an elliptic curve over a field F, and P an F-rational point on E of order N. For $1 \leqslant s \leqslant N-1$, let $f_{P,s}$ in $F(E)^\times$ be a function with divisor $(f_{P,s}) = N(sP) - N(O)$.

In $K_2^T(E)$ define

$$T_{P,s,t} = \left\{ \frac{f_{P,s}}{f_{P,s}(tP)}, \frac{f_{P,t}}{f_{P,t}(sP)} \right\} \quad (s \neq t)$$

$$S_{P,s} = \{f_{P,s}, -f_{P,s}\} + \sum_{t=1, t\neq s}^{N-1} T_{P,s,t} \quad (1 \leq s \leq N-1)$$

Remark

$S_{P,s} + S_{P,N-s} = 0$

Let F be a field and $N \geq 4$. A pair (E, P) with E an elliptic curve E over F and an F-rational point P of order N admits a unique Weierstraß model Tate normal form

$$E : y^2 + (1-g)xy - fy = x^3 - fx^2$$

with f in $F^\times$, g in F, and where $P = (0,0)$.

N	f	g	Δ
7	$t^3 - t^2$	$t^2 - t$	$t^7(t-1)^7(t^3 - 8t^2 + 5t + 1)$
8	$2t^2 - 3t + 1$	$\frac{2t^2-3t+1}{t}$	$\frac{(t-1)^8(2t-1)^4(8t^2-8t+1)}{t^4}$
10	$\frac{2t^5-3t^4+t^3}{(t^2-3t+1)^2}$	$\frac{-2t^3+3t^2-t}{t^2-3t+1}$	$\frac{t^{10}(t-1)^{10}(2t-1)^5(4t^2-2t-1)}{(t^2-3t+1)^{10}}$

Joint work with Brunault, Liu, Villegas (continued)

Theorem

If $E = E_t$ is an elliptic curve over a number field F, in Tate normal form for $N = 7, 8$ or 10, and $P = (0,0)$, then $2P$ hits the 0-component in each fibre of the minimal regular model over $\mathcal{O}_F$ if

- $t, 1-t$ *are in* $\mathcal{O}_F^\times$, *for* $N = 7$
- $\frac{1}{t} - 1, \frac{1}{t} - 2$ *are in* $\mathcal{O}_F^\times$, *for* $N = 8$
- $\frac{1}{t} - 1, 1 - 2t$ *are in* $\mathcal{O}_F^\times$, *for* $N = 10$

In that case:

- *each* $S_{P,s}$ *is in* $K_2^T(E)_{\mathrm{int}}$ *for* $N = 7$;
- $2S_{P,s}$ *is in* $K_2^T(E)_{\mathrm{int}}$ *for* $N = 8, 10$.

Joint work with Brunault, Liu, Villegas (continued)

Lemma

For every integer a, and all $\varepsilon, \varepsilon'$ in $\{\pm 1\}$

$$f_a(X) = X^3 + aX^2 - (a + \varepsilon + \varepsilon' + 1)X + \varepsilon$$

is irreducible in $\mathbb{Q}[X]$.

A cubic field F has an element u such that $F = \mathbb{Q}(u)$ and both u and $1 - u$ are in $\mathcal{O}_F^\times$ precisely when u is a root of some $f_a(X)$.

$F/\mathbb{Q}$ is cyclic if and only if $\varepsilon = \varepsilon' = 1$ or $|2a - \varepsilon + \varepsilon' + 3| = 7$.

- For $\varepsilon = \varepsilon' = 1$ we get the simplest cubic fields (Shanks).
- $u \mapsto 1 - u$ and $u \mapsto u^{-1}$ generate some identifications; we end up with two 'half' families.
- There are similarly 40 families of quartic fields $\mathbb{Q}(u)$ with both u and $\frac{u-1}{u+1}$ units (identifications under a dihedral group of order 8); the Galois closure in a family almost always has group S_4 (28×), D_4 (10×), C_4 (1×) simplest quartic fields (Gras), $C_2 \times C_2$ (1×).

Joint work with Brunault, Liu, Villegas (continued)

Theorem

Define fields F, with element t, parametrised by an integer a.

