Mastermath **Topics in Number Theory**
Exam 8th June 2010 (13:00-16:00)

Use separate sheets for the two parts of the course

If you cannot do a part of a question, you may still use its conclusion later on.

p-adic numbers

(1) Let p be a prime number, and $f(X) = \sum_{n=1}^{\infty} p^{(n-2)^2} X^n$ in $\mathbb{Q}_p[[X]]$.
 (a) Find the radius of convergence of $f(X)$ and of $f'(X)$.

By Strassman's theorem, $f(X) = 0$ has at most two solutions in $\mathbb{Z}_p$. Clearly, 0 is one, and we shall show there is another solution in $-p + p^2\mathbb{Z}_p$.
 (b) Show that, if α is in $-p + p^2\mathbb{Z}_p$, then $f'(\alpha)$ is in $-p + p^2\mathbb{Z}_p$.
 (c) Show that $f(\alpha + \beta) \equiv f(\alpha) + \beta f'(\alpha)$ modulo $p^{m+2}\mathbb{Z}_p$ if α is in $\mathbb{Z}_p$ and β is in $p^m\mathbb{Z}_p$ with $m \geq 2$.
 (d) Let $\alpha_1 = -p$. For $m \geq 2$, let $\alpha_m = \alpha_{m-1} - f(\alpha_{m-1})/f'(\alpha_{m-1})$. Show that, for all $m \geq 2$, $\alpha_m - \alpha_{m-1}$ is in $p^m\mathbb{Z}_p$ and $f(\alpha_m)$ is in $p^{m+2}\mathbb{Z}_p$.
 (e) Explain why this shows there is a zero of $f(X)$ in $-p + p^2\mathbb{Z}_p$.

(2) Let $K = \mathbb{Q}_3(\sqrt{3})$, an extension of $\mathbb{Q}_3$ of degree 2. The extension of $|\cdot|_3$ on $\mathbb{Q}_3$ to K is given by the formula $|a + b\sqrt{3}| = \sqrt{|a^2 - 3b^2|_3}$ for a, b in $\mathbb{Q}_3$.
 (a) Show that the valuation ring $\mathcal{O}_K$ is $\{a + b\sqrt{3}$ with $a, b \in \mathbb{Z}_3\}$, the valuation ideal $\mathfrak{P}_K$ is $\sqrt{3}\mathcal{O}_K$, and that $\mathcal{O}_K/\mathfrak{P}_K$ has three elements.

We shall now show that the only roots of unity in K are 1 and -1.
 (b) Show that if α is in $\mathfrak{P}_K$ and m is a positive integer not divisible by 3, then $|(1+\alpha)^m - 1| = |\alpha|$.
 (c) Let m be a positive integer not divisible by 3. If ζ in K is a root of $X^m - 1 = 0$ then show that $\zeta = \pm 1$. (Hint: Show that either $\zeta - 1$ or $-\zeta - 1$ is in $\mathfrak{P}_K$.)
 (d) Conclude that -1 is not a square in K, and that K contains no root of X^2+X+1. (Hint: if α were a root, consider $\frac{2\alpha+1}{\sqrt{3}}$.)

L-functions

(3) Recall that the Möbius function is defined by:

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n = p_1 p_2 \dots p_r \text{ is square-free,} \\ 0 & \text{otherwise.} \end{cases}$$

 (a) Given a positive integer k let

$$q_k(n) = \sum_{d^k | n} \mu(d)$$

Show that $q_k(n)$ is a multiplicative function which satisfies:

$$q_k(n) = \begin{cases} 1 & \text{if } n \text{ is } k\text{-th power free,} \\ 0 & \text{otherwise.} \end{cases}$$

 (b) Show that

$$\sum_{n \geq 1} \frac{q_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}$$

(4) (a) For $s \in \mathbb{C}$ with $Re(s) > 1$ let

$$f(s) = \zeta(s) - \frac{1}{1-s}.$$

Show that $f(s)$ extends to a holomorphic function in the region

$$\{s \in \mathbb{C} \mid Re(s) > 0\}.$$

(b) Show that for $s \in \mathbb{C}$ with $Re(s) > 1$ the identity:

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1}$$

holds.

(c) Conclude from (a) and (b) that

$$\Phi(s) = \sum_p \frac{\log p}{p^s}$$

extends to a meromorphic function on

$$\{s \in \mathbb{C} \mid Re(s) > \frac{1}{2}\}.$$

and find its poles in this region.

Grading scheme			
1a: 5	2a: 6	3a: 9	4a: 9
1b: 5	2b: 4	3b: 9	4b: 9
1c: 5	2c: 6		4c: 9
1d: 7	2d: 4		
1e: 3			
Maximum score = 90			
Grade = 1 + Score/10			