On the Equations of Nonlinear Single-Phase Poroelasticity

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Corresponding literature

Detailed proofs of the results from the talk are in the article

[*] C.J. van Duijn, A. Mikelic, Mathematical Theory of Nonlinear Single-Phase Poroelasticity, https://www.darcycenter.org/wp-content/uploads/2019/06/Duijn-and-Mikelic-2019-5.pdf.

Related article:

[**] C.J. van Duijn, A. Mikelić, T. Wick: A monolithic phase-field model of a fluid-driven fracture in a nonlinear poroelastic medium, *Mathematics and Mechanics of Solids*, Vol. 24 (5) (2019), 1530–1555.

[***] C.J. van Duijn, A. Mikelić, T. Wick: Mandel's problem as a benchmark for the multidimensional nonlinear poroelasticity, in preparation, 2019.

Introduction

– The elastic quasi-static deformation of a fluid saturated porous medium received much attention in the civil engineering literature because of its relevance to many problems of practical interest.
– In the framework of consolidation in soil mechanics: the physical loading of soil layers or the effect of soil subsidence due to groundwater withdrawal for drinking water supply or industrial and agricultural purposes. See monographs by Coussy, Lewis and Schrefler and Verruijt. They build on the classical theory of Terzaghi and the pioneering approach of Biot.
— Recently, other examples of elastic deformation of porous media arise in the context of industrial and biomedical applications, such as paper printing, bone regeneration, blood flow and car filters.

A simple linear model

In its simplest form, assuming both the fluid and the porous material (grains) to be incompressible and assuming the porous medium to be homogeneous and linear elastic with small strains, the mathematical formulation reads

$$\text{div }\partial_t \mathbf{u} + \text{ div }\left(\frac{\mathbb{K}}{\eta_f}(\rho_f \mathbf{g} - \nabla p)\right) = q \tag{1}$$

and

$$-\text{ div }\sigma = \mathbf{F}, \tag{2}$$

where

$$\sigma = \mathcal{G}e(\mathbf{u}) - \alpha p \mathbb{I}, \tag{3}$$

with

$$\mathcal{G}E = 2\mu E + \lambda \text{Tr}(E)\mathbb{I}, \quad \text{for symmetric matrices } E. \tag{4}$$

Glossary

In these equations, $\mathbf{u}$ [m] denotes skeleton displacement, $\mathbb{K}$ [m^2] intrinsic permeability (a symmetric positive definite rank-2 tensor), η_f [Pa s] fluid viscosity, p [Pa] fluid pressure and q [$1/s$] sources/sinks. Further, σ [Pa] is the total stress, $\mathbf{F}$ a given body force (generally linked to gravitational effects), $\mathcal{G}$ the symmetric, positive-definite, rank-4 Gassmann tensor, $e(\mathbf{u})$ the linearized strain tensor and $\alpha \in (0, 1]$ Biot's effective stress parameter. Finally, μ [Pa] and λ [Pa] are Lamé's parameters. Using for $\mathcal{G}$ the specific form (4), i.e. Hooke's law, assumes that the skeleton is mechanically isotropic.

The linear quasi-static Biot system, as well as its dynamical analogue, was also derived by means of a multiscale approach, where the starting point is the linear fluid-structure interaction at the pore level.

Nonlinear mixture of two incompressible phases

In the engineering literature one writes $\alpha = 1 - K/K_g$, where K is the drained bulk modulus of the porous skeleton and K_g the bulk modulus of the grains. Since it is assumed that $K_g = +\infty$, we will set $\alpha = 1$ in (3).

Equations (1)-(4) were studied by Ženišek 1984, who was one of the first to demonstrate existence and uniqueness, and by Showalter 2000 in the dynamic case.

Later, Cao et al (DCDS Ser. B, 2013) considered a nonlinear extension of (1), by replacing the permeability tensor $\mathbb{K}$ by the product $\mathbb{K}k(\text{div}\,\mathbf{u})$. The function $k(\cdot)$ is a relative permeability depending on the volumetric strain div $\mathbf{u}$.

The quasi-static equations= lack of the time derivatives.
Some authors circumvent this by introducing a time dependence in (2)-(4) as well. For instance Bociu L, Guidoboni G et al (ARMA 2016) replace $\mathbf{u}$ in (3) by $\mathbf{u} + \delta\partial_t\mathbf{u}$, $(\delta > 0)$, i.e. introduce a viscoelastic effect.

A general nonlinear fluid phase mass balance

A regularization proposed by Murad and Cushman (Internat. J Engrg Sci., 1996):

$$\sigma = 2\mu e(\mathbf{u}) + (\lambda \mathrm{div}\ \mathbf{u} + \lambda^* \mathrm{div}\ \partial_t \mathbf{u} - \alpha p)\mathbb{I}, \tag{5}$$

with $\lambda^* > 0$. This form arises in the non-equilibrium theory, where the fluid pressure and the solid pressure differ by λ^* div $\partial_t \mathbf{u}$.

OUR MODEL

We propose to study the quasi-static formulation in which we replace equation (1) by the nonlinear fluid phase mass balance based on the mixture theory of Bedford and Drumheller, see the monograph by Lewis and Schrefler 1998:

$$n\partial_t \rho + \rho \mathrm{div}\ \partial_t \mathbf{u} + \mathrm{div}\ \mathbf{j} = Q, \tag{6}$$

where $\mathbf{j}$ denotes the Darcy mass flux

$$\mathbf{j} = \frac{\mathbb{K} k(n)\rho}{\eta_f}(\rho \mathbf{g} - \nabla p). \tag{7}$$

Nonlinear coefficients

n denotes porosity, $\rho = \rho_f$ [kg/m^3] fluid density, k relative permeability and Q [kg/m^3 s] sources/sinks.
In equations (6)-(7), the porosity n is a given function of the volumetric strain: i.e.

$$n = n(\text{div } \mathbf{u}). \tag{8}$$

Next, assuming weak compressibility we write

$$\rho = \rho(p) = \rho_0(1 + \beta(p - p_0)). \tag{9}$$

Further the relative permeability k depends the porosity $k = k(n)$.
The relative permeability in satisfies

$$k \in C^1[0, 1], \quad k(0) > 0 \quad \text{and} \quad k' > 0 \text{ in } ([0, 1). \tag{10}$$

A well-known example is the Kozeny-Carman formula

$$k(n) = k_0 \frac{n^3}{(1 - n)^2} \quad (k_0 > 0), \tag{11}$$

in a realistic porosity interval, bounded away from $n = 0$ and $n = 1$.

Problem formulation

– Let $\Omega \subset \mathbb{R}^m$ (m=2,3) denote a bounded domain, occupied by a linear elastic skeleton. The skeleton material (grains) is assumed incompressible: i.e. the bulk modulus of the grains is infinitely large. The voids in the porous structure are completely filled with a slightly compressible fluid, in the sense that the fluid pressure p and density ρ are related by (9).

– For given $\xi \in \Omega$, let $\mathbf{x}(\xi, t)$ denote the location of a solid particle at time $t > 0$, that started at $\mathbf{x}(\xi, 0) = \xi,$. Then the skeleton velocity $\mathbf{v}_s$ is given by $\mathbf{v}_s = \partial_t \mathbf{x}|_\xi$.

– Restricting themselves to small displacements $\mathbf{u}$ (within the elastic regime), Rutquist et al and Lewis and Schrefler argue that in the mass balance equation for the fluid and solid, the material derivative $\dfrac{D}{Dt} = \partial_t + \mathbf{v}_s \cdot \nabla$ can be replaced by the partial derivative ∂_t. This is made explicit by a scaling argument in van Duijn et al [**].