- *Let u be a root of an $f_a(X)$ defining a special cubic field $F = \mathbb{Q}(u)$, and put $t = u$ ($N = 7$) or $t = 1/(u+1)$ ($N = 8$).*
- *Let u be a root of an $f_a(X)$ defining a special quartic field $F = \mathbb{Q}(u)$, and put $t = \frac{1-u}{2}$ for $N = 10$.*

If the Tate normal form for (N, t) defines an elliptic curve E/F, then, with $P = (0,0)$:

- *the $\gcd(2, N) \cdot S_{P,s}$ for $s = 1, \ldots, N-1$ are in $K_2^T(E)_{\mathrm{int}}$;*
- *for the Beilinson regulator $R(a)$ of the first $\lfloor \frac{N-1}{2} \rfloor$ we have*

$$\lim_{|a|\to\infty} \frac{R(a)}{\log^{\lfloor \frac{N-1}{2} \rfloor} |a|} = C_N \cdot \left| \det \left(\frac{N^4}{3} B_3 \left(\left\{ \frac{ij}{N} \right\} \right) \right)_{1 \le i,j \le \lfloor \frac{N-1}{2} \rfloor} \right| \neq 0$$

($B_3(X) = X^3 - \frac{3}{2}X^2 + \frac{1}{2}X$: third Bernoulli polynomial; $\{x\}$: the fractional part of x; $C_7 = 1$, $C_8 = C_{10} = 4$)

- d: discriminant of F
- c: conductor norm c of E

$N = 7, 8$: F defined by $f_a(X) = X^3 + aX^2 - (a+1)X + 1$, $a \geq 0$
F is cubic non-Abelian for $a \neq 3$

$N = 7$: we list the rational number $\widetilde{Q}$ for $S_{P,1}, S_{P,2}, S_{P,3}$
$N = 8$: we list the rational number $\widetilde{Q}$ for $2S_{P,1}, 2S_{P,2}, 2S_{P,3}$

$N = 10$: F defined by $f_a(X) = X^4 + aX^3 - aX + 1$, a in $\mathbb{Z} \setminus \{\pm 3\}$
- Galois group of splitting field: D_4 for $a \neq 0$
- two complex places for $a = -2, \ldots, 2$, otherwise totally real
- we list the rational number $\widetilde{Q}$ for $2S_{P,1}, 2S_{P,2}, 2S_{P,3}, 2S_{P,4}$,

Joint work with Brunault, Liu, Villegas (continued)

Some of our data for $N = 7$ red: F not totally real

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
2	-23	$2^3 \cdot 7^2$	3.20759739648506351	7^{-6}
3	7^2	$13 \cdot 29$	14.5301315201187081	7^{-5}
4	257	$2^3 \cdot 41$	235.760168840014734	7^{-4}
5	$17 \cdot 41$	239	1671.96067772426875	$2 \cdot 3 \cdot 5 \cdot 7^{-5}$
6	1489	$2^3 \cdot 13$	4051.92834496448134	7^{-3}
7	2777	83	-6590.94375552556550	$-2 \cdot 5 \cdot 7^{-5} \cdot 11$
8	4729	$2^3 \cdot 41$	114693.828270615380	$2^3 \cdot 3^3 \cdot 7^{-4}$
9	7537	$7^2 \cdot 13$	520366.913326434323	$2 \cdot 3 \cdot 7^{-4} \cdot 137$
10	$7^2 \cdot 233$	$2^3 \cdot 127$	-1485239.71027494934	$-2 \cdot 3^2 \cdot 7^{-4} \cdot 113$
11	$17 \cdot 977$	1471	5790649.98684165696	$2^4 \cdot 3 \cdot 5^2 \cdot 7^{-5} \cdot 41$
12	$97 \cdot 241$	$2^3 \cdot 251$	17255203.9121322960	$2^4 \cdot 3^2 \cdot 7^{-4} \cdot 131$
13	32009	2633	28504752.7830982117	$2^8 \cdot 3 \cdot 7^{-4} \cdot 37$
14	$47 \cdot 911$	$2^3 \cdot 419$	93361926.2369695039	$2^3 \cdot 3 \cdot 7^{-4} \cdot 3571$
15	$73 \cdot 769$	$43 \cdot 97$	192572866.057081271	$2^3 \cdot 3^2 \cdot 7^{-4} \cdot 43 \cdot 53$

