Ancient Solutions to Mean Curvature Flow

Joost is jarig!

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Joint work with

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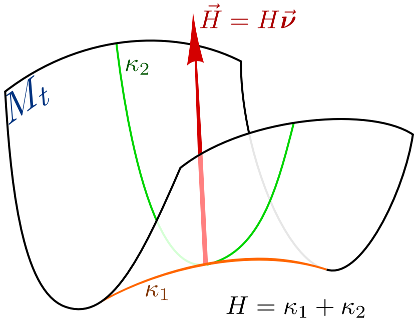
Mean Curvature Flow



$$\text{MCF} \iff V = H$$

$$\iff \left(\frac{\partial \vec{X}}{\partial t}\right)^{\perp} = \Delta \vec{X}$$

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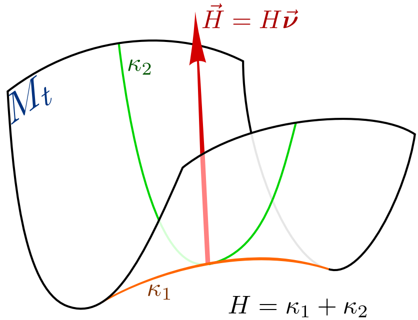

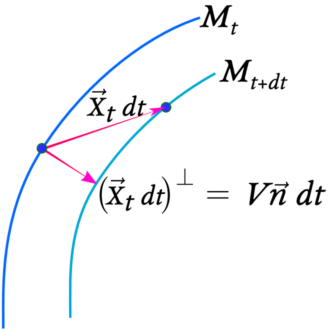

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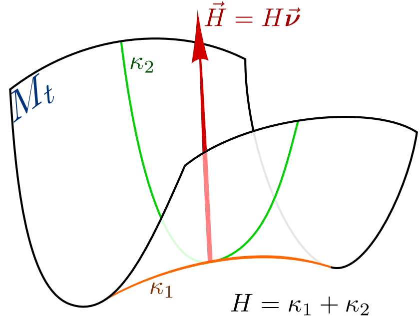

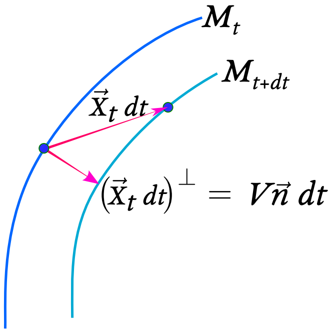

$$\text{MCF} \iff V = H$$

$$\iff \left(\frac{\partial \vec{X}}{\partial t}\right)^\perp = \Delta \vec{X}$$

$$\Delta \vec{X} \stackrel{\text{def}}{=} \left(g^{ij} \frac{\partial^2 \vec{X}}{\partial \xi_i \partial \xi_j}\right)^\perp \qquad g_{ij} = \vec{X}_{\xi_i} \cdot \vec{X}_{\xi_j}$$

Ancient solutions

Short time existence: M_* smooth immersed hypersurface of $\mathbb{R}^{n+1}$

$$\implies \exists T > 0 \; \exists \text{ smooth solution } \{M_t : 0 \le t < T\} \text{ with } M_0 = M_*.$$

Solutions immediately become real analytic hypersurfaces

Ancient solutions

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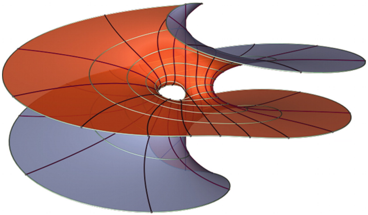
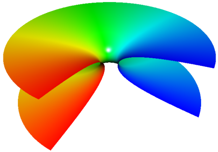
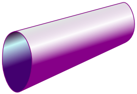
Examples: minimal surfaces ($H = 0$)



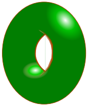


Examples: shrinking solitons



Cylinder
$S^k \times \mathbf{R}^{n-k}$

Sphere S^n



Shrinking Doughnut (Gregynog)