Balance equations

The resulting mass balances reads:

$$n\partial_t \rho + \rho \mathrm{div}\ \mathbf{v}_s + \mathrm{div}\ \mathbf{j} = Q \quad \text{(fluid phase)} \tag{12}$$

and

$$\partial_t (1-n) + (1-n) \mathrm{div}\ \mathbf{v}_s = 0 \quad \text{(solid phase)}, \tag{13}$$

where $\mathbf{j}$ is mass flux (7).
Within the same approximation one may write

$$\mathrm{div}\ \mathbf{v}_s = \partial_t \mathrm{div}\ \mathbf{u} \Rightarrow$$

$$n\partial_t \rho + \rho \partial_t \mathrm{div}\ \mathbf{u} + \mathrm{div}\ \mathbf{j} = Q \tag{14}$$

and

$$\partial_t (1-n) + (1-n) \mathrm{div}\ \partial_t \mathbf{u} = 0. \tag{15}$$

Integrating (15) in time from $t = 0$, say, to $t > 0$, we have

$$1 - n = (1 - n_0) e^{-\mathrm{div}\ (\mathbf{u} - \mathbf{U}_0)} \quad \text{for } t > 0.. \tag{16}$$

Summary of the equations

For small displacements $\mathbf{u}-\mathbf{U}_0$, expression (16) is approximated by

$$n = n_0 + (1-n_0)\text{div }(\mathbf{u}-\mathbf{U}_0). \tag{17}$$

We set

$$\mathbf{u} := \mathbf{u}-\mathbf{U}_0, \tag{18}$$

where $\mathbf{U}_0 \in H^1_0(\Omega)^m \cap H^2(\Omega)^m$ is the initial displacement. With

$$\mathbf{F} := \mathbf{F} + \text{div }(\mathcal{G}e(\mathbf{U}_0)), \tag{19}$$

we obtain for the fluid pressure p and the skeleton displacement $\mathbf{u}$

$$n\partial_t\rho + \rho\ \text{div }\partial_t\mathbf{u} + \text{ div }\left(\frac{\mathbb{K}k(n)\rho}{\eta_f}(\rho\mathbf{g}-\nabla p)\right) = Q, \tag{20}$$

$$-\text{ div }(\mathcal{G}e(\mathbf{u}) - p\mathbb{I}) = \mathbf{F}, \tag{21}$$

$$\rho = \rho(p) = \rho_0(1+\beta(p-p_0)), \tag{22}$$

$$n = n(\text{div }\mathbf{u}) = 1-(1-n_0)e^{-\text{div }\mathbf{u}} \tag{23}$$

$$\approx n_0 + (1-n_0)\text{div }\mathbf{u} \quad \text{(small strains)}. \tag{24}$$

Negative porosity

We consider a simplified version of the linear problem (1)-(4) and show that div $\mathbf{u}$ can attain values for which the **porosity** from (23)-(24) becomes **negative**.

Let $\Omega = (0, L)^2$ for some $L > 0$. We suppose, as in the rest of this paper, that div $\mathbf{u}|_{t=0} = 0$. Further we set $\mathbf{F} = 0$ in (21). Then

$$H = (2\mu + \lambda)\mathrm{div}\ \mathbf{u} - p$$

is harmonic in Ω. We prescribe

$$\begin{cases} \{x_1 = 0,\ L\} :\ u_2 = 0,\ \sigma_{11} = \Sigma^{1,0} \text{ (or resp. } \Sigma^{1,L}),\ p = 0; \\ \{x_2 = 0,\ L\} :\ u_1 = 0,\ \sigma_{22} = \Sigma^{2,0} \text{ (or resp. } \Sigma^{2,L}),\ p = 0; \end{cases} \tag{25}$$

Then $H|_{\partial\Omega} = \Sigma^b$ and we have

Proposition 1 Let $\mathcal{E} = \mathrm{div}\ \mathbf{u}$ denote the volumetric stress and let $n(\mathcal{E})$ be given by (23). Suppose there exists a constant $\Sigma > 0$ such that $\Sigma^b \leq -\Sigma$. Then for Σ sufficiently large, there exists a $T_p = T_p(\Sigma) > 0$ such that

$$n(\mathcal{E}(x,t)) < 0 \quad \text{for } t > T_p \text{ and } x \in \overline{\Omega}. \tag{26}$$

Modification of balance equations

In a number of steps we modify equation (20) so that it becomes well-posed in a mathematical sense and reduces to its original form in the physical range of the unknowns.
First, to satisfy the natural bounds, we replace the porosity approximation (24) by a smooth increasing function $\overline{n} : \mathbb{R} \to \mathbb{R}$ such that

$$\overline{n}(\mathcal{E}) = \begin{cases} \lim\limits_{\mathcal{E}\to+\infty} \overline{n}(\mathcal{E}) = 1, \\ n_0 + (1-n_0)\mathcal{E}, \ \text{ for } \ \mathcal{E}_* \le \mathcal{E} \le \mathcal{E}^*; \\ \lim\limits_{\mathcal{E}\to-\infty} \overline{n}(\mathcal{E}) = \delta_0 > 0. \end{cases} \tag{27}$$

Next we set

$$p = p(\rho) := p_0 + \frac{\rho - \rho_0}{\beta \rho_0}. \tag{28}$$

When considering (20), one clearly has in mind that ρ takes values near the reference ρ_0. However the mathematical nature of the equations does not guarantee this behaviour.

2nd modification of balance equations

Hence a second modification is needed, now for ρ in the second and third term of the left-hand side of (20). Disregarding gravity, we replace (20) by the modified fluid mass balance equation

$$\overline{n}(\mathcal{E})\partial_t \rho + d(\rho)\partial_t \mathcal{E} - \operatorname{div}\left(k(\mathcal{E})\mathcal{D}(\rho)\mathbb{K}\nabla\rho\right) = Q, \tag{29}$$

where $\overline{n}(\mathcal{E})$ is given by (27) and $k(\mathcal{E}) = k(\overline{n}(\mathcal{E}))$. Further, $d,\ \mathcal{D} : \mathbb{R} \to \mathbb{R}$ are chosen such that

$$\left.\begin{aligned} d(\rho) &= \rho, \\ \mathcal{D}(\rho) &= \frac{\rho}{\eta_f \beta \rho_0}, \end{aligned}\right\} \quad \text{for} \quad |\rho - \rho_0| \le \rho_0 - \rho_*, \tag{30}$$

where $\rho_* \in (0, \rho_0)$ is a small constant. Outside this range we take for d and $\mathcal{D}$ extensions that suit the mathematical analysis. We clarify this at a later point.

Summary of the modified equations

The balance of forces (21) is modified by adding the regularizing term $\lambda^*\partial_t\mathcal{E}$, as in the expression (5). This gives

$$-\operatorname{div}\left(\mathcal{G}e(\mathbf{u}) + (\lambda^*\partial_t\mathcal{E} - \alpha p)\mathbb{I}\right) = \mathbf{F}, \tag{31}$$

where we have $\lambda^* \geq 0$.
As initial conditions we have

$$\mathcal{E}|_{t=0} = 0 \quad \text{and} \quad \rho|_{t=0} = \rho^0 \quad \text{in} \quad \Omega, \tag{32}$$

where $\rho^0 : \Omega \to (0, +\infty)$ is taken near the reference value ρ_0.
Along the boundary we prescribe

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \nabla\rho\cdot\nu|_{\partial\Omega} = 0, \quad \text{for} \quad 0 < t \leq T. \tag{33}$$

where ν is the outward unit normal at $\partial\Omega$.

Energy equality

Now we derive an expression for the free energy which acts as a Lyapunov functional for system (29), (31). This a generalization of the free energy introduced originally by Biot.

Let $\{\mathbf{u}, \rho\}$ be a smooth solution of equations (29), (31) that satisfies conditions (32) and (33). Further, let $g : \mathbb{R} \to \mathbb{R}$ be a smooth, strictly increasing and globally Lipschitz function satisfying $g(\rho_0) = 0$. We set $G(\rho) = \int_{\rho_0}^{\rho} g(z)\ dz$.