Some of our data for $N = 8$

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
2	-23	$5 \cdot 137$	5.97110504152047155	2^{-21}
3	7^2	$7 \cdot 113$	31.2948786232840397	2^{-18}
4	257	3^3	25.2202129687784361	$2^{-18} \cdot 3^{-1}$
5	$17 \cdot 41$	$11 \cdot 41$	3130.70411060858445	$2^{-15} \cdot 3$
6	1489	$7 \cdot 13$	3377.15438740388289	2^{-13}
7	2777	$3^3 \cdot 5 \cdot 7$	-110191.314028644712	$-2^{-10} \cdot 3$
8	4729	$17 \cdot 127$	806249.659144856084	$2^{-13} \cdot 11 \cdot 13$
9	7537	$19 \cdot 199$	-3399020.63508445448	$-2^{-12} \cdot 257$
10	$7^2 \cdot 233$	$3^3 \cdot 7 \cdot 31$	9860642.47040826474	$2^{-11} \cdot 3 \cdot 109$
11	$17 \cdot 977$	$23 \cdot 367$	-38313626.2137679483	$-2^{-13} \cdot 4547$
12	$97 \cdot 241$	$5 \cdot 463$	22214626.7118122391	$2^{-14} \cdot 4787$
13	32009	$3^3 \cdot 7$	2759510.81590883242	$2^{-13} \cdot 3 \cdot 7 \cdot 13$
14	$47 \cdot 911$	$7 \cdot 29 \cdot 97$	-549654076.156923184	$-2^{-12} \cdot 3^4 \cdot 311$
15	$73 \cdot 769$	$17 \cdot 31 \cdot 47$	1205314746.12464172	$2^{-9} \cdot 5 \cdot 1289$

Joint work with Brunault, Liu, Villegas (continued)

Data for $N = 10$ red: F one complex place blue: $F = \mathbb{Q}(\zeta_8)$

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
-7	$2^3 \cdot 41^2$	$2^2 \cdot 23^2$	67284.5712909244205	$2^{-11} \cdot 5^{-5}$
-6	$2^6 \cdot 7^2 \cdot 37$	$3^4 \cdot 7^2$	12809909.2599370080	$2^{-9} \cdot 5^{-4} \cdot 13$
-5	$2^3 \cdot 13 \cdot 17^2$	$2^2 \cdot 19^2$	321613.252539691824	$2^{-10} \cdot 5^{-4}$
-4	$2^8 \cdot 17$	17^2	1308.96784301967823	$2^{-10} \cdot 5^{-7}$
-2	$2^6 \cdot 5$	13^2	3.90265959107592883	$2^{-14} \cdot 5^{-9}$
-1	$2^3 \cdot 7^2$	$2^2 \cdot 11^2$	18.1524378610645748	$2^{-14} \cdot 5^{-8}$
0	2^8	3^4	1.29080207928400602	$2^{-14} \cdot 3^{-2} \cdot 5^{-8}$
1	$2^3 \cdot 7^2$	$2^2 \cdot 7^2$	7.41655915683319223	$2^{-15} \cdot 5^{-8}$
2	$2^6 \cdot 5$	5^2	0.604505751430063810	$2^{-14} \cdot 5^{-10}$
4	$2^8 \cdot 17$	7^2	211.227406732423650	$2^{-11} \cdot 5^{-7}$
5	$2^3 \cdot 13 \cdot 17^2$	2^2	825.817965343090665	$2^{-11} \cdot 5^{-7}$
6	$2^6 \cdot 7^2 \cdot 37$	3^4	272030.854985666477	$2^{-9} \cdot 3^2 \cdot 5^{-6}$
7	$2^3 \cdot 41^2$	$2^2 \cdot 5^4$	111421.646021166774	$2^{-10} \cdot 5^{-5}$

