Well-posedness for a moving boundary model of an evaporation front in a porous medium

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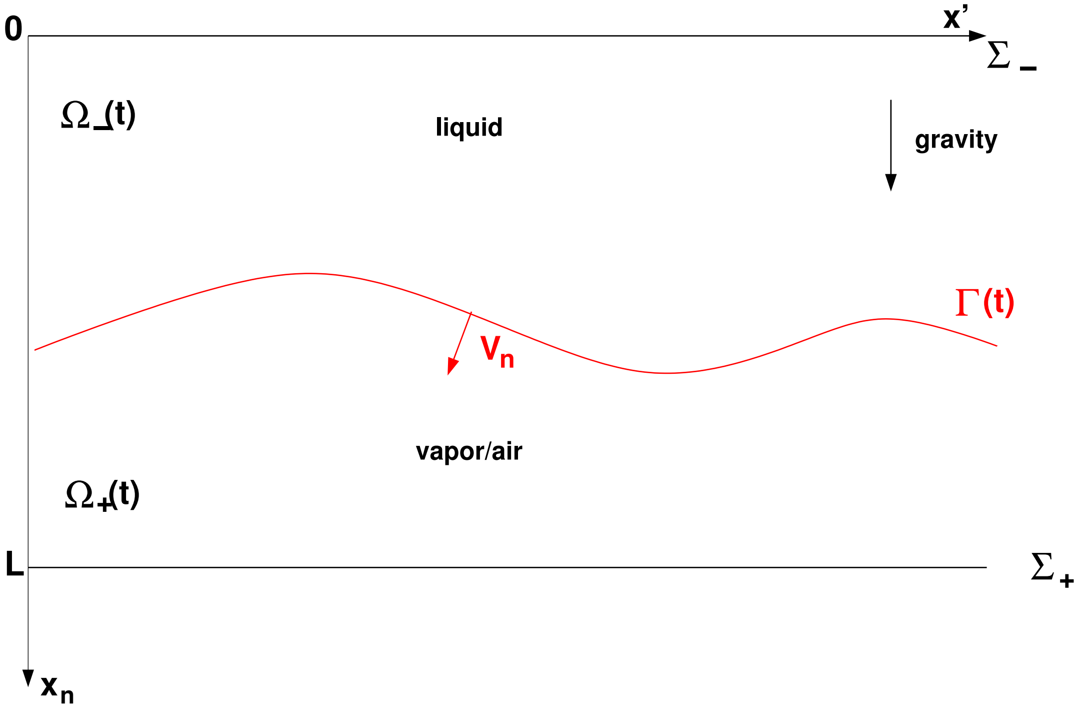
The problem

(Il'ichev / Tsypkin '08, Il'ichev / Shargatov '13)
Downward flow with evaporation in a porous medium above an underground construction:



The model: liquid phase $\Omega_-(t)$

- main variable: hydrodynamic pressure P
- Darcy's law & mass conservation
- prescribed pressure at upper and lower boundary

$$\left.\begin{array}{rcll} \Delta P &=& 0 & \text{in } \Omega_-(t), \\ P &=& P_a + P_c & \text{on } \Gamma(t), \\ P &=& P_0 & \text{on } \Sigma_- \end{array}\right\}$$

$P_{a,c}$: atmospheric / capillary pressure

The model: vapor phase $\Omega_+(t)$

- main variable: humidity $\nu \approx \frac{\rho_v}{\rho_a}$
- linear diffusion & mass conservation
- phase change at evaporation humidity ν^* (fixed temperature)
- prescribed humidity at lower boundary

$$\left.\begin{array}{rcll} (\partial_t - D\Delta)\nu &=& 0 & \text{in } \Omega_+(t), \\ \nu &=& \nu^* & \text{on } \Gamma(t), \\ \nu &=& \nu_a & \text{on } \Sigma_+ \end{array}\right\}$$