Examples: shrinking solitons

Cylinder
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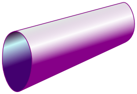

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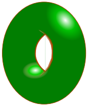

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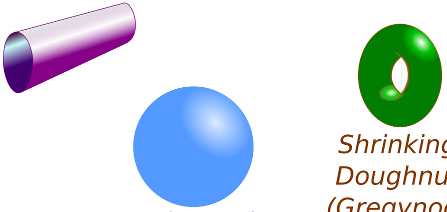

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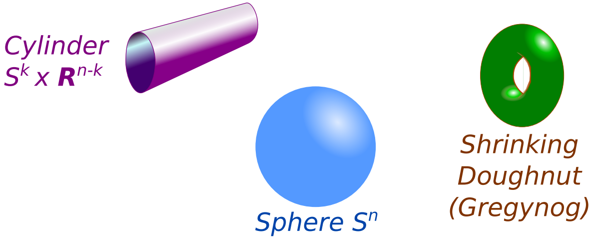

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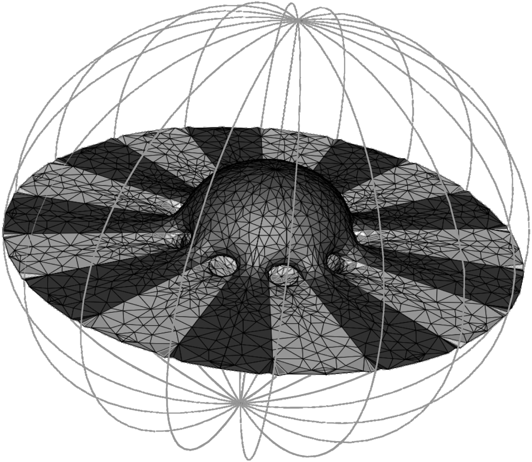
Shrinking Donuts: $M_t = \sqrt{-t} \cdot M_*$, with $M_* \approx S^1 \times S^{n-1} \subset \mathbb{R}^{n+1}$

And many more!

A non convex shrinking soliton in $\mathbb{R}^3$



Existence suggested by Tom Ilmanen ~ 1994
Rigorous construction by Xuan Hien Nguyen, and also
Kleen-Møller-Kapouleas ~ 2012

Curve Shortening–ancient solitons

Curve Shortening is MCF with $n = 1$: $V = k$

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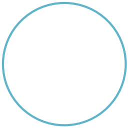
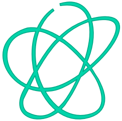
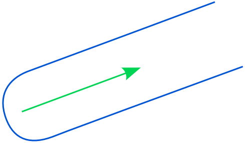
shrinkers



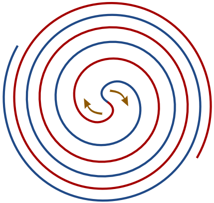
Circle



Abresch-Langer



translating soliton

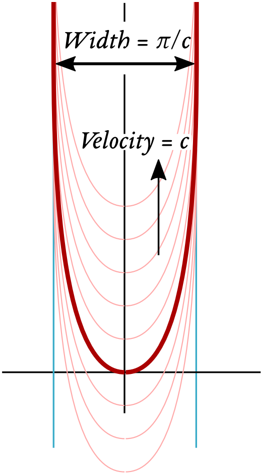


rotating soliton

Translating solitons

The Grim Reaper

$$y - ct = -\frac{1}{c}\log\cos cx$$



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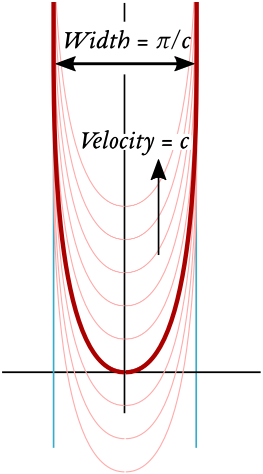
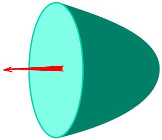


