Topics in number theory: p-adic numbers
Homework 3

Due **Tuesday**, the 6th of April, 2010

- You are allowed to work in pairs. If you are collaborating with another person, hand in only one copy of solutions with both the names mentioned on it.
- You can hand in the homework in the class. Alternatively, you can either email a copy to `tejaswi@few.vu.nl` or (if convenient) put a copy of the homework in the mailbox (of Navilarekallu) in the science building in VU (Room S-322). If you are submitting by email or in the mailbox, the deadline is 2 p.m. on the due date.
- If you have any questions, email `tejaswi@few.vu.nl`.

(1) Show that, for n in $\mathbb{Z}_p$ and x in $2p\mathbb{Z}_p$, $|\exp(n\log(1+x))-1|_p = |xn|_p$. (Note $\exp(n\log(1+x))$ is the definition of $(1+x)^n$.)

(2) Let $f(X) = \sum_{n\geq 1} \frac{X^n}{n^2}$. Verify that $x \mapsto f(x)$ is a function on $p\mathbb{Z}_p$, and use the corollary to Strassman's theorem to bound the number of its zeroes in the balls $p^m\mathbb{Z}_p$ $(m \geq 1)$.

(3) Let $K = \mathbb{Q}_2(\sqrt{2})$, an extension of $\mathbb{Q}_2$ of degree. $|\cdot|_2$ on $\mathbb{Q}_2$ extends to a non-archimedean absolute value on K by the formula $|a + b\sqrt{2}| = \sqrt{|a^2 - 2b^2|_2}$ for a, b in $\mathbb{Q}_2$.
   (a) Show that K does not contain fourth roots of unity other than ± 1.
   (b) Show that the valuation ring of K is $\mathcal{O}_K = \{a + b\sqrt{2}$ with $a, b \in \mathbb{Z}_2\}$ with valuation ideal $\mathfrak{P}_K = \sqrt{2}\mathcal{O}_K$, and describe the residue field $\mathcal{O}_K/\mathfrak{P}_K$ explicitly.
   (c) Suppose that $\zeta \in K \setminus \{1\}$ satisfies $\zeta^p = 1$ for some prime $p \neq 2$. Use the identity $(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{p-1}) = p$, and the absolute values of ζ and $1-\zeta$ to show that there are no non-trivial p-th roots of unity in K for any odd prime p.

(4) We let $L = \mathbb{Q}_2(\sqrt{5})$ and $M = \mathbb{Q}_2(\sqrt{7})$. They are extensions of degree 2 of $\mathbb{Q}_2$ because 5 and 7 are not squares in $\mathbb{Q}_5$. $|\cdot|_2$ on $\mathbb{Q}_2$ extends to a non-archimedean absolute value on each by the formula $|a + b\sqrt{D}| = \sqrt{|a^2 - Db^2|_2}$ for a, b in $\mathbb{Q}_2$, and $D = 5$ or 7.
   (a) For a α in $L^\times$ or $M^\times$, define $v(\alpha)$ in $\frac{1}{2}\mathbb{Z}$ by $|\alpha| = 2^{-v(\alpha)}$. Show that $v(\alpha) = v_2(\alpha)$ if α is in $\mathbb{Q}_2^\times$. Moreover, show that for M, the image of v is $\frac{1}{2}\mathbb{Z}$, but that for L the image is $\mathbb{Z}$. (Hint: For L, first show that if a, b are in $\mathbb{Z}_2^\times$, then $v(a + b\sqrt{5}) = 1$.)
   (b) Show that $X^8 - 1$ has exactly 4 roots in M, namely those of $X^4 - 1$.
   (c) Show that for M, the valuation ring is $\mathcal{O}_M = \{a + b\sqrt{7}$ with $a, b \in \mathbb{Z}_2\}$ with valuation ideal $\mathfrak{P}_M = (1 + \sqrt{7})\mathcal{O}_M$, and describe the residue field $\mathcal{O}_M/\mathfrak{P}_M$ explicitly.
   (d) Determine the valuation ring $\mathcal{O}_L$ of L and its valuation ideal $\mathfrak{P}_L$, and describe the residue field $\mathcal{O}_L/\mathfrak{P}_L$ explicitly.