Topics in number theory: p-adic numbers

Problems.

(1) Let $K = \mathbb{Q}_5(\sqrt{2})$ and $L = \mathbb{Q}_5(\sqrt{5})$. Find $e(K/\mathbb{Q}_5)$ and $e(L/\mathbb{Q}_5)$. Show that $X^8 - 1 = 0$ has eight roots in K (in fact, $X^{24} - 1 = 0$ has 24 roots in K). Find uniformizers for K and L. Describe $\mathcal{O}_K/\mathfrak{P}_K$.

(2) Let K be a totally ramified extension of $\mathbb{Q}_p$ of degree e. Show that there are elements β in K and α in $\mathbb{Z}_p$ with $v_p(\alpha) = 1$ such that $|\beta^e - \alpha|_p < 1/p$.

(3) If K is tamely totally ramified (that is $e = e(K/\mathbb{Q}_p) = [K : \mathbb{Q}_p]$ and $(e, p) = 1$) then show that there are elements β in K and α in $\mathbb{Z}_p$ with $v_p(\alpha) = 1$ such that $\beta^e = \alpha$ and $K = \mathbb{Q}_p(\beta)$.

(4) Show that $x^5 - p^2x^3 + p^2$ is irreducible over $\mathbb{Q}_p[x]$. More generally, if $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}_p[x]$ is such that $a_n = 1$, $v_p(a_i) \geq n - i$ for $1 \leq i \leq n-1$, $r = v_p(a_0) < n$ and $(r, n) = 1$, then show that $f(x)$ is irreducible over $\mathbb{Q}_p[x]$.