Ancient Solutions to Mean Curvature Flow

Joost is jarig!

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Joint work with

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Mean Curvature Flow

\[ \vec{H} = H \vec{\nu} \]

\[ H = \kappa_1 + \kappa_2 \]

\[ \text{MCF} \iff V = H \]

\[ \iff \left( \frac{\partial \vec{X}}{\partial t} \right)^\perp = \Delta \vec{X} \]
Mean Curvature Flow

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Mean Curvature Flow

\[
\vec{H} = H \vec{v}
\]

\[
H = \kappa_1 + \kappa_2
\]

\[
\text{MCF} \iff V = H
\]

\[
\iff \left( \frac{\partial \vec{X}}{\partial t} \right) \perp = \Delta \vec{X}
\]

\[
\Delta \vec{X} \overset{\text{def}}{=} \left( g_{ij} \frac{\partial^2 \vec{X}}{\partial \xi_i \partial \xi_j} \right) \perp
\]

\[
g_{ij} = \vec{X}_{\xi_i} \cdot \vec{X}_{\xi_j}
\]
Ancient solutions

Short time existence: $M_\ast$ smooth immersed hypersurface of $\mathbb{R}^{n+1}$

$$\implies \exists T > 0 \ \exists \text{smooth solution} \ \{M_t : 0 \leq t < T\} \ \text{with} \ M_0 = M_\ast.$$  

Solutions immediately become real analytic hypersurfaces
Ancient solutions

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**Backward nonexistence:** for most $M_0$ there is no solution for $t < 0$. 
Ancient solutions

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**Ancient Solution**: a solution \( \{M_t\} \) that is defined \textit{for all} \( t < 0 \).
Ancient solutions

**Short time existence:** $M_*$ smooth immersed hypersurface of $\mathbb{R}^{n+1}$

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Solutions immediately become real analytic hypersurfaces

**Backward nonexistence:** for most $M_0$ there is no solution for $t < 0$. 

**Ancient Solution:** a solution $\{M_t\}$ that is defined *for all* $t < 0$. 

*Examples: minimal surfaces ($H = 0$)*
Examples: shrinking solitons

Cylinder \( S^k \times \mathbb{R}^{n-k} \)

Sphere \( S^n \)

Shrinking Doughnut (Gregynog)
Examples: shrinking solitons

Shrinking Spheres: \( S^n_{r(t)} \subset \mathbb{R}^{n+1} \) with radius \( r(t) = \sqrt{-2nt} \)
Examples: shrinking solitons

Shrinking Spheres: $S^n_{r(t)} \subset \mathbb{R}^{n+1}$ with radius $r(t) = \sqrt{-2nt}$

Shrinking Cylinders: $S^k_{r(t)} \times \mathbb{R}^{n-k}$ with radius $r(t) = \sqrt{-2kt}$. 
Examples: shrinking solitons

- Shrinking Spheres: $S^n_{r(t)} \subset \mathbb{R}^{n+1}$ with radius $r(t) = \sqrt{-2nt}$

- Shrinking Cylinders: $S^k_{r(t)} \times \mathbb{R}^{n-k}$ with radius $r(t) = \sqrt{-2kt}$.

- Shrinking Donuts: $M_t = \sqrt{-t} \cdot M_*$, with $M_* \approx S^1 \times S^{n-1} \subset \mathbb{R}^{n+1}$

And many more!
A non convex shrinking soliton in $\mathbb{R}^3$

Existence suggested by Tom Ilmanen ~ 1994
Rigorous construction by Xuan Hien Nguyen, and also
Kleen-Møller-Kapouleas ~ 2012
Curve Shortening–ancient solitons

Curve Shortening is MCF with $n = 1$: $V = k$
Curve Shortening–ancient solitons

Curve Shortening is MCF with $n = 1$: $V = k$

- **shrinkers**
- **Circle**
- **Abresch-Langer**
- **translating soliton**
- **rotating soliton**
Translating solitons

The Grim Reaper

\[ y - ct = -\frac{1}{c} \log \cos cx \]
Translating solitons

The Grim Reaper

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Bowl Soliton

For \( n + 1 \geq 3 \) in \( \mathbb{R}^{n+1} \)

translators are like paraboloids

(no nice formula for translators when \( n \geq 2 \))
Curve Shortening—non self similar ancient solutions

\[ k_t = k^2 k_{\theta\theta} + k^3 \]