Bowl Soliton

For $n + 1 \geq 3$ in $\mathbb{R}^{n+1}$ translators are like paraboloids

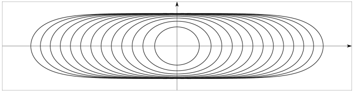


(no nice formula for translators when $n \geq 2$)

Curve Shortening–non self similar ancient solutions

$$k_t = k^2 k_{\theta\theta} + k^3$$

The Paper Clip



$$k(\theta, t) = \sqrt{\cos 2\theta - \coth 2t}$$

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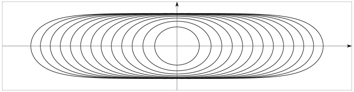

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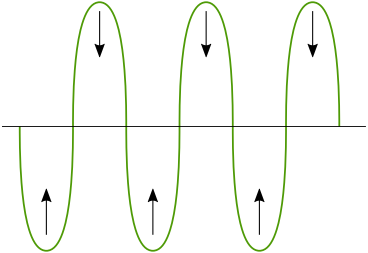
Ancient Sine



$$k(\theta, t + \tfrac{\pi}{2} i) = \sqrt{\cos 2\theta - \tanh 2t}$$

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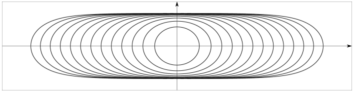

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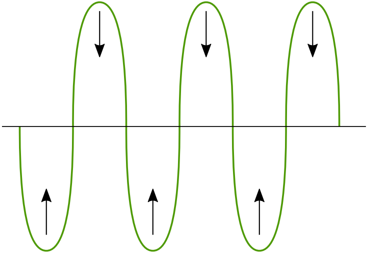


$$k(\theta, t + \tfrac{\pi}{2} i) = \sqrt{\cos 2\theta - \tanh 2t}$$

Paperclip: A-1992 (Gregynog)

Paperclip&Ancient Sine Curve: Wadati, Iizuki, Nakayama-1994

"Curve Lengthening"

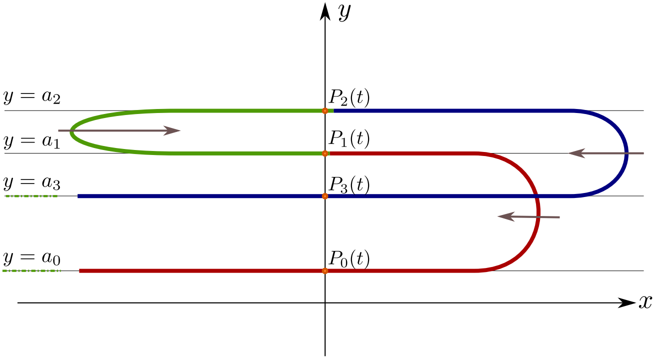
Galaktionov found similar solutions for $u_t = u^p u_{xx} + u^q$ in the 1980ies

Classification of Ancient Solutions of Curve Shortening

Theorem (Daskalopoulos, Hamilton, Sesum–2012).
The only ancient solutions of plane Curve Shortening that are:
compact, embedded, and convex,
are:
the circle *and* ***the paper clip****.*

Other ancient solutions to plane Curve Shortening

Qian You (Thesis, Madison 2014)



Ancient Trombones

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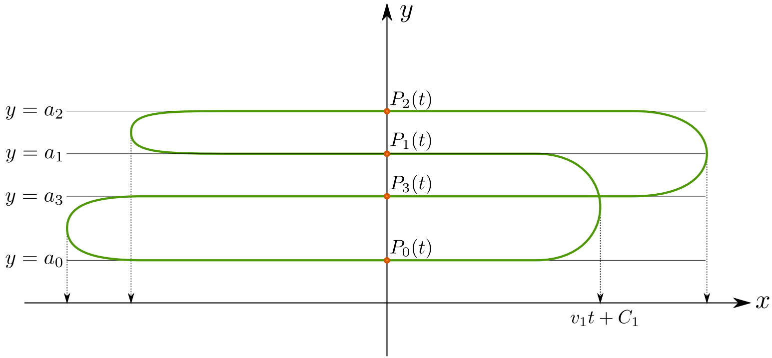