We first multiply equation (31) by $\partial_t \mathbf{u}$ and integrate the result in Ω. Next we multiply (29) by $g(\rho)$ and integrate the result in Ω. Adding the resulting expressions yields

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\mathcal{G}e(\mathbf{u}) : e(\mathbf{u}) + \overline{n}(\mathcal{E})G(\rho) - \mathbf{F}\cdot\mathbf{u}\right)\ dx + \lambda^* \int_\Omega (\partial_t \mathcal{E})^2\ dx$$
$$+ \int_\Omega k(\mathcal{E})\mathcal{D}(\rho)g'(\rho)\mathbb{K}\nabla\rho\cdot\nabla\rho\ dx - \int_\Omega Qg(\rho)\ dx +$$
$$\int_\Omega \Big\{d(\rho)g(\rho) - \overline{n}'(\mathcal{E})G(\rho) - p(\rho)\Big\}\partial_t\mathcal{E}\ dx = \int_\Omega \partial_t\mathbf{F}\cdot\mathbf{u}\ dx. \quad (34)$$

Lyapunov functional in the linear setting

Before considering the general nonlinear case described by this expression, we first show its implication for the simplified linear setting. Then we use in (29) and (34)

$$\overline{n}(\mathcal{E}) = n_0, \quad d(\rho) = \rho_0, \quad k(\mathcal{E}) = 1 \quad \text{and} \quad \mathcal{D} = \frac{1}{\eta_f \beta}.$$

After simplifications due to the linearization, (34) yields

$$\frac{d}{dt}\int_\Omega \left\{\frac{1}{2}\mathcal{G}e(\mathbf{u}) : e(\mathbf{u}) + \frac{n_0}{\beta\rho_0^2}(\rho-\rho_0)^2 - \mathbf{F}\cdot\mathbf{u}\right\} dx + \lambda^* \int_\Omega (\partial_t \mathcal{E})^2 \, dx + \int_\Omega \frac{1}{\eta_f \beta^2 \rho_0^2}\mathbb{K}\nabla\rho\cdot\nabla\rho \, dx = \int_\Omega Qg(\rho)\, dx - \int_\Omega \partial_t \mathbf{F}\cdot\mathbf{u}\, dx. \quad (35)$$

Hence

$$\mathcal{L}(\mathbf{u},\rho) = \int_\Omega \left(\frac{1}{2}\mathcal{G}e(\mathbf{u}) : e(\mathbf{u}) + \frac{n_0}{2\beta\rho_0^2}(\rho-\rho_0)^2 - \mathbf{F}\cdot\mathbf{u}\right) dx \quad (36)$$

acts as a Lyapunov functional for the linear form of system (29), (31). Expression (36) coincides with Biot's original free energy.

Lyapunov functional in the nonlinear setting

Next we return to the nonlinear case (34). We need that

$$\int_\Omega \Big\{ d(\rho)g(\rho) - (1-n_0)G(\rho) - p(\rho) \Big\} \partial_t \mathcal{E} \; dx = 0.$$

In the interval $|\rho - \rho_0| < \overline{\rho} := \rho_0 - \rho_*$ where $d(\rho) = \rho$, it gives

$$g(\rho) = \frac{1}{\beta n_0 \rho_0} \big(1 - (\frac{\rho_0}{\rho})^{n_0}\big). \tag{37}$$

$$G(\rho) = \int_{\rho_0}^{\rho} g(\xi)\; d\xi = \frac{1}{\beta n_0 (1-n_0)\rho_0} \Big((1-n_0)\rho - \rho_0^{n_0} \rho^{1-n_0} + n_0 \rho_0 \Big). \tag{38}$$

When $|\rho - \rho_0| > \overline{\rho}$, the function $d(\rho)$ has not yet been defined. We do this by first extending $g(\rho)$ for $|\rho - \rho_0| > \overline{\rho}$ and then by setting

$$d(\rho) = \frac{(1-n_0)G(\rho) + p(\rho) - p_0}{g(\rho)}. \tag{39}$$

Construction of the free energy

Clearly, (37) cannot be used for $\rho \leq 0$. Instead we extend (37) in a linear C^1-manner for $|\rho - \rho_0| > \overline{\rho}$. With $\tilde{\rho} = \rho_0 + \overline{\rho} = 2\rho_0 - \rho_*$. $G(\rho)$ will now be quadratic in ρ. the desired extension for $d(\rho)$ when $|\rho - \rho_0| > \overline{\rho}$. Thus

$$d(\rho) = \begin{cases} \rho & \text{for } |\rho - \rho_0| \leq \overline{\rho}, \\ \text{(39) with } g \text{ and } G \text{ given by the extension} & \text{for } |\rho - \rho_0| > \overline{\rho}. \end{cases} \tag{40}$$

Hence with the triple $\{g(\rho), G(\rho), d(\rho)\}$ constructed above, the cross-term drops from expression (34).

Next we introduce a **second modification** to deal with a porosity satisfying (27). Again we search to cancel the cross-term. This integral vanishes if

$$d(\rho)g(\rho) - \overline{n}'(\mathcal{E})G(\rho) = p(\rho) - p_0. \tag{41}$$

2nd modification in the construction of the free energy

Keeping g and G as above, we now modify $d(\rho)$, calling it $D(\rho, \mathcal{E})$, such that

$$D(\rho,\mathcal{E}) = \frac{\overline{n}'(\mathcal{E})}{g(\rho)} G(\rho) + \frac{p(\rho) - p_0}{g(\rho)}. \tag{42}$$

Using (39) in this expression gives

$$D(\rho,\mathcal{E}) = d(\rho) + (\overline{n}'(\mathcal{E}) - (1 - n_0)) \frac{G(\rho)}{g(\rho)}. \tag{43}$$

Clearly, for $|\rho - \rho_0| < \overline{\rho}$ and $\mathcal{E}_* < \mathcal{E} < \mathcal{E}^*$, this expression reduces to

$$D(\rho,\mathcal{E}) = \rho.$$

Finally we use in the Darcy term from equation (29)

$$\mathcal{D}(\rho) = \frac{1}{\eta_f \rho_0 \beta} \begin{cases} \tilde{\rho}, & \text{for } \rho > \tilde{\rho}; \\ \rho, & \text{for } \rho_* < \rho < \tilde{\rho}; \\ \rho_*, & \text{for } \rho < \rho_*. \end{cases} \tag{44}$$

Summary of the Lyapunov functional

Thus in the end we consider the "second" modified fluid mass balance

$$\overline{n}(\text{div } \mathbf{u})\partial_t \rho + D(\rho, \text{div } \mathbf{u}) \text{ div } \partial_t \mathbf{u} = \text{ div } \Big(k(\overline{n}(\text{div } \mathbf{u}))\mathcal{D}(\rho)\mathbb{K}\nabla \rho \Big) + Q. \tag{45}$$

The function $D(\rho, \mathcal{E})$ in (45) generalizes the fluid density. It is chosen so that

$$J(\mathbf{u}, \rho) = \frac{1}{2} \int_\Omega \mathcal{G}e(\mathbf{u}) : e(\mathbf{u}) \; dx + \int_\Omega \overline{n}(\text{div } \mathbf{u}) G(\rho) \; dx - \int_\Omega \mathbf{F} \cdot \mathbf{u} \; dx \tag{46}$$

acts as a Lyapunov functional for the system. The function $G : \mathbb{R} \to \mathbb{R}$ satisfies $G(\rho_0) = 0$, $G(\rho) > 0$ if $\rho \neq \rho_0$ and G is strictly convex, with quadratic behavior for large values of $|\rho|$.