The Paper Clip

\[ k(\theta, t) = \sqrt{\cos 2\theta - \coth 2t} \]
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**The Paper Clip**

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**Ancient Sine**

\[ k(\theta, t + \frac{\pi}{2} i) = \frac{\sqrt{\cos 2\theta - \tanh 2t}}{\sqrt{\cos 2\theta - \coth 2t}} \]
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Paperclip: A-1992 (Gregynog)

Paperclip & Ancient Sine Curve: WADATI, IIZUKI, NAKAYAMA-1994

“Curve Lengthening”

GALAKTIONOV found similar solutions for \( u_t = u^p u_{xx} + u^q \) in the 1980ies
Classification of Ancient Solutions of Curve Shortening

Theorem (Daskalopoulos, Hamilton, Sesum–2012).

The only ancient solutions of plane Curve Shortening that are: compact, embedded, and convex, are:

the circle and the paper clip.
Other ancient solutions to plane Curve Shortening
Qian You (Thesis, Madison 2014)

Ancient Trombones
Conjecture: the only ancient solutions of plane Curve Shortening with finite total curvature are:

▶ self shrinkers (circle, Abresch-Langer curves)
▶ Ancient Trombones.
Conjecture: the only ancient solutions of plane Curve Shortening with finite total curvature are:

- self shrinkers (circle, Abresch-Langer curves)
- Ancient Trombones.
Proof ingredient

$A(t)$: unsigned area between the two curves $C_1(t)$ and $C_2(t)$

$$\frac{dA}{dt} \leq \int_{C_1(t) \cup C_2(t)} |V - k| \, ds$$
Unbounded total curvature

Compact, embedded, not convex

existence proof: Zhang, Olson, Khan, & A. (2019)
Higher dimensions
MCF: ancient convex solutions

\( M_t^n \subset \mathbb{R}^{n+1} \) evolves by \( V = H \) for \( t < 0 \).

Parabolic blow-up \( N_\tau \) of \( M_t \):

\[
M_t = \sqrt{-t} N_\tau, \quad \tau = -\log(-t).
\]

\( N_\tau \) evolves by \( V = H + \frac{1}{2} \langle X, \nu \rangle \) (rescaled MCF).
MCF: ancient convex solutions

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Theorem. (Huisken, 1986). If $M_{t_0}$ is compact and convex then $M_t$ shrinks to a point and $N_\tau$ converges to the sphere as $\tau \to \infty$. Moreover,

$$\frac{d}{d \tau} \left\{ \frac{1}{(4\pi)^{n/2}} \int_{N_\tau} e^{-\|X\|^2/4} dA \right\} \leq 0$$
MCF: ancient convex solutions

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$\mathcal{H}(N^n) \overset{\text{def}}{=} \frac{1}{(4\pi)^{n/2}} \int_{N^n} e^{-\|X\|^2/4} dA$
Convex self similar solutions

\[ H(N^n) \overset{\text{def}}{=} \frac{1}{(4\pi)^{n/2}} \int_N e^{-\|X\|^2/4} dA \]

In general there is no classification of self similar solutions to MCF:
Convex self similar solutions

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For \textit{convex} embedded shrinkers Huisken showed that the only ones are

\[ S^n \quad S^{n-1} \times \mathbb{R} \quad S^{n-2} \times \mathbb{R}^2 \quad \ldots \quad S^1 \times \mathbb{R}^{n-1} \quad S^0 \times \mathbb{R}^n \]

where each sphere \( S^k \) has radius \( \sqrt{2k} \).
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where each sphere \( S^k \) has radius \( \sqrt{2k} \).

Note: \( \mathcal{H}(N \times \mathbb{R}^k) = \mathcal{H}(N) \).

also: \( S^0 = \partial[-1, 1] = \{-1, +1\} \)
MCF analogs of the Paper Clip

\[ S^n \quad S^{n-1} \times \mathbb{R} \quad S^{n-2} \times \mathbb{R}^2 \quad \ldots \quad S^1 \times \mathbb{R}^{n-1} \quad S^0 \times \mathbb{R}^n \]

Shrinking disks of type \( S^0 \times \mathbb{R}^n \) — the BOURNI-LANGFORD-TINAGLIA pancakes

Height is \( \pi \)

Radius shrinks according to

\[ \frac{dR}{dt} = -1 - \frac{n - 1}{R} \]

\[ R(t) = -t - (n - 1) \log(-t) + O(1) \]
All ancient convex solutions in $\mathbb{R}^3$??