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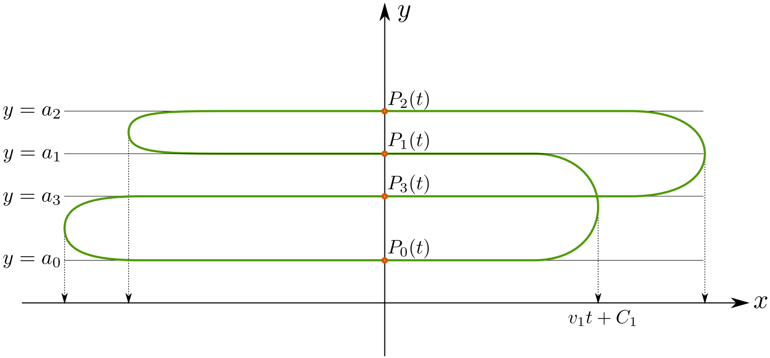
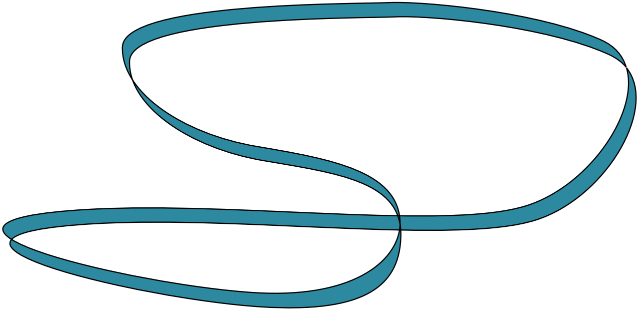


Conjecture: the only ancient solutions of plane Curve Shortening with finite total curvature are:

- self shrinkers (circle, Abresch-Langer curves)
- Ancient Trombones.

Proof ingredient



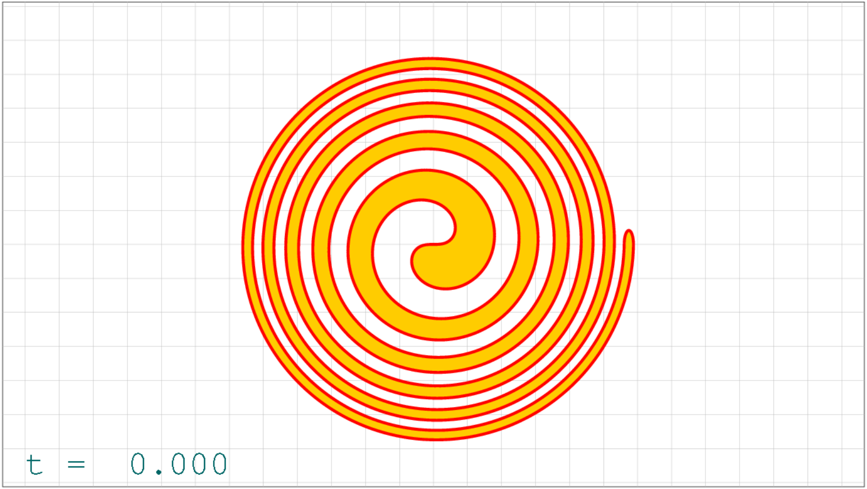
$A(t)$: unsigned area between the two curves $C_1(t)$ and $C_2(t)$

$$\frac{dA}{dt} \leq \int_{C_1(t) \cup C_2(t)} |V - k| ds$$

Unbounded total curvature

Compact, embedded, not convex

existence proof: Zhang, Olson, Khan, & A. (2019)



Higher dimensions

MCF: ancient convex solutions

$M_t^n \subset \mathbb{R}^{n+1}$ evolves by $V = H$ for $t < 0$.