ν_a: atmospheric humidity
D: vapor diffusivity

The model: motion of interface $\Gamma(t)$

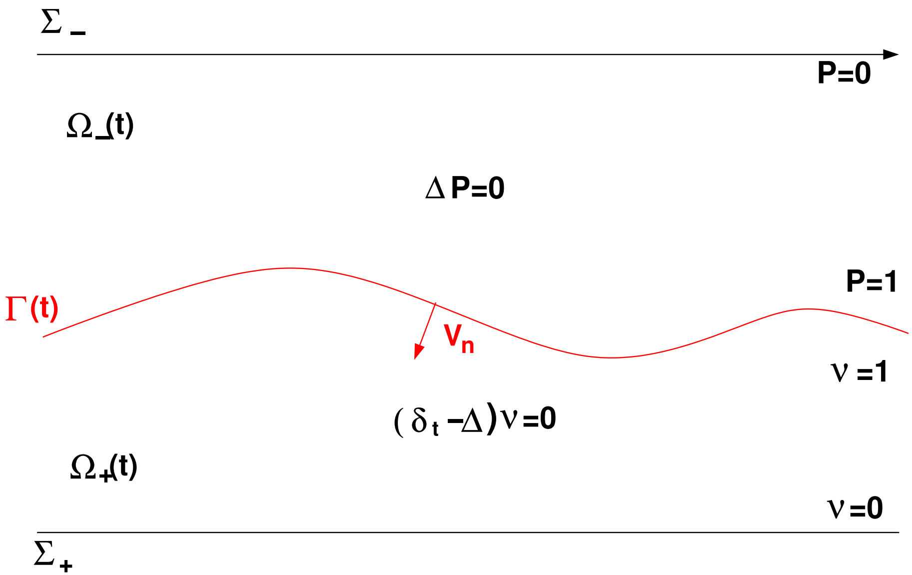
determined by mass balance of water:

$$\left(1 - \frac{\rho_v}{\rho_w}\right) V_n = -\frac{k}{m\mu_w}\partial_{n(t)}(P - \rho_w g x_n) + D\frac{\rho_a}{\rho_w}\partial_{n(t)}\nu \quad \text{on } \Gamma(t).$$

m: porosity
k: permeability to water
μ_w: viscosity of water
$\rho_{v,w,a}$: density of vapor, water, air
g: gravity

Mathematical structure

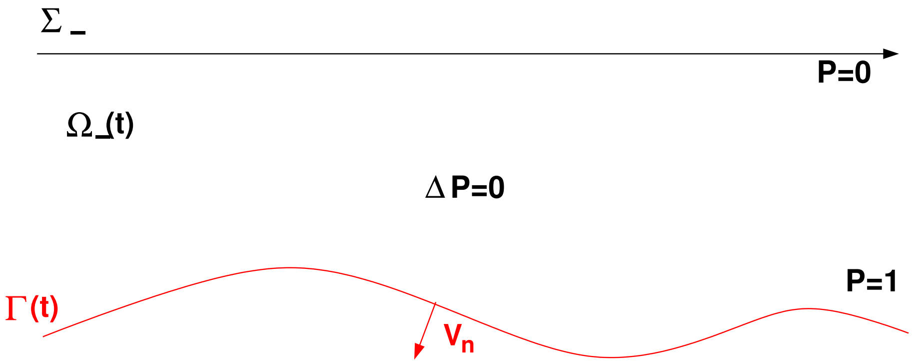
After nondimensionalization and rescaling:



Boundary motion: $\mu V_n = \partial_{n(t)}(-\alpha P + \beta \nu + x_n)$

Mathematical structure

Liquid phase only: Hele-Shaw problem!

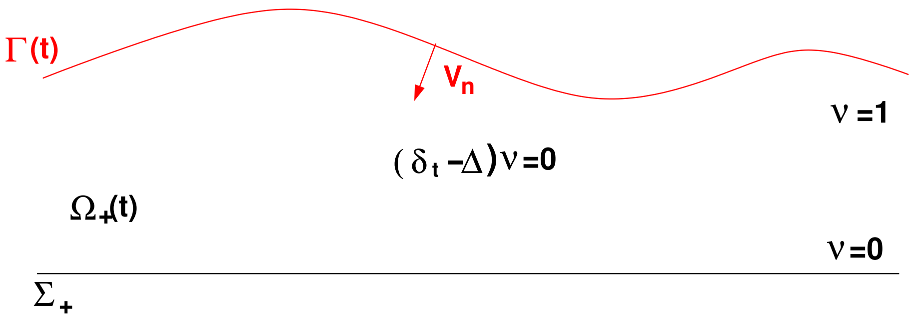


Boundary motion: $\mu V_n = \partial_{n(t)}(-\alpha P + x_n)$

Well-posedness condition: $-\operatorname{sgn}(\alpha)\partial_n P < 0$.