Graphs of the free energy in the linear and the nonlinear models

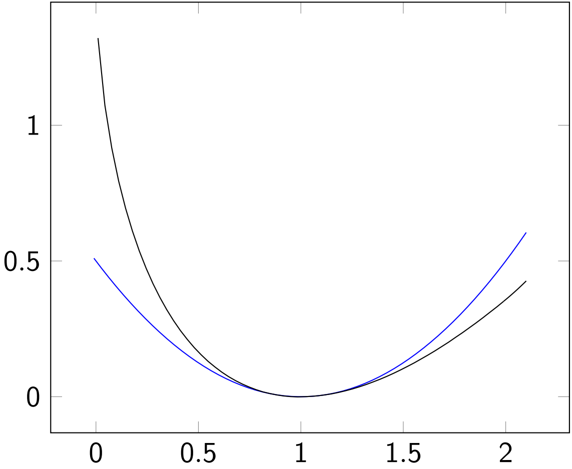

Figure: Sketch of the free energy $\beta G(\rho/\rho_0)$. The linear case is in blue. The nonlinear case with $n_0 = 1/3$ and $\rho_*/\rho_0 = 0.01$, is in black.

Summary of equations and weak formulation

The problem describing the nonlinear poroelastic behavior of a fluid saturated porous medium is to find the displacement $\mathbf{u} : \overline{Q}_T \to \mathbb{R}^m$ and the fluid density $\rho : \overline{Q}_T \to \mathbb{R}$ satisfying

(i) the balance equations

$$\overline{n}(\mathcal{E})\partial_t\rho + D(\rho,\mathcal{E})\partial_t\mathcal{E} = \text{ div } \Big(k(\mathcal{E})\mathcal{D}(\rho)\mathbb{K}\nabla\rho\Big) + Q, \tag{47}$$

$$-\text{ div } \Big(\mathcal{G}e(\mathbf{u}) + \lambda^*\partial_t\mathcal{E}\mathbb{I} - p(\rho)\mathbb{I}\Big) = \mathbf{F}, \tag{48}$$

in $Q_T = (0,T)\times\Omega$ and

(ii) the initial-boundary conditions (32)-(33).
The coefficients in equations (47)-(48) were introduced in this section. Specifically,
$\overline{n}(\mathcal{E})$ and $k(\mathcal{E})$ satisfy (27),
$D(\rho,\mathcal{E})$, $\mathcal{D}(\rho)$ and $p(\rho)$ are given by (42), (44) and (28),
and $\lambda^* \geq 0$.

A weak free energy formulation

We recast this classical formulation in the following weak form.

Definition 1 We call a triple $(\mathbf{u}, \mathcal{E}, \rho) \in L^\infty(0,T;H^1(\Omega)^m) \times L^\infty(0,T;H^1_{loc}(\Omega)) \times \big(L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))\big)$, $\partial_t\mathcal{E} \in L^2(Q_T) \cap L^\infty(0,T;H^1_{loc}(\Omega))$ a **weak free energy** solution if

(i)

$$
\begin{aligned}
&-\int_0^T\int_\Omega \rho\overline{n}(\mathcal{E})\partial_t\Phi\ dxdt - \int_\Omega n_0\rho^0(x)\Phi(x,0)\ dx + \\
&\qquad \int_0^T\int_\Omega \partial_t\mathcal{E}\Big(D(\rho,\mathcal{E}) - \rho\overline{n}'(\mathcal{E})\Big)\ \Phi\ dxdt + \\
&\qquad\qquad \int_0^T\int_\Omega k(\mathcal{E})\mathcal{D}(\rho)\mathbb{K}\nabla\rho\cdot\nabla\Phi\ dxdt \\
&= \int_0^T\int_\Omega Q\Phi\ dxdt, \quad \forall\Phi \in H^1(Q_T),\ \Phi|_{t=T} = 0; \qquad (49)
\end{aligned}
$$

(ii) $\mathcal{E} = \operatorname{div}\mathbf{u}$;

A weak free energy formulation, 2nd part

(iii)

$$\int_\Omega \mathcal{G}e(\mathbf{u}) : e(\xi)\ dx + \lambda^*\partial_t \int_\Omega \mathcal{E}\ \text{div}\ \xi\ dx - \int_\Omega p(\rho)\text{div}\ \xi\ dx =$$

$$\int_\Omega \mathbf{F}\cdot\xi\ dx, \quad \forall \xi \in H_0^1(\Omega)^3 \text{ and for almost all } t \in (0,T]; \qquad (50)$$

(iv) $\mathcal{E}|_{t=0} = 0 \quad \text{in} \quad \Omega.$

(v) For every $t_1, t_2 \in [0,T],\ t_1 < t_2,$

$$\int_\Omega \left(\frac{1}{2}\mathcal{G}e(\mathbf{u}(t_2)) : e(\mathbf{u}(t_2)) + \overline{n}(\mathcal{E})(t_2))G(\rho(t_2)) - \mathbf{F}(t_2)\cdot\mathbf{u}(t_2)\right)dx +$$

$$\int_{t_1}^{t_2}\int_\Omega \left(\lambda^*(\partial_t\mathcal{E})^2 + k(\mathcal{E})\mathcal{D}(\rho)g'(\rho)\mathbb{K}\nabla\rho\cdot\nabla\rho - Qg(\rho) + \partial_t\mathbf{F}\cdot\mathbf{u}\right)dxdt \leq$$

$$\int_\Omega \left(\frac{1}{2}\mathcal{G}e(\mathbf{u}(t_1)) : e(\mathbf{u}(t_1)) + \overline{n}(\mathcal{E}(t_1))G(\rho(t_1)) - \mathbf{F}(t_1)\cdot\mathbf{u}(t_1)\right)dx. \qquad (51)$$

Here $\rho^0 \in L^2(\Omega)$, $Q \in C([0,T]; L^2(\Omega))$ and $\mathbf{F} \in H^1(0,T; L^2(\Omega)^m)$.

Weak free energy solutions

In Definition 1 we explicitly incorporate energy inequality (51). When dealing with classical solutions, equations (47)-(48) imply the energy balance

$$\partial_t J(\mathbf{u},\rho) + \int_\Omega \lambda^*(\partial_t \mathcal{E})^2 \; dx + \int_\Omega k(\mathcal{E})\mathcal{D}(\rho)g'(\rho)\mathbb{K}\nabla\rho\cdot\nabla\rho \; dx = \int_\Omega Qg(\rho)\; dx - \int_\Omega \partial_t \mathbf{F}\cdot\mathbf{u}\; dx. \tag{52}$$

However, in the weak formulation (49)-(50) we cannot use $\Phi = g(\rho)$ and $\xi = \partial_t \mathbf{u}$, due to lack of smoothness. Therefore (v) has to be added explicitly. Hence we consider only those weak solutions satisfying additionally (51). Therefore they are called **weak free energy** solutions.

In a number of steps we prove existence of weak solutions when $\lambda^* > 0$. We achieve this by first considering the incremental formulation. In this approximation, we obtain existence results which hold for all $\lambda^* \geq 0$.

The "entropy" unknown

Now we study in this section the time discretized form of (47), (48).
In doing so we use the function $g = g(\rho)$ as the primary unknown.
This is allowed since $g : \mathbb{R} \to \mathbb{R}$ is smooth and strictly increasing.
The switch to g is done for mathematical convenience, because it allows us to obtain Lyapunov functional estimates in a straightforward way. Let

$$p(g) := p(\rho(g)) \quad \text{and} \quad \mathcal{D}(g) := \mathcal{D}(\rho(g))\rho'(g). \tag{53}$$

Further,

$$G(\rho(z)) = \int_{\rho_0}^{\rho(z)} g(\xi)\, d\xi = \int_0^z \zeta\rho'(\zeta)\, d\zeta, \quad z \in \mathbb{R}, \tag{54}$$

$$G(g) := \int_0^g \zeta\rho'(\zeta)\, d\zeta,\ D(g,\mathcal{E}) = \frac{\overline{n}'(\mathcal{E})}{g} G(g) + \frac{p(g) - p_0}{g}. \tag{55}$$

$D(g,\mathcal{E})$ is bounded with respect to $\mathcal{E}$ and grows linearly in g for large $|g|$.