Parabolic blow-up N_τ of M_t:

$$M_t = \sqrt{-t}\, N_\tau, \qquad \tau = -\log(-t).$$

N_τ evolves by $V = H + \frac{1}{2}\langle \boldsymbol{X}, \boldsymbol{\nu} \rangle$ (rescaled MCF).

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Theorem. (Huisken, 1986). *If M_{t_0} is* ***compact and convex*** *then M_t shrinks to a point and N_τ converges to the sphere as $\tau \to \infty$. Moreover,*

$$\frac{d}{d\tau}\left\{\frac{1}{(4\pi)^{n/2}}\int_{N_\tau} e^{-\|\boldsymbol{X}\|^2/4}\, dA\right\} \leq 0$$

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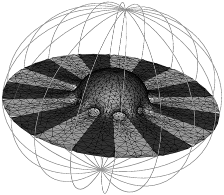
$$\mathcal{H}(N^n) \stackrel{\text{def}}{=} \frac{1}{(4\pi)^{n/2}} \int_{N^n} e^{-\|\boldsymbol{X}\|^2/4} dA$$

Convex self similar solutions

$\mathcal{H}(N^n) \stackrel{\text{def}}{=} \frac{1}{(4\pi)^{n/2}} \int_N e^{-\|\boldsymbol{X}\|^2/4} dA$

In general there is no classification of self similar solutions to MCF:



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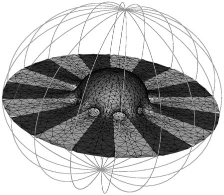

For ***convex*** embedded shrinkers Huisken showed that the only ones are

$$S^n \qquad S^{n-1} \times \mathbb{R} \qquad S^{n-2} \times \mathbb{R}^2 \qquad \cdots \qquad S^1 \times \mathbb{R}^{n-1} \qquad S^0 \times \mathbb{R}^n$$

where each sphere S^k has radius $\sqrt{2k}$.

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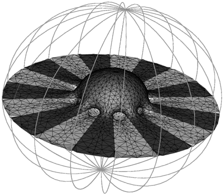

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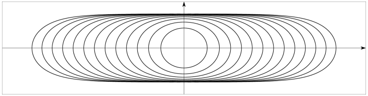
Note: $\mathcal{H}(N \times \mathbb{R}^k) = \mathcal{H}(N)$.

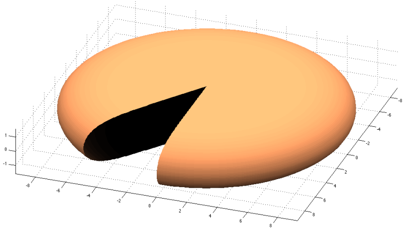
also: $S^0 = \partial[-1, 1] = \{-1, +1\}$

MCF analogs of the Paper Clip

S^n $\quad S^{n-1}\times\mathbb{R}$ $\quad S^{n-2}\times\mathbb{R}^2$ $\quad \cdots$ $\quad S^1\times\mathbb{R}^{n-1}$ $\quad S^0\times\mathbb{R}^n$

Shrinking disks of type $S^0\times\mathbb{R}^n$— the BOURNI-LANGFORD-TINAGLIA pancakes