Mathematical structure

Vapor phase only: Stefan problem!

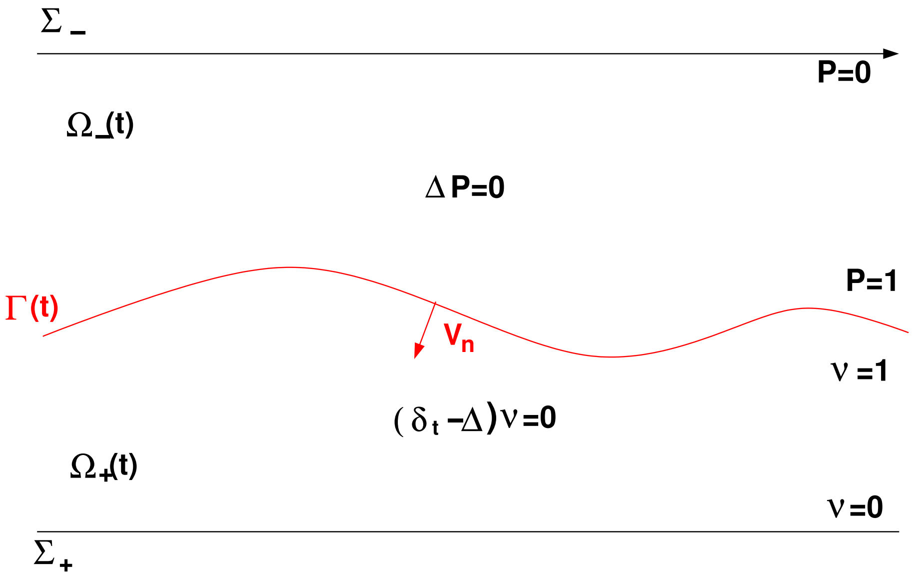


Boundary motion: $\mu V_n = \partial_{n(t)}(\beta \nu + x_n)$

Well-posedness condition: $\partial_n \nu < 0$.

Mathematical structure

Both phases:



Boundary motion: $\mu V_n = \partial_{n(t)}(-\alpha P + \beta \nu + x_n)$

Well-posedness condition: $\partial_n(-\alpha P + \beta \nu) < 0$.

Remarks:

MBP with elliptic-parabolic systems in the bulk:

- mostly with surface tension term in boundary (Escher '04, Lippoth / P. '16,...)

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- without surface tension: Bazalii / Degtyarev '89

Approach

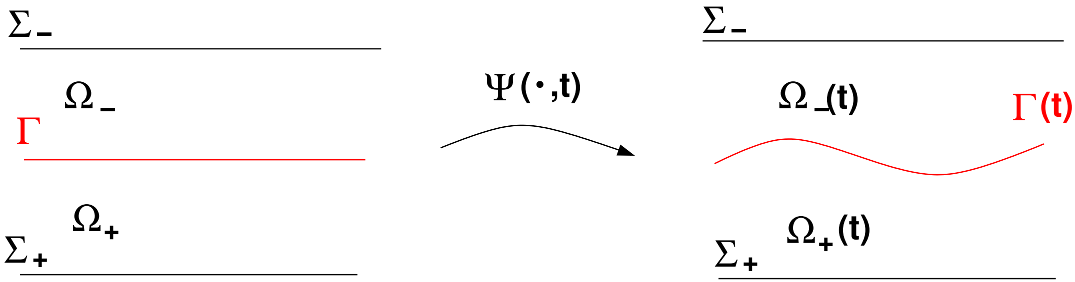
extends L^p-based results for the Stefan problem
(Solonnikov / Frolova '00) by "adding" the elliptic phase

- transform to fixed domain by unknown diffeomorphism

$\Rightarrow$ coupled nonlinear elliptic-parabolic system

- obtain maximal regularity results for its linearization
- show short-time well-posedness by Banach fixed point iteration

Transformation



$$\Psi(z,t) = z + (\sigma + \phi)(z,t)e_n$$

σ: fixed, encodes initial interface.