The time discretization

Using these definitions in (47) and (48), we find in terms of g

$$\overline{n}(\mathcal{E})\partial_t\rho(g) + D(g,\mathcal{E})\partial_t\mathcal{E} = \operatorname{div}\Big(k(\mathcal{E})\mathcal{D}(g)\mathbb{K}\nabla g\Big) + Q, \tag{56}$$

$$-\operatorname{div}\Big(\mathcal{G}e(\mathbf{u}) + \lambda^*\partial_t\mathcal{E}\mathbb{I} - p(g)\mathbb{I}\Big) = \mathbf{F}, \tag{57}$$

in Q_T.

Next we turn to the **time discretized** form of equations (56)-(57). Let $\tau \in (0,1)$ denote the time discretization step and $N \in \mathbb{N}$ a large integer such that $N\tau = T$. At each discrete time $t_j = j\tau$, with $j = 0, 1, \ldots, N$, we set

$$\mathbf{F}^j(x) = \mathbf{F}(x, j\tau),\ Q^j(x) = Q(x, j\tau), \quad x \in \Omega.$$

$$\mathbf{u}^{j-1}(x) = \mathbf{u}(x, t_{j-1}),\ g^{j-1}(x) = g(x, t_{j-1}), \quad x \in \Omega.$$

Then $\mathbf{u}$ and g at time t_j are obtained as solutions of the incremental problem (writing $\mathbf{U} = \mathbf{u}^{j-1}$, $\Xi = g^{j-1}$ and $V = H_0^1(\Omega)^m \times H^1(\Omega))$:

The incremental problem

Problem (**PD**): Given $(\mathbf{U},\ \Xi) \in V$, find $(\mathbf{u},\ g) \in V$ such that

$$\int_\Omega \frac{\overline{n}(\text{div }\mathbf{U})}{\tau}(\rho(g)-\rho(\Xi))\psi\ dx + \int_\Omega D_\tau(g, \text{div }\mathbf{u}, \text{div }\mathbf{U})\ \text{ div }\frac{\mathbf{u}-\mathbf{U}}{\tau}\psi dx$$
$$+ \int_\Omega k(\text{div }\mathbf{u})\mathcal{D}(g)\mathbb{K}\nabla g\cdot\nabla\psi\ dx = \int_\Omega Q^j\psi dx,\ \ \forall\psi\in H^1(\Omega); \tag{58}$$

$$\int_\Omega \mathcal{G}e(\mathbf{u}) : e(\xi)\ dx + \frac{\lambda^*}{\tau}\int_\Omega \text{div }(\mathbf{u}-\mathbf{U})\text{ div }\xi\ dx -$$
$$\int_\Omega p(g)\text{div }\xi\ dx = \int_\Omega \mathbf{F}^j\cdot\xi\ dx,\quad \forall\xi\in H_0^1(\Omega)^m. \tag{59}$$

The coefficient D_τ in equation (58) is given by

$$D_\tau(g,\ \text{div }\mathbf{u},\ \text{div }\mathbf{U}) = \frac{\overline{n}(\text{ div }\mathbf{u}) - \overline{n}(\text{ div }\mathbf{U})}{\text{div }\mathbf{u} - \text{ div }\mathbf{U}}\frac{G(g)}{g} + \frac{p(g)-p_0}{g}. \tag{60}$$

This expression results from $D(g,\mathcal{E})$ in (55), when the derivative $\overline{n}'(\mathcal{E})$ is replaced by the finite difference $\dfrac{\overline{n}(\text{ div }\mathbf{u}) - \overline{n}(\text{ div }\mathbf{U})}{\text{div }\mathbf{u} - \text{ div }\mathbf{U}}$.
The specific choice of (60) appears convenient in the estimates concerning the time discrete Lyapunov functional.

Using the weak topology of the space $H_0^1(\Omega)^m \times H^1(\Omega)$, serious difficulties arise with the coefficients $\overline{n}$, D_τ and k depending on div $\mathbf{u}$. To remedy this, we introduce a Friedrichs mollifier Υ_ε, where ε is a small positive parameter, and replace div $\mathbf{u}$ in the nonlinearities by the convolution div $\mathbf{u} \star \Upsilon_\varepsilon = -\mathbf{u} \star \nabla \Upsilon_\varepsilon$. Using this substitution one can treat nonlinear coefficients containing div $\mathbf{u}$ as lower order terms in the equations. This allows us to use the theory of pseudo-monotone operators.

The regularized incremental problem

Applying this convolution, the regularized form of problem (**PD**) reads:

Problem $(\mathbf{PD})_\varepsilon$: Given $(\mathbf{U},\ \Xi) \in V$, find $(\mathbf{u}_\varepsilon,\ g_\varepsilon) \in V$ such that, with $\mathcal{E}_\varepsilon = -\mathbf{u}_\varepsilon \star \nabla \Upsilon_\varepsilon$,

$$\int_\Omega \frac{\overline{n}(\text{div }\mathbf{U})}{\tau}(\rho(g_\varepsilon) - \rho(\Xi))\psi\ dx + \int_\Omega \left(\frac{\overline{n}(\mathcal{E}_\varepsilon) - \overline{n}(\text{div }\mathbf{U})}{\tau g_\varepsilon} G(g_\varepsilon) + \frac{p(g_\varepsilon) - p_0}{\tau g_\varepsilon} \right) \text{div }(\mathbf{u}_\varepsilon - \mathbf{U})\psi\ dx$$

$$+ \int_\Omega k(\mathcal{E}_\varepsilon)\mathcal{D}(g_\varepsilon)\mathbb{K}\nabla g_\varepsilon \cdot \nabla\psi\ dx = \int_\Omega Q^j \psi dx,\ \forall\psi \in H^1(\Omega), \quad (61)$$

$$\int_\Omega \mathcal{G}e(\mathbf{u}_\varepsilon) : e(\xi)\ dx + \frac{\lambda^*}{\tau}\int_\Omega \text{div }(\mathbf{u}_\varepsilon - \mathbf{U})\ \text{div }\xi\ dx - \int_\Omega p(g_\varepsilon)\text{div }\xi\ dx = \int_\Omega \mathbf{F}^j \cdot \xi\ dx, \quad \forall \xi \in H_0^1(\Omega)^m. \quad (62)$$

Existence of a solution to the incremental problem

Proposition 2 Let $\varepsilon > 0$ be a small positive constant. Under the assumptions of Definition 1, problem $(\mathbf{PD})_\varepsilon$ admits at least one solution $(\mathbf{u}_\varepsilon,\ g_\varepsilon) \in V$.

Proof: Following e.g. monograph *Roubiček T, Nonlinear Partial Differential Equations with Applications , Springer 2005* we establish the *pseudo monotonicity* of the corresponding operator. Furthermore, using the Lyapunov functional property the operator is *coercive* and applying Brézis thm we conclude existence of at least one solution for problem $(\mathbf{PD})_\varepsilon$. For details we refer to [*].

Theorem 1 Problem $(\mathbf{PD})$ admits at least one solution $(\mathbf{u}, g) \in V$.

Proof: From the coercivity part of the proof of Proposition 2, it follows that

$$||\mathbf{u}_\varepsilon||_{H_0^1(\Omega)^m} + ||g_\varepsilon||_{H^1(\Omega)} \leq C, \tag{63}$$

The discretized Lyapunov functional

Since $p(g_\varepsilon)$ is bounded in $H^1(\Omega)$, uniformly with respect to ε, we conclude that

$$||\mathbf{u}_\varepsilon||_{H^2(\Omega)^m} \leq C. \tag{64}$$

Using the weak compactness, we get a solution for Problem (**PD**).

To complete the study of the incremental problem, we need to estimate the behavior of solutions after at least $O(1/\tau)$ times steps.