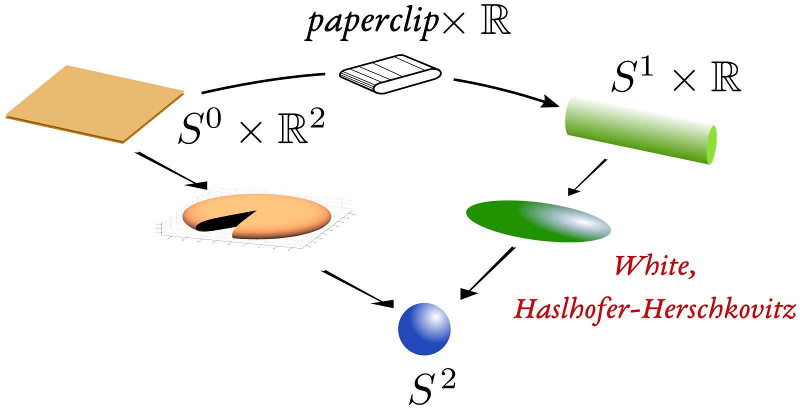


Height is π

Radius shrinks according to $\dfrac{dR}{dt} = -1 - \dfrac{n-1}{R}$

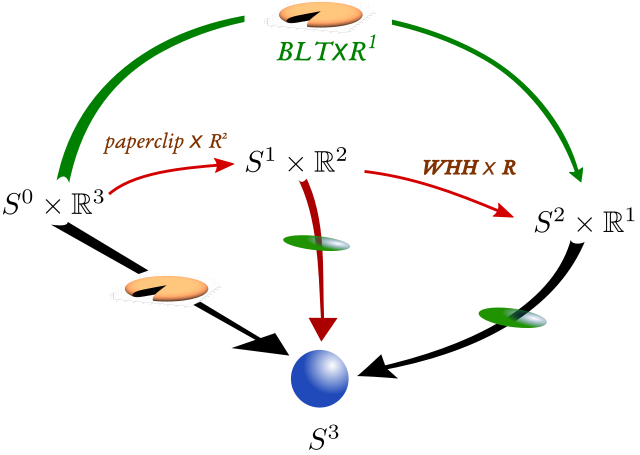
$$R(t) = -t - (n-1)\log(-t) + O(1)$$

All ancient convex solutions in $\mathbb{R}^3$??



White; Haslhofer-Hershkovitz

in $\mathbb{R}^4$

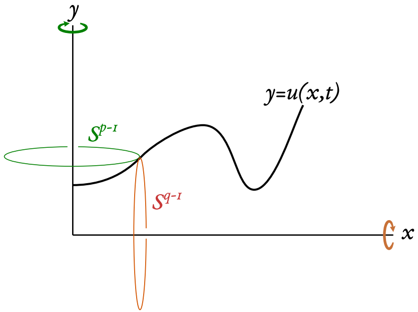


Solutions with $O(p) \times O(q)$ symmetry

Theorem. (White, Haslhofer-Hershkovitz) *There is a compact convex solution of rescaled MCF that connects $S^p \times \mathbb{R}^{n-p}$ with S^n.*

Q: how many are there, what do they look like?

Assume double rotational symmetry.



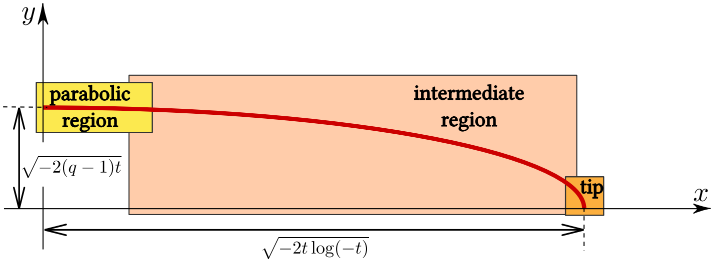
$$N_\tau = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^n : \|Y\| = u(x, \tau)\}$$

$$u_\tau = \frac{u_{xx}}{1 + u_x^2} - \frac{x}{2} u_x + \frac{u}{2} - \frac{q-1}{u}$$

.

