Differentiation = linear approximation

X, Y normed vector spaces, $f : X \to Y$, e.g. $X = \mathbb{R}^n, Y = \mathbb{R}^m$. p denotes some (fixed) point in X (I use p instead of x_0 now).

Definition. f is called differentiable in $p \in X$ if there exists $A : X \to Y$ linear and continuous such that $R : X \to Y$ defined implicitly by

$$f(x) = f(p) + A(x-p) + R(x), \text{ satisfies } \lim_{x \to p} \frac{|R(x)|}{|x-p|} = 0.$$

Here the vertical bars $||$ denote the norm or length of the quantitity in between.

Simplest case. $X = \mathbb{R}, Y = \mathbb{R}$. In this case $A(x-p) = f'(p)(x-p)$ and it seems we are nitpicking. Why not use the (equivalent) definition

$$\frac{df}{dx}(p) = f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x-p} \text{ if the limit exists,}$$

and call f differentiable in $p \in \mathbb{R}$ if this happens to be the case?

Answer. Because this only works for $X = \mathbb{R}$ and does not take us very far.

Second simplest case. $X = \mathbb{R}^n, Y = \mathbb{R}^m$. In this (Calculus 2) case A may be seen as a matrix. The first row of the matrix has the partial derivatives of the first component of f as entries, the second row the partial derivatives of the second component of f, etc, so

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(p),$$

the matrix of all partial derivatives (Jacobian matrix) in p. In other words, the first row is the gradient of f_1, the second of f_2, etc. We write ∂ instead of d because some smart person decided to do so. **Warning.** The existence of all these partial derivatives means nothing without:

Main theorem for second simplest case. If all the partial derivatives are continuous in p (what does this, implicitly, mean?) then f is differentiable in $p \in X$ and A is given by the Jacobian matrix as above.

Special second simplest case. $X = \mathbb{R}^2, Y = \mathbb{R}^2$. Here we usually write (x, y) instead of (x_1, x_2), (p, q) instead of (p_1, p_2), and (u, v) instead of (f_1, f_2), to allow for reinterpretation of $f : \mathbb{C} \to \mathbb{C}$ below. In this case

$$A = \begin{pmatrix} \frac{\partial u}{\partial x}(p,q) & \frac{\partial u}{\partial y}(p,q) \\ \frac{\partial v}{\partial x}(p,q) & \frac{\partial v}{\partial y}(p,q) \end{pmatrix}$$

Also of interest

In between simplest and second simplest case. $X = \mathbb{R}^n, Y = \mathbb{R}$. Now, provided f is differentiable in p, $A = 0$ corresponds to necessary (not sufficient) conditions for f to have an extremum in p.

A physical example, classical mechanics. f is difference between kinetic and potential energy, $A = 0$ is equivalent to the equations of motion.

Complex differentiation

Complex functions $f : \mathbb{C} \to \mathbb{C}$, s denotes some (fixed) point in $\mathbb{C}$ (I use s instead of z_0 now).

Definition. f is called differentiable in $s \in \mathbb{C}$ if there exists $\alpha \in \mathbb{C}$ such that $R : \mathbb{C} \to \mathbb{C}$ defined implicitly by

$$f(z) = f(s) + \alpha(z - s) + R(z), \text{ satisfies } \lim_{z \to s} \frac{|R(z)|}{|z - s|} = 0$$

Now the vertical bars | | denote the absolute value, which is the length of the corresponding vector in $\mathbb{R}^2$. We have, as for $f : \mathbb{R} \to \mathbb{R}$, that

$$f'(s) = \lim_{z \to s} \frac{f(z) - f(s)}{z - s} = \alpha,$$

which works fine for polynomials like $f(z) = z^3 - z + 1$, giving what we (should) expect, but what if we only know u and v, for instance $f(z) = \exp(z)$?

Writing $\alpha = a + ib$, $z = x + iy$, $s = p + iq$, $f = u + iv$, to compare to $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$, we find that the **2x2 matrix** A **must have a special form**, namely

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

simply **because**

$$\alpha(z - s) \in \mathbb{C}$$

rewrites as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x - p \\ y - q \end{pmatrix} \in \mathbb{R}^2.$$

Combining the main theorem above for $f : \mathbb{R}^2 \to \mathbb{R}^2$, with the special form of A, **a sufficient condition for complex differentiability** of $f = u + iv$

in $s = p + iq$ is the **continuity of all four partial derivatives in** (p, q), **plus the Cauchy-Riemann equations** in $s = p + iq$ which characterise the special form of A, **i.e.**

$$\frac{\partial u}{\partial x}(p,q) = \frac{\partial v}{\partial y}(p,q), \quad \frac{\partial u}{\partial y}(p,q) = -\frac{\partial v}{\partial x}(p,q).$$

Exercise, or see the book. Verify directly that complex differentiability implies the Cauchy Riemann equations.

Remark. $f = u + iv : \mathbb{C} \to \mathbb{C}$ differentiable in $s = p + iq$ is equivalent to $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ differentiable in (p, q) combined with the Cauchy-Riemann equations in (p, q).
N.B. $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ differentiable in (p, q) is often best verifiable using the main theorem above.

Exercises

1. Verify, both by means of the limit definition, as well as by using the Cauchy-Riemann equations, that $f(z) = z^2$ is differentiable in every $z \in \mathbb{C}$. Determine $f'(z)$.

2. Verify, both by means of the limit definition, as well as by using the Cauchy-Riemann equations, that $f(z) = \frac{1}{z}$ is differentiable in every $0 \neq z \in \mathbb{C}$. Determine $f'(z)$.

2. Verify, using the Cauchy-Riemann equations, that $f(z) = \exp(z)$ is differentiable in every $z \in \mathbb{C}$. Determine $f'(z)$.