K-theory of a curve

Let C be an irreducible regular curve over a field k and $F = k(C)$. We have the exact localisation sequence

$$\begin{aligned} \cdots \to \coprod_{P \in C} K_4(k(P)) &\to K_4(C) \to K_4(F) \\ \xrightarrow{\partial} \coprod_{P \in C} K_3(k(P)) &\to K_3(C) \to K_3(F) \\ \to \coprod_{P \in C} K_2(k(P)) &\to K_2(C) \to K_2(F) \\ \xrightarrow{T} \coprod_{P \in C} k(P)^* &\to K_1(C) \to F^* \\ \xrightarrow{\mathrm{div}} \coprod_{P \in C} \mathbb{Z} &\to K_0(C) \to \mathbb{Z} \to 0 \end{aligned}$$

with the tame symbol $T : K_2(F) \to \coprod_{P \in C} k(P)^*$ given by

$$T_P(\{f, g\}) = (-1)^{\mathrm{ord}_P(f)\mathrm{ord}_P(g)} \frac{f^{\mathrm{ord}_P(g)}}{g^{\mathrm{ord}_P(f)}}|_P$$

We try to approximate $K_2(C)$ as $\ker(T)$ and $K_4(C)$ as $\ker(\partial)$ using

$$\cdots \to \coprod_{P\in C} K_2(k(P)) \to K_2(C) \to K_2(F) \xrightarrow{T} \coprod_{P\in C} k(P)^* \to \cdots$$

$$\cdots \to \coprod_{P\in C} K_4(k(P)) \to K_4(C) \to K_4(F) \xrightarrow{\partial} \coprod_{P\in C} K_3(k(P)) \to \cdots$$

Fact If k is a number field then
(1) all $K_{2n}(k(P))$ $(n \geq 1)$ are infinite torsion groups;
(2) $K_3(k(P))$ is torsion if and only if $k(P)$ is totally real.

Conjecture (Beilinson)

If C is complete, regular and geometrically irreducible, and k is a number field, then
(a) $\dim_{\mathbb{Q}} K_4(C)_{\mathbb{Q}} = [k:\mathbb{Q}] \cdot \textit{genus}(C) \stackrel{\text{def}}{=} r$ *no integrality condition*
(b) $R_4(C) = qL^{(r)}(C,-1) \neq 0$*, where* $q \in \mathbb{Q}^*$*,* $R_4(C)$ *is a Beilinson regulator of* $K_4(C)_{\mathbb{Q}}$ *and we assume* $L(C,s)$ *satisfies the expected functional equation.*

The localisation sequence with weights

After tensoring the exact localisation sequence with $\mathbb{Q}$, it decomposes as a direct sum of exact sequences ($j = 0, 1, 2, \dots$)

$$\begin{aligned} \cdots \to \textstyle\coprod_{P \in C} K_4^{(j-1)}(k(P)) &\to K_4^{(j)}(C) \to K_4^{(j)}(F) \\ \to \textstyle\coprod_{P \in C} K_3^{(j-1)}(k(P)) &\to K_3^{(j)}(C) \to K_3^{(j)}(F) \\ \to \textstyle\coprod_{P \in C} K_2^{(j-1)}(k(P)) &\to K_2^{(j)}(C) \to K_2^{(j)}(F) \\ \to \textstyle\coprod_{P \in C} K_1^{(j-1)}(k(P)) &\to K_1^{(j)}(C) \to K_1^{(j)}(F) \\ \to \textstyle\coprod_{P \in C} K_0^{(j-1)}(k(P)) &\to K_0^{(j)}(C) \to K_0^{(j)}(F) \to 0 \end{aligned}$$

In particular, $0 \to K_2^{(2)}(C) \to K_2^{(2)}(F) \to \coprod_P K_1^{(1)}(k(P))$ and $\coprod_P K_4^{(2)}(k(P)) \to K_4^{(3)}(C) \to K_4^{(3)}(F) \to \coprod_P K_3^{(2)}(k(P))$ are exact, where one expects $K_4^{(2)}(L) = 0$ for any field L.