Asymptotics of the WHH ancient ovals

Formal asymptotics



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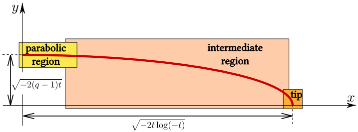

In the parabolic region:

$$u_\tau = \frac{u_{yy}}{1+u_y^2} - \frac{y u_y}{2} + \frac{u}{2} - \frac{n-1}{u}$$

$$u = \sqrt{2(n-1)}(1+v) \implies v_\tau = v_{yy} - \frac{y}{2} v_y + v - \frac{1}{2} v^2 + \cdots$$

$$\implies v = \frac{y^2-2}{4\tau} + o\big(|\tau|^{-1}\big)$$

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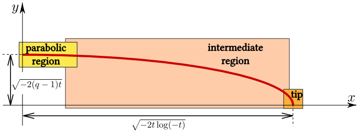


In the intermediate region:

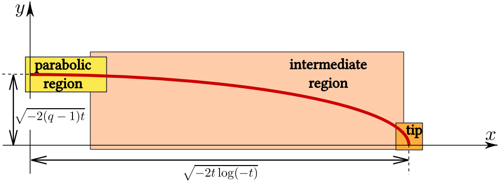
$$y = \frac{z}{\sqrt{|\tau|}} \implies \frac{\partial u}{\partial \tau} = \frac{u_{zz}}{|\tau| + u_z^2} - \frac{z}{2}\left(1 - \frac{1}{\tau}\right) u_z + \frac{u}{2} - \frac{n-1}{u}.$$

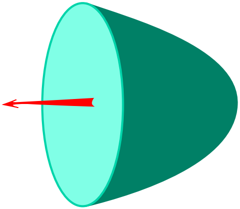
$$\implies u = \sqrt{n-1}\sqrt{2 - z^2} + o(1) \qquad (\tau \to -\infty)$$

Most of the surface is an ellipsoid!

Asymptotics of the WHH ancient ovals

Formal asymptotics





In the tip region: p_t = tip of $M_t \subset \mathbb{R}^{n+1}$,

$$\lambda_t\big(M_t - p_t\big) \longrightarrow \mathcal{B}$$

where $\lambda_t := H(p_t, t) = H_{\max}(t)$ and $\mathcal{B}$ is unique Bowl Soliton, i.e. the unique rotationally symmetric translating soliton with velocity one.

Asymptotics and Uniqueness of WHH ancient ovals

Asymptotics Theorem (Daskalopoulos, Sesum, A–2015) *Every convex rotationally symmetric ancient solution that converges to the cylinder as* $\tau \to -\infty$ *satisfies the formal asymptotic description.*

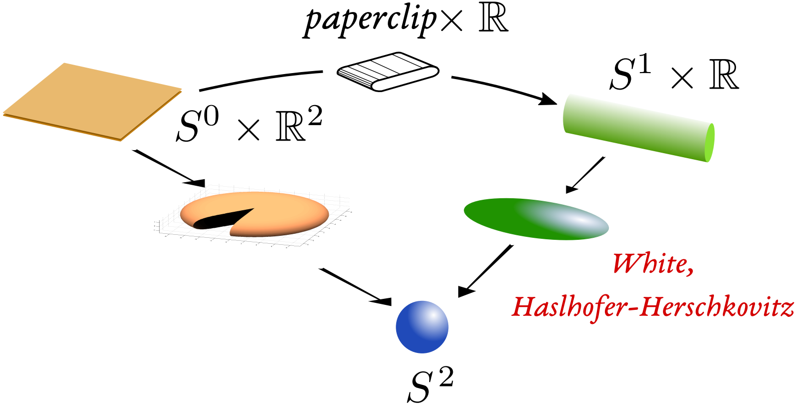
Asymptotics and Uniqueness of WHH ancient ovals

Asymptotics Theorem (Daskalopoulos, Sesum, A–2015) *Every convex rotationally symmetric ancient solution that converges to the cylinder as* $\tau \to -\infty$ *satisfies the formal asymptotic description.*

Uniqueness Theorem (Daskalopoulos, Sesum, A–2019) *Every convex ancient solution that converges to the cylinder is rotationally symmetric.*