In problem (**PD**), where the discrete time step τ enters as parameter, one finds after one step $(\mathbf{u}^1, g^1)$ from the initial values $(\text{div } \mathbf{u}, \rho)|_{t=0} = (0, \rho^0)$. The idea is to repeat this procedure for an arbitrary number of steps. If $M \in \mathbb{N}$, $M \leq N = T/\tau$, then $(\mathbf{u}^M, g^M)$ denotes the time discretized approximation of the original quasi-static equation, at $t = t_M = M\tau$.

The corresponding Lyapunov functional at $t = t_M$ reads

$$J^M = \int_\Omega \left(\frac{1}{2} \mathcal{G} e(\mathbf{u}^M) : e(\mathbf{u}^M) - \mathbf{F}^M \cdot \mathbf{u}^M + \overline{n}(\text{div } \mathbf{u}^M) G(g^M) \right) dx. \tag{65}$$

The Lyapunov functional property

Theorem 2 For each $M \in \mathbb{N}$, $M \leq N = T/\tau$, we have

$$J^M + \tau \sum_{j=1}^{M} \int_\Omega \Bigg(\lambda^* \Big(\frac{\operatorname{div}\, (\mathbf{u}^j - \mathbf{u}^{j-1})}{\tau} \Big)^2 + \frac{\mathbf{F}^j - \mathbf{F}^{j-1}}{\tau} \cdot \mathbf{u}^{j-1} +$$

$$k(\operatorname{div}\, \mathbf{u}^j) \mathcal{D}(g^j) \mathbb{K} \nabla g^j \cdot \nabla g^j - Q^j g^j \Bigg) \, dx \leq J^0. \qquad (66)$$

Here

$$J^0 = n_0 \int_\Omega G(g^0) \, dx, \quad g^0 = g(\rho^0).$$

Proof: At time $t = t_j$, with $j = 1, \ldots, N$, the equations in problem (**PD**) read

$$\int_\Omega \mathcal{G} e(\mathbf{u}^j) : e(\xi) \, dx + \frac{\lambda^*}{\tau} \int_\Omega \operatorname{div}\, (\mathbf{u}^j - \mathbf{u}^{j-1}) \operatorname{div}\, \xi \, dx -$$

$$\int_\Omega p(g^j) \operatorname{div}\, \xi \, dx = \int_\Omega \mathbf{F}^j \cdot \xi \, dx, \ \forall \xi \in H_0^1(\Omega)^m, \qquad (67)$$

Proof of the Lyapunov property

$$\int_\Omega \left(\frac{\overline{n}(\text{div } \mathbf{u}^{j-1})}{\tau} (\rho(g^j) - \rho(g^{j-1})) + \frac{\overline{n}(\text{div } \mathbf{u}^j) - \overline{n}(\text{div } \mathbf{u}^{j-1})}{\tau g^j} G(g^j) \right) \psi \, dx$$

$$+ \int_\Omega \frac{p(g^j) - p_0}{\tau g^j} \text{div } (\mathbf{u}^j - \mathbf{u}^{j-1}) \psi \, dx + \int_\Omega k(\text{div } \mathbf{u}^j) \mathcal{D}(g^j) \mathbb{K} \nabla g^j \cdot \nabla \psi \, dx$$

$$= \int_\Omega Q^j \psi \, dx, \quad \forall \psi \in H^1(\Omega). \tag{68}$$

We take $\xi = (\mathbf{u}^j - \mathbf{u}^{j-1})/\tau$ in (67) and $\psi = g^j$ in (68). The resulting two equalities are added and summed-up with respect to j up from $j = 1$ to $j = M$.

A priori estimates, uniform in the time step τ

Having established existence for the discrete problem (**PD**) in Theorem 1 and a Lyapunov estimate in Theorem 2, we are now in a position to obtain estimates that are uniform in the time step τ.

Proposition 3 There exists a constant $C > 0$ such that

$$||\mathbf{u}^M||^2_{H^1(\Omega)^m} + ||g^M||^2_{L^2(\Omega)} \leq C, \tag{69}$$

$$\tau \sum_{j=1}^{M} \int_\Omega \left(\lambda^* (\frac{\text{div } (\mathbf{u}^j - \mathbf{u}^{j-1})}{\tau})^2 + |\nabla g^j|^2 \right) \, dx \leq C, \tag{70}$$

for all M and τ such that $1 \leq M \leq N = T/\tau$, with τ sufficiently small.

Proof: We combine expression (65) for J^M and inequality (66)and for δ and τ sufficiently small, we obtain for the combination

$$\mathcal{U}_j = ||\mathbf{u}^j||^2_{H^1(\Omega)^m} + ||g^j||^2_{L^2(\Omega)}, \quad j = 0, \ldots, M,$$

proof of the a priori estimates, uniform in the time step

the inequality

$$\mathcal{U}_M \leq C_1 + C_2\tau \sum_{j=0}^{M-1} \mathcal{U}_j,$$

where C_1 and C_2 do not dependent on τ and M. Next we apply the *discrete Gronwall inequality* to find

$$\mathcal{U}_M \leq C_1 e^{C_2(M-1)\tau} < C_1 e^{C_2 T} \quad \text{for all } 1 \leq M \leq N.$$

The second estimate follows directly from Theorem 2.

However, to pass to the limit $\tau \to 0$ in the nonlinearities, one needs more information on the behavior of the ratios $\{$ div $(\mathbf{u}^j - \mathbf{u}^{j-1})/\tau\}$ and $\{(g^j - g^{j-1})/\tau\}$. In fact, we must establish relative compactness of the sequences $\{\text{div } \mathbf{u}^j\}$ and $\{g^j\}$. We start with a local H^1-estimate for $\mathcal{E}^j = \text{div } \mathbf{u}^j$.

A priori estimates, uniform in the time step for the volumetric strain

Lemma 1 Let $\varphi \in C_0^\infty(\Omega)$ and $\tau > 0$ sufficiently small. Then there exists a constant $C = C(\varphi)$ such that

$$\tau \sum_{j=1}^{N} ||\varphi \mathcal{E}^j||^2_{H^1(\Omega)} + \frac{\lambda^*}{2\mu + \lambda} \max_{1 \le M \le N} ||\varphi \mathcal{E}^M||^2_{H^1(\Omega)} \le C. \qquad (71)$$

Proof: We use that

$$L^j = (2\mu + \lambda)\mathcal{E}^j - p(g^j) + \lambda^* \frac{\text{div}\ (\mathbf{u}^j - \mathbf{u}^{j-1})}{\tau}, \ j = 1, \ldots M. \qquad (72)$$

satisfies

$$-\Delta L^j = \text{div}\ \mathbf{F}^j \quad \text{in} \quad \Omega. \qquad (73)$$

Then using the interior elliptic regularity and previous estimates we obtain (71).

A priori estimates, uniform in the time step for the density

Finally, we look for an estimate for $\mathcal{N}^j = \overline{n}(\mathcal{E}^j)\rho(g^j)$. With the results of Proposition 3 and Lemma 1, the space-time compactness of $\mathcal{N}$ will imply the same property of g.