Theorem

Let F be a field of characteristic zero (for simplicity). There exist $\mathbb{Q}$-vector spaces $\widetilde{M}_{(n)}(F)$ $(n \geq 2)$ generated by symbols $[f]_n$ with f in $F^\flat = F \setminus \{0,1\}$, satisfying certain (unknown) relations, such that

(1) for the cohomological complex in degrees 1, 2

$$\widetilde{\mathcal{M}}_{(2)}(F) : \widetilde{M}_{(2)}(F) \xrightarrow{\mathrm{d}} \bigwedge^2 F^*_{\mathbb{Q}}$$
$$[f]_2 \mapsto (1-f) \wedge f$$

there exist

- *an injective homomorphism $H^1(\widetilde{\mathcal{M}}_{(2)}(F)) \to K_3^{(2)}(F)$ an isomorphism if F is a number field*
- *an isomorphism $H^2(\widetilde{\mathcal{M}}_{(2)}(F)) \to K_2^{(2)}(F)$*

Theorem

(2) for the cohomological complex in degrees 1, 2, 3

$$\widetilde{\mathcal{M}}_{(3)}(F) : \widetilde{M}_{(3)}(F) \xrightarrow{\mathrm{d}} \widetilde{M}_{(2)}(F) \otimes F^*_{\mathbb{Q}} \xrightarrow{\mathrm{d}} \bigwedge^3 F^*_{\mathbb{Q}}$$

$$[f]_2 \otimes g \mapsto (1-f) \wedge f \wedge g$$

$$[f]_3 \mapsto [f]_2 \otimes f$$

there exist

- *an isomorphism* $H^1(\widetilde{\mathcal{M}}_{(3)}(F)) \to K_5^{(3)}(F)$ *if* F *is a number field*
- *a homorphism* $H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to K_4^{(3)}(F)$
- *an isomorphism* $H^3(\widetilde{\mathcal{M}}_{(3)}(F)) \to K_3^{(3)}(F)$

Relation with ∂

Theorem

Let C be a regular curve over a number field k and $F = k(C)$. For P in C there exists

$$\begin{aligned}\delta_P : \widetilde{M}_{(2)}(F) \otimes F^*_{\mathbb{Q}} &\to \widetilde{M}_{(2)}(k(P)) \\ [f]_2 \otimes g &\mapsto \operatorname{ord}_P(g)[f(P)]_2 \\ &\qquad [0]_2 = [1]_2 = [\infty]_2 = 0\end{aligned}$$

$\delta = \prod_P \delta_P$ *induces a commutative (up to a universal sign) diagram*

$$\begin{tikzcd} H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \arrow[r] \arrow[d, "2\delta"'] & K_4^{(3)}(F) \arrow[d, "\partial"] \\ \coprod_{P \in C} H^1(\widetilde{\mathcal{M}}_{(2)}(k(P))) \arrow[r] & \coprod_{P \in C} K_3^{(2)}(k(P)) \end{tikzcd}$$

Some examples (elliptic curves over $\mathbb{Q}$)

Simple example 1 For $E : (y - \frac{1}{2})^2 = x^3 + \frac{1}{4}$ ($N = 27$) we have $y(1-y) = (-x)^3$ so for $\beta = [y]_2 \otimes (1-x) - 3[x]_2 \otimes y$

$$\mathrm{d}(\beta) = (1-y) \wedge y \wedge (1-x) - (1-x) \wedge x^3 \wedge y = 0 \,.$$

$\delta(\beta)$ is supported in $E(\mathbb{Q}(\sqrt{5}))$ so is trivial. So β gives $\tilde{\beta}$ in $K_4^{(3)}(E)$. Numerically, $R_4(\tilde{\beta}) = -\frac{6}{5} L'(E, -1)$.

Simple example 2 For $E : (y - \frac{1}{2})^2 = x^3 - x^2 + \frac{1}{4}$ ($N = 11$) we have $y(1-y) = x^2(1-x)$ and $\beta = [x]_2 \otimes y + [y]_2 \otimes x$ satisfies $\mathrm{d}(\beta) = 0$. $\delta(\beta)$ is supported in $E(\mathbb{Q})$ so is trivial. So β gives $\tilde{\beta}$ in $K_4^{(3)}(E)$. Numerically, $R_4(\tilde{\beta}) = -3L'(E, -1)$.

François Brunault (preprint 2022) used Siegel modular units to construct, for each elliptic curve E over $\mathbb{Q}$, elements in $H^2(\widetilde{\mathcal{M}}_{(3)}(\mathbb{Q}(E)))$; if E has conductor at most 50, there is an element such that, numerically, it is the kernel of δ, and has non-zero regulator that relates as expected in Beilinson's conjecture to $L'(E, -1)$. Triệu Thu Hà (2024) provided some isolated examples for elliptic curves over $\mathbb{Q}$ (related to Mahler measures).