Notation:

$$\begin{aligned} \phi^{\pm} &:= \phi|_{\Omega_{\pm}} \\ u^{+} &:= \nu \circ \Psi - U^{+}, \\ u^{-} &:= P \circ \Psi - U^{-} \end{aligned}$$

$U^{\pm}$: solutions to transformed Laplace and Heat eq. for $\phi \equiv 0$ such that

$$(\phi^{\pm}, u^{\pm})|_{t=0} = 0.$$

The nonlinear system

$$
\begin{aligned}
\tilde{L}^{\pm}u^{\pm} - \tfrac{U^{\pm}_{z_n}}{1+\sigma_{z_n}} L^{\pm}\phi^{\pm} &= K^{\pm}\phi^{\pm} + R^{\pm}(\phi^{\pm}, u^{\pm}) && \text{in } \Omega_{\pm} \times (0,T),\\
\partial_t \phi - \alpha^{\pm}\partial_{z_n}\phi^{\pm} & \\
-\tilde{\alpha}^{\pm}\partial_{z_n}u^{\pm} + \zeta \cdot \nabla'\phi &= g_0 + R^{B}(\phi^{\pm}, u^{\pm}) && \text{on } \Gamma \times (0,T),\\
\phi_{\pm} - \phi &= 0 && \text{on } \Gamma \times (0,T),\\
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\end{aligned}
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Initial data: $(\phi^{+}, u^{+}, \phi)|_{t=0} = 0$.

$\tilde{L}^{\pm}$,$L^{\pm}$: transformed versions of Laplace /heat operator
$K^{\pm}$: first order differential operator
$R^{\pm,B}$: “higher order” terms in $(\phi^{\pm}, u^{\pm})$.

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Use the freedom in the choice of ϕ to simplify!

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This is a quasilinear elliptic-parabolic system
with dynamic boundary condition.

Core: The linear problem for $(\phi^{\pm}, \phi)$

$$\begin{aligned} L^{\pm}\phi^{\pm} &= 0 \quad \text{in } \Omega_{\pm} \times (0,T), \\ \partial_t \phi - \alpha^{\pm}\partial_{z_n}\phi^{\pm} + \zeta \cdot \nabla'\phi &= g \quad \text{on } \Gamma \times (0,T), \\ \phi_{\pm} - \phi &= 0 \quad \text{on } \Gamma \times (0,T), \\ \phi_{\pm} &= 0 \quad \text{on } \Sigma_{\pm} \times (0,T) \end{aligned}$$

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Maximal regularity:

1. halfspace model problem: Laplacian / heat operator

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Maximal regularity:

1. halfspace model problem: Laplacian / heat operator
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3. Freezing of coefficients $\Longrightarrow$ domain, variable coefficients

   All for $g|_{t=0} = 0$.
4. Inverse trace theorem (Denk / Saal / Seiler '08)

   $\Longrightarrow$ extend to $g|_{t=0} \neq 0$.

Model problem

$$\begin{aligned} \Delta\phi^- &= 0 \quad \text{in } \mathbb{H}^n_- \times (0,\infty), \\ (\partial_t - \Delta)\phi^+ &= 0 \quad \text{in } \mathbb{H}^n_+ \times (0,\infty), \\ \partial_t - \alpha^\pm \partial_{z_n}\phi^\pm + c\cdot\nabla'\phi &= g \quad \text{on } \mathbb{R}^{n-1} \times (0,\infty), \\ \phi^\pm - \phi &= 0 \quad \text{on } \mathbb{R}^{n-1} \times (0,\infty), \end{aligned}$$

Fourier transform in $\mathbb{R}^{n-1}$ and Laplace transform in time:

$$P(\lambda, i\xi)\hat{\phi}(\lambda,\xi) = \hat{g}(\lambda,\xi)$$

Symbol

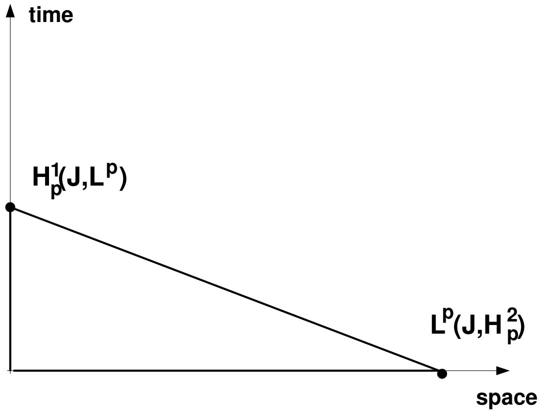
$$P(\lambda, z) = \lambda + \alpha^-|z|_- + \alpha^+\sqrt{\lambda + |z|_-^2} - c\cdot z, \qquad |z|_- := \sqrt{-\sum z_k^2}$$