Proposition 4 For given $\tau > 0$ and $j = 1, \dots, N,$, let $(\mathbf{u}_\tau(t_j), g_\tau(t_j)) \in V$ denote a solution of problem (**PD**). Then

$$\max_{1\le j\le N} \left(||\mathbf{u}_\tau(t_j)||_{H^1(\Omega)^m} + ||g_\tau(t_j)||_{L^2(\Omega)} \right) \le C, \tag{74}$$

$$\tau \sum_{j=1}^{N} \int_\Omega \left(\lambda^* \Big(\frac{\text{div } (\mathbf{u}_\tau(t_j) - \mathbf{u}_\tau(t_{j-1}))}{\tau}\Big)^2 + |\nabla g_\tau(t_j)|^2 \right) dx \le C, \tag{75}$$

$$\tau \sum_{j=1}^{N} ||\varphi \text{ div } \mathbf{u}_\tau(t_j)||^2_{H^1(\Omega)} + \lambda^* \max_{1\le j\le N} ||\varphi \text{div } \ \mathbf{u}_\tau(t_j)||^2_{H^1(\Omega)} \le C, \tag{76}$$

$$\tau \sum_{j=1}^{N} \Big(||\frac{\mathcal{N}^j - \mathcal{N}^{j-1}}{\tau}||^2_{H^{-2}(\Omega)} + ||\varphi \mathcal{N}^j||^2_{H^1(\Omega)} \Big) \le C, \tag{77}$$

where $\mathcal{N}^j = \overline{n}(\text{div } \mathbf{u}_\tau(t_j))\rho(g_\tau(t_j))$ and where $\varphi \in C_0^\infty(\Omega)$.

Existence for continuous time problem

In Proposition 4, where the time step τ enters as a parameter, one finds $\{(\mathbf{u}_\tau(t_j), g_\tau(t_j))\}_{j=1,\dots,N}$ from the "initial value" div $\mathbf{u}(0) = 0$ and $g(0) = g^0$. Here $N = O(1/\tau)$ and $g^0 = g(\rho^0)$. This procedure yields a time discretized approximation of the original quasi-static equations.

Now we investigate the limit $\tau \searrow 0$. Here a crucial role is played by the parameter λ^*, which is needed to control the behaviour in time of $\mathcal{E} = \text{div } \mathbf{u}$.

Existence for continuous time problem with $\lambda^* > 0$: a priori estimates

Using the discrete solution $(\mathbf{u}_\tau(t_j), g_\tau(t_j))$, we construct two approximations that hold for all $0 \le t \le T$. The first is the piecewise constant approximation

$$(\overline{\mathbf{u}}_\tau(t), \overline{g}_\tau(t)) = (\mathbf{u}_\tau(t_j), g_\tau(t_j)) \quad \text{for} \quad j\tau \le t < (j+1)\tau. \tag{78}$$

The second is the Rothe interpolant, which is the piecewise linear time-continuous approximation

$$\begin{aligned}(\tilde{\mathbf{u}}_\tau(t), \tilde{g}_\tau(t)) = \big(j + 1 - \frac{t}{\tau}\big)(\mathbf{u}_\tau(t_j), g_\tau(t_j)) + \\ \big(\frac{t}{\tau} - j\big)(\mathbf{u}_\tau(t_{j+1}), g_\tau(t_{j+1})), \\ \text{for} \quad j\tau \le t \le (j+1)\tau.\end{aligned} \tag{79}$$

In (78) and (79) the index j runs from $j = 0$ to $j = N - 1$.

Existence for continuous time problem with $\lambda^* > 0$: summary of a priori estimates

Applying Proposition 4, yields for both approximations, with $\natural \in \{^-, ^\sim\}$,

$$\max_{0\le t\le T} \left(||\mathbf{u}^\natural{}_\tau(t)||^2_{H^1(\Omega)^m} + ||g^\natural_\tau(t)||^2_{L^2(\Omega)}\right)\ dt \le C, \tag{80}$$

$$\int_0^T \int_\Omega |\nabla g^\natural{}_\tau(t)|^2\ dxdt \le C, \tag{81}$$

$$\int_0^T ||\varphi\ \mathcal{E}^\natural{}_\tau(t)||^2_{H^1(\Omega)}\ dt \le C, \tag{82}$$

$$\lambda^* \max_{0\le t\le T} ||\varphi \mathcal{E}^\natural{}_\tau(t)||^2_{H^1(\Omega)} \le C, \tag{83}$$

$$\int_0^T ||\varphi \mathcal{N}^\natural{}_\tau(t)||^2_{W^{1,3/2}(\Omega)}) \le C, \tag{84}$$

where $\mathcal{E}^\natural{}_\tau = \operatorname{div} \mathbf{u}^\natural{}_\tau$, $\overline{\mathcal{N}}_\tau = \overline{n}(\overline{\mathcal{E}}_\tau)\rho(\overline{g}_\tau)$ and $\tilde{\mathcal{N}}_\tau(t) = (j + 1 - t/\tau)\mathcal{N}^j + (t/\tau - j)\mathcal{N}^{j+1}$.

Existence for continuous time problem with $\lambda^* > 0$: summary of a priori estimates no 2

Further we have

$$\partial_t \tilde{\mathcal{N}}_\tau = \frac{\mathcal{N}^{j+1} - \mathcal{N}^j}{\tau} \quad \text{and} \quad \partial_t \tilde{\mathcal{E}}_\tau = \frac{\mathcal{E}^{j+1} - \mathcal{E}^j}{\tau},$$
$$\text{for} \quad t_j \leq t \leq t_{j+1} \quad \text{and} \quad j = 0, \ldots, N-1.$$

$$\int_0^T \int_\Omega \lambda^* |\partial_t \tilde{\mathcal{E}}_\tau(t)|^2 \; dxdt + \int_0^T ||\partial_t \tilde{\mathcal{N}}_\tau(t)||^2_{H^{-2}(\Omega)} \; dt \leq C. \quad (85)$$

Since the piecewise constant approximation $(\overline{\mathbf{u}}_\tau(t), \overline{g}_\tau(t))$ is discontinuous in time, its time derivative is only a measure.

$$||\partial_t \overline{\mathcal{E}}_\tau||_{\mathcal{M}(0,T;L^2(\Omega))} + ||\partial_t \overline{\mathcal{N}}_\tau||_{\mathcal{M}(0,T;H^{-2}(\Omega))} \leq C, \quad (86)$$

where $\mathcal{M}(0,T;H^{-2}(\Omega))$ is the dual space of $C([0,T];H^2_0(\Omega))$.

Existence for continuous time problem with $\lambda^* > 0$: weak convergences no 1

From estimates (80)-(85) and the well-known weak and weak* compactness theorems, we conclude that there exists a quadruple $\{\tilde{\mathbf{u}}, \tilde{g}, \tilde{\mathcal{E}}, \tilde{\mathcal{N}}\}$ such that along a subsequence $\tau \searrow 0$ we have

$$\tilde{\mathbf{u}}_\tau \to \tilde{\mathbf{u}} \quad \text{weak}^* \text{ in} \quad L^\infty(0, T; H_0^1(\Omega)^m), \tag{87}$$

$$\tilde{g}_\tau \rightharpoonup \tilde{g} \quad \text{weakly in} \quad L^2(0, T; H^1(\Omega)), \tag{88}$$

$$\tilde{\mathcal{E}}_\tau \rightharpoonup \tilde{\mathcal{E}} \quad \text{weakly in} \quad L^2(0, T; H^1(\omega)), \tag{89}$$

$$\partial_t \tilde{\mathcal{E}}_\tau \rightharpoonup \partial_t \tilde{\mathcal{E}} \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)), \tag{90}$$

$$\tilde{\mathcal{N}}_\tau \rightharpoonup \tilde{\mathcal{N}} \quad \text{weakly in} \quad L^2(0, T; W^{1,3/2}(\omega)), \tag{91}$$

$$\partial_t \tilde{\mathcal{N}}_\tau \rightharpoonup \partial_t \tilde{\mathcal{N}} \quad \text{weakly in} \quad L^2(0, T; H^{-2}(\Omega)). \tag{92}$$

Existence for continuous time problem with $\lambda^* > 0$: weak convergences no 2

Concerning the convergence of $(\overline{\mathbf{u}}_\tau, \overline{g}_\tau)$, we use estimates (80)-(84), now combined with (86). We use that the spaces

$$W^{1,2,\mathcal{M}}(0,T;H^1(\omega),L^2(\omega)) = \{z \in L^2(0,T;H^1(\omega)) \mid \frac{dz}{dt} \in \mathcal{M}(0,T;L^2(\omega))\}$$

and $W^{1,2,\mathcal{M}}(0,T;W^{1,3/2}(\omega),H^{-2}(\omega))$ are compactly embedded in $L^2(0,T;L^2(\omega))$, for any smooth bounded subset ω of Ω. The result is that there exists $(\overline{\mathbf{u}}, \overline{g}, , \overline{\mathcal{E}}, \overline{\mathcal{N}})$ such that along a subsequence $\tau \searrow 0$ one has the same convergence as in (87)-(89) and (91). The convergence in (90) and (92) is now replaced by weak$-*$ convergence in $\mathcal{M}(0,T;L^2(\Omega))$ for $\partial_t \overline{\mathcal{E}}_\tau$ and in $\mathcal{M}(0,T;H^{-2}(\Omega))$ for $\partial_t \overline{\mathcal{N}}_\tau$.