$S^0 \times \mathbb{R}^2$ 

$paperclip \times \mathbb{R}$

$S^1 \times \mathbb{R}$

White, Haslhofer-Herschkovitz

$S^2$
White; Haslhofer-Hershkovitz
in $\mathbb{R}^4$
Solutions with $O(p) \times O(q)$ symmetry

**Theorem.** (White, Haslhofer-Hershkovitz) *There is a compact convex solution of rescaled MCF that connects $S^p \times \mathbb{R}^{n-p}$ with $S^n$.*

Q: how many are there, what do they look like?

Assume double rotational symmetry.

\[ N_\tau = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^n : \|Y\| = u(x, \tau)\} \]

\[ u_\tau = \frac{u_{xx}}{1 + u_x^2} - \frac{x}{2} u_x + \frac{u}{2} - \frac{q - 1}{u} \]
Asymptotics of the WHH ancient ovals

Formal asymptotics
Asymptotics of the WHH ancient ovals

Formal asymptotics

In the parabolic region:

\[
\begin{align*}
\eta_x &= \frac{u_{yy}}{1 + u_x^2} - \frac{yu_y}{2} + \frac{u}{2} - \frac{n - 1}{u} \\
\eta &= \sqrt{2(n-1)(1 + v)} \implies \nu_x = \nu_{yy} - \frac{y}{2} \nu_y + \nu - \frac{1}{2} \nu^2 + \cdots \\
\implies \nu &= \frac{y^2 - 2}{4\tau} + o\left(|\tau|^{-1}\right)
\end{align*}
\]
Asymptotics of the WHH ancient ovals

Formal asymptotics

In the intermediate region:

\[ y = \frac{z}{\sqrt{|\tau|}} \implies \frac{\partial u}{\partial \tau} = \frac{u_{zz}}{|\tau| + u_z^2} - \frac{z}{2} \left(1 - \frac{1}{\tau}\right) u_z + \frac{u}{2} - \frac{n-1}{u}. \]

\[ \implies u = \sqrt{n-1} \sqrt{2 - z^2} + o(1) \quad (\tau \to -\infty) \]

Most of the surface is an ellipsoidal!
Asymptotics of the WHH ancient ovals

Formal asymptotics

In the tip region: \( p_t = \text{tip of } M_t \subset \mathbb{R}^{n+1} \),

\[
\lambda_t (M_t - p_t) \to B
\]

where \( \lambda_t := H(p_t, t) = H_{\text{max}}(t) \) and \( B \) is unique Bowl Soliton, i.e. the unique rotationally symmetric translating soliton with velocity one.
Asymptotics and Uniqueness of WHH ancient ovals

Asymptotics Theorem (Daskalopoulos, Sesum, A–2015) Every convex rotationally symmetric ancient solution that converges to the cylinder as \( \tau \to -\infty \) satisfies the formal asymptotic description.
Asymptotics and Uniqueness of WHH ancient ovals

Asymptotics Theorem (Daskalopoulos, Sesum, A–2015) Every convex rotationally symmetric ancient solution that converges to the cylinder as $\tau \to -\infty$ satisfies the formal asymptotic description.

Uniqueness Theorem (Daskalopoulos, Sesum, A–2019) Every convex ancient solution that converges to the cylinder is rotationally symmetric. Two convex ancient solutions that converge to the cylinder as $\tau \to -\infty$ differ only by a translation in time, and a parabolic rescaling in space time.
All ancient convex solutions in $\mathbb{R}^3$??
Proof ingredients
incomplete self shrinkers
Proof ingredients

The minimizing foliation for the Huisken functional

\[
\text{div} \left( e^{-\|X\|^2/4 \nu} \right) = 0.
\]
\[ 0 = \int \vec{\nabla} \cdot (e^{-\frac{||X||^2}{4}} \vec{v}) \, dA \]

\[ = -\int_{\Sigma_{L\infty}} e^{-\frac{||X||^2}{4}} \, dA + \int_{\Gamma_{L\infty}} e^{-\frac{||X||^2}{4}} (\vec{N} \cdot \vec{v}) \, dA + \int_{\Delta_{L}} e^{-\frac{||X||^2}{4}} \, dA. \]

On the other hand: \( \mathcal{H}(\Sigma) \geq \mathcal{H}(\Gamma) \), i.e. \( \int_{\Sigma} e^{-\frac{||X||^2}{4}} \, dA \geq \int_{\Gamma} e^{-\frac{||X||^2}{4}} \, dA \).