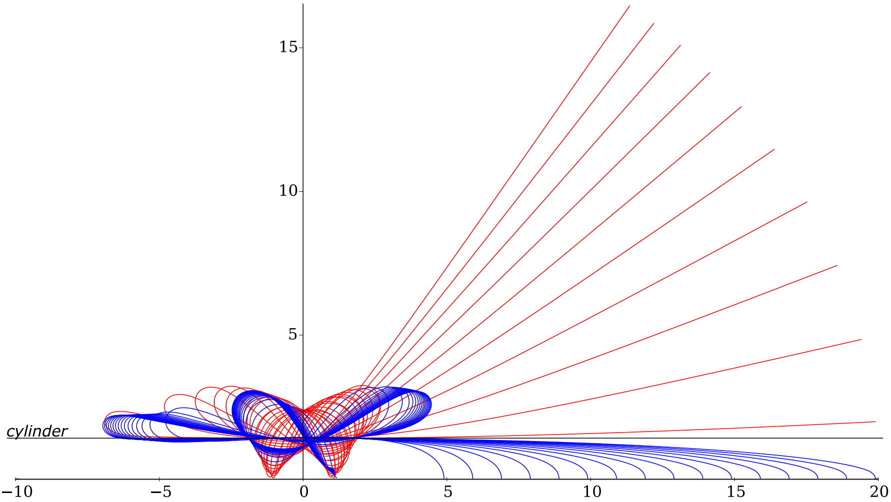
Two convex ancient solutions that converge to the cylinder as $\tau \to -\infty$ *differ only by a translation in time, and a parabolic rescaling in space time.*

All ancient convex solutions in $\mathbb{R}^3$??



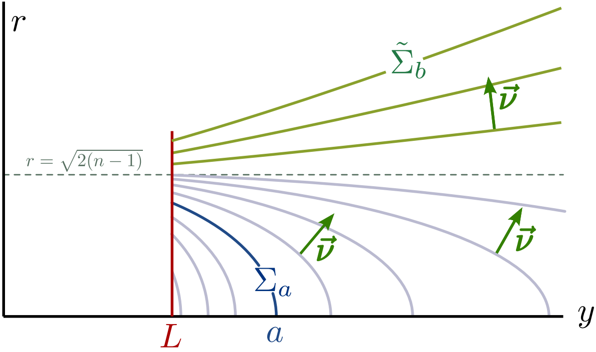
Proof ingredients

incomplete self shrinkers

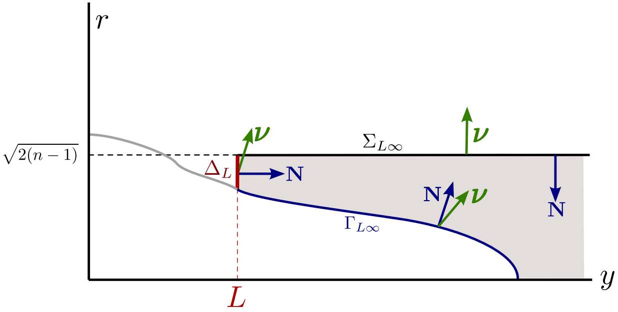


Proof ingredients

The minimizing foliation for the Huisken functional



$$\mathrm{div}\left(e^{-\|X\|^2/4}\vec{\nu}\right) = 0.$$



$$0 = \int_{\dots} \vec{\nabla} \cdot \left(e^{-\|X\|^2/4} \vec{v}\right) dA$$
$$= -\int_{\Sigma_{L\infty}} e^{-\|X\|^2/4} dA + \int_{\Gamma_{L\infty}} e^{-\|X\|^2/4} (\vec{N} \cdot \vec{v}) dA + \int_{\Delta_L} e^{-\|X\|^2/4} dA.$$

On the other hand : $\mathcal{H}(\Sigma) \geq \mathcal{H}(\Gamma)$, i.e. $\int_\Sigma e^{-\|X\|^2/4} dA \geq \int_\Gamma e^{-\|X\|^2/4} dA$