Furthermore, the estimates allow us to conclude

$$\overline{\mathcal{E}}_\tau \to \overline{\mathcal{E}} \text{ strongly in } L^2((0,T)\times\omega) \text{ and (a.e) on } (0,T)\times\omega, \quad (93)$$

$$\overline{\mathcal{N}}_\tau \to \overline{\mathcal{N}} \text{ strongly in } L^2((0,T)\times\omega) \text{ and (a.e) on } (0,T)\times\omega. \quad (94)$$

Existence for continuous time problem with $\lambda^* > 0$: strong convergences

As a consequence

$$\rho(\overline{g}_\tau) = \frac{\overline{\mathcal{N}}_\tau}{\overline{n}(\overline{\mathcal{E}}_\tau)} \to \frac{\overline{\mathcal{N}}}{\overline{n}(\overline{\mathcal{E}})} \tag{95}$$

$$\overline{g}_\tau = \rho^{-1}\left(\frac{\overline{\mathcal{N}}_\tau}{\overline{n}(\overline{\mathcal{E}}_\tau)}\right) \to \rho^{-1}\left(\frac{\overline{\mathcal{N}}}{\overline{n}(\overline{\mathcal{E}})}\right) = \overline{g}. \tag{96}$$

strongly in $L^2((0,T)\times\omega)$ and a.e. on $(0,T)\times\omega$. $\Rightarrow$

$$\begin{cases} \rho(\overline{g}_\tau) \to \rho(\overline{g}) \ \text{ strongly in } \ L^2((0,T)\times\omega); \\ \mathcal{D}(\overline{g}_\tau) \to \mathcal{D}(\overline{g}) \ \text{ strongly in } \ L^2((0,T)\times\omega). \end{cases} \tag{97}$$

Inherited from $\overline{\mathcal{E}}_\tau = \operatorname{div} \overline{\mathbf{u}}_\tau$, the convergence properties imply

$$\overline{\mathcal{E}} = \operatorname{div} \overline{\mathbf{u}} \quad \text{a.e. in} \quad (0,T)\times\Omega. \tag{98}$$

Finally,

$$\int_0^T ||\overline{\mathcal{E}}_\tau(t) - \tilde{\mathcal{E}}_\tau(t)||^2_{L^2(\Omega)}\, dt + \int_0^T ||\overline{\mathcal{N}}_\tau(t) - \tilde{\mathcal{N}}_\tau(t)||^2_{H^{-2}(\Omega)}\, dt = C\tau^2.$$

Existence theorem with $\lambda^* > 0$

From this point on we denote the limit, as $\tau \searrow 0$, by the quadruple $(\mathbf{u}, g, \mathcal{E}, \mathcal{N})$, where $\mathcal{E} = \text{div}\ \mathbf{u}$ and $\mathcal{N} = \overline{n}(\mathcal{E})\rho(g)$.
We are now in a position to prove the main existence result for a weak solution of the time continuous case.

Theorem 3 Let $\lambda^* > 0$. Then there exists at least one **weak free energy solution** $(\mathbf{u}, \mathcal{E}, \rho)$ satisfying Definition 1.

Proof: We **first** consider the momentum balance equation (59).
For any $\xi \in H_0^1(\Omega)^m$

$$\int_\Omega \mathcal{G}e(\overline{\mathbf{u}}_\tau) : e(\xi)\ dx = \int_\Omega \mathcal{G}e(\mathbf{u}^j) : e(\xi)\ dx = -\frac{\lambda^*}{\tau}\int_\Omega (\mathcal{E}^j - \mathcal{E}^{j-1})\ \text{div}\ \xi\ dx$$
$$= \int_\Omega p(g^j)\text{div}\ \xi\ dx + \int_\Omega \mathbf{F}^j \cdot \xi\ dx = -\lambda^* \int_\Omega \partial_t \tilde{\mathcal{E}}_\tau(t-\tau)\ \text{div}\ \xi\ dx +$$
$$\int_\Omega p(\overline{g}_\tau)\text{div}\ \xi\ dx + \int_\Omega \mathbf{F}_\tau \cdot \xi\ dx \qquad (100)$$

Passing to the limit $\tau \searrow 0$ is without difficulties.

Proof of the existence theorem with $\lambda^* > 0$

In the **second** step we write the discretization of problem **(PD)**, for any $\psi \in C_0^\infty(\Omega)$ and $\alpha \in C^\infty[0,T]$, as

$$\int_\tau^T \int_\Omega \Big(\partial_t \tilde{\mathcal{N}}_\tau(t-\tau) - \partial_t \tilde{\nu}_\tau(t-\tau)\big(\rho(\overline{g}_\tau) - \frac{G(\overline{g}_\tau)}{\overline{g}_\tau}\big) + \partial_t \tilde{\mathcal{E}}_\tau(t-\tau)\frac{p(\overline{g}_\tau) - p_0}{\overline{g}_\tau}\Big)\psi(x)\alpha(t)\ dxdt - \int_\tau^T \int_\Omega Q_\tau \psi(x)\alpha(t)\ dxdt + \int_\tau^T \int_\Omega k(\overline{\mathcal{E}}_\tau)\mathcal{D}(\overline{g}_\tau)\mathbb{K}\nabla \overline{g}_\tau \cdot \nabla\psi(x)\alpha(t)\ dxdt = 0, \tag{101}$$

where

$$\tilde{\nu}_\tau(t) = \big(j + 1 - \frac{t}{\tau}\big)\overline{n}(\mathcal{E}^j) + \big(\frac{t}{\tau} - j\big)\overline{n}(\mathcal{E}_{j+1}),$$

and

$$Q_\tau(t) = Q(t_j) = Q^j$$
$$\text{for } j\tau \le t < (j+1)\tau. \tag{102}$$

Next

$$\partial_t \tilde{\nu}_\tau \rightharpoonup \partial_t \overline{n}(\mathcal{E}) \quad \text{weakly in} \quad L^2((0,T)\times\Omega) \tag{103}$$

and

$$\tilde{\nu}_\tau \to \overline{n}(\mathcal{E}) \quad \text{strongly in} \quad L^2((0,T)\times\omega) \text{ and a.e. in} \quad (0,T)\times\omega. \tag{104}$$

We are now in position to pass to the limit $\tau \searrow 0$ in (101) and obtain

$$\partial_t\Big(\overline{n}(\mathcal{E})\rho(g)\Big) - \partial_t\overline{n}(\mathcal{E})\rho(g) + D(\rho,\mathcal{E})\partial_t\mathcal{E} - \text{ div }\Big(k(\mathcal{E})\mathcal{D}(g)\mathbb{K}\nabla g\Big) = Q \text{ in } \quad \mathcal{D}'((0,T)\times\Omega), \tag{105}$$

In the final step we justify the initial and boundary conditions and the energy inequality (51).

Some open questions

- Nonisothermal nonlinear poroelasticity.
- Unsaturated nonlinear poroelasticity
- Multiphase nonlinear poroelasticity
- Is it possible to obtain equations of the nonlinear poroelasticity by upscaling the fluid-structure problem at the microscopic (pore) level?