P is not quasihomogeneous in (λ, z).

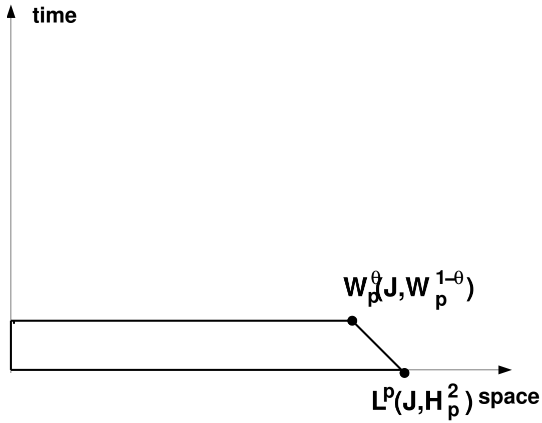
But: Maximal regularity results of Denk / Kaip '13 apply.

Function spaces

Parabolic phase (ϕ^+, u^+):

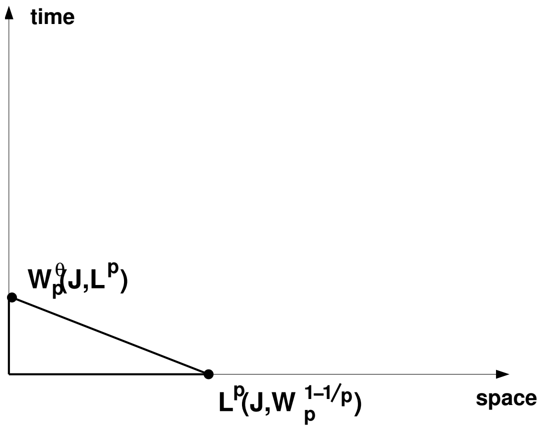


Elliptic phase (ϕ^-, u^-):

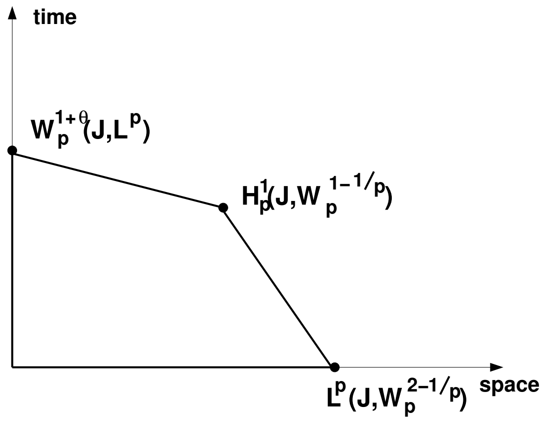


Function spaces

Boundary data (g):



Solution ϕ on the boundary:



The nonlinear problem

- written in fixed-point form:

$$\mathcal{U} = C^{-1}(\mathcal{F}(\mathcal{U}) + \mathcal{G}_0) = \Phi(\mathcal{U})$$

where C encodes the linearization.

- Contraction from embeddings with small constants as T is small, and inital data are zero.

- Additional trick for elliptic phase: Work with different values of θ!

Further questions:

- Stability of horizontal equilibria

- parabolic smoothing (“parameter trick”)

- Existence of nontrivial equilibria

- Justification of approximations near loss of stability (?)

Further questions:

- Stability of horizontal equilibria
  (Linear stability: BSc project A. Cotino
  Long- time existence near equlibria: MSc project O. Çaylak)
- parabolic smoothing ("parameter trick")
  (MSc project M. Huveneers)
- Existence of nontrivial equilibria
- Justification of approximations near loss of stability (?)