Spectral Theory

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1 Hilbert spaces: selfadjoint operators

In many applications it is important to understand the spectral properties of a linear operator $T : X \to X$, where X is some vector space over $\mathbb{R}$ or $\mathbb{C}$. In the finite dimensional (complex) case linear operators may be characterised as matrices and the Jordan normal form theorem applies, providing a basis of generalised eigenvectors. If, in addition, T is normal (i.e. T and T^* commute) with respect to an inner product, then the basis is orthogornal and consists of eigenvectors only.

For spectral theory it is often convenient to work in complex spaces. For symmetric operators however the real numbers are just fine. The simplest theorem for the infinite dimensional case may be formulated and proved in the real setting.

Theorem 1.1 *Let H be a real Hilbert space, $T : H \to H$ a compact symmetric linear operator. Then T has a finite or infinite sequence of eigenvectors $\{x_1, x_2, \ldots\}$ with $(x_i, x_j) = \delta_{ij}$, corresponding to nonzero eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots \in \mathbb{R}$ with, if the sequence is infinite,*

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \downarrow 0.$$

Moreover

$$|\lambda_1| = \max_{0 \neq x \in H} \left|\frac{(Tx, x)}{(x, x)}\right| = |(Tx_1, x_1)|,$$

and, more generally,

$$|\lambda_{n+1}| = \max_{\substack{0 \neq x \in H \\ (x, x_1) = \cdots = (x, x_n) = 0}} \left|\frac{(Tx, x)}{(x, x)}\right| = |(Tx_{n+1}, x_{n+1})|.$$

The null space $N(T)$ of T is the orthoplement of the (closed) subspace generated by the x_j's.

Proof. The proof we give here closely resembles a proof of the finite dimensional case. It is based on the symmetry of T (i.e. $(Tx, y) = (x, Ty)$) and the observation that the supremum

$$s = \sup_{0 \neq x \in H} \left|\frac{(Tx, x)}{(x, x)}\right|$$

is attained because (i): the operator is compact, which means that bounded sets are mapped to precompact sets, and (ii): bounded sequences in H have weakly convergent subsequences.

To see that the supremum is a maximum we choose a sequence $y_n \in H$ with $(y_n, y_n) = 1$ and $(Ty_n, y_n) \to \pm s$. Then there is a subsequence, denoted again by y_n, such that $y_n \rightharpoonup y \in H$, meaning that $(x, y_n) \to (x, y)$ for all $x \in H$. This is called weak convergence of y_n to y.

Remark. We do not go into the details of weak topologies here but it is good to recall that in reflexive Banach spaces bounded sequences have weakly convergent subsequences. If in addition the space is separable, the relative weak topology on the closed ball is a metric topology, so that there is no difference between compact and sequentially compact. Our Hilbert space is not assumed to be separable, but the convergent subsequence argument is still valid. This can be seen by first restricting attention to the the closed subspace spanned by the sequence x_n.

Continuing with the proof, there is a further subsequence such that $Ty_n \to Ty$ because T is compact. Hence

$$(Ty_n, y_n) = (Ty, y_n) + (Ty_n - Ty, y_n) \to (Ty, y).$$

Thus s is attained.

The rest of the proof is as the "finite dimensional" proof. If $s = 0$ then it is easily seen that $T = 0$ so suppose $s > 0$. Changing to $-T$ if necessary we may assume that

$$s = \max_{0 \neq x \in H} \frac{(Tx, x)}{(x, x)} = (Tx_1, x_1),$$

with $(x_1, x_1) = 1$. For x_1 we can take the weak limit y above. We claim that $Tx_1 = sx_1$. Indeed, $Tx_1 = (Tx_1, x_1)x_1 + Tx_1 - (Tx_1, x_1)x_1 = sx_1 + y$, and $y = Tx_1 - (Tx_1, x_1)x_1$ satisfies $(y, x_1) = 0$ whence $z_\varepsilon = x_1 + \varepsilon y$ has $(z_\varepsilon, z_\varepsilon) = 1 + \varepsilon^2 (y, y)$. But $(Tz_\varepsilon, z_\varepsilon) = (Tx_1, x_1) + \varepsilon(Tx_1, y) + \varepsilon(y, Tx_1) + \varepsilon^2(Ty, Ty) = (Tx_1, x_1) + 2\varepsilon(Tx_1, y) + \varepsilon^2(Ty, Ty)$ with $(Tx_1, y) = (sx_1 + y, y) = (y, y)$. If $y \neq 0$ we can thus make the quotient smaller than s by varying ε, a contradiction.

Thus $Tx_1 = sx_1$ and s is an eigenvalue of T (or $-T$ if we changed to $-T$). This provides us with λ_1. We then repeat the argument with the restriction of T to $\{x \in H : (x, x_1) = 0\}$, which is invariant under T because T is symmetric. In this fashion we either get a finite sequence of real nonzero eigenvalues $\lambda_1, \ldots, \lambda_n$ with orthogornal eigenvectors $x_1, \ldots x_n$ and $s = 0$ in the $(n+1)$-th step, or we get an infinite sequence. In the latter case the compactness of T implies that $\lambda_j \to 0$. This completes the proof.

The formulation and proof of a generalisation of this theorem to general unbounded closed or bounded operators are more difficult. The following exercices are intended to indicate what's going on.

Exercise 1 Let T be a bounded operator on a real or complex Hilbert space. If T is symmetric, i.e. $(Tx, y) = (x, Ty)$ for all $x, y \in H$, show that (Tx, x) is always real and that

$$||T|| = \sup_{||x||=||y||=1} |(Tx, y)| = \sup_{||x||=1} |(Tx, x)|.$$

Hint: consider $(T(x+y), x+y) - (T(x-y), x-y)$ and use $||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$.

Exercise 2 Let T be a bounded symmetric operator on a real or complex Hilbert space. Show that distinct eigenvalues have mutually orthogornal eigenvectors. If $M \subset H$ is an invariant subspace ($TM \subset M$), show that also $M^{\perp} = \{x \in H : \forall y \in M\ (x, y) = 0\}$ is an invariant subspace.

Exercise 3 Let T be a compact symmetric operator on a real or complex Hilbert space. If $\lambda \neq 0$ is an eigenvalue of T, show that $N(T - \lambda) = \{x \in H : Tx = \lambda x\}$ is finite dimensional. Explain why in Theorem 1.1, if the sequence is infinite, $\lambda_n \to 0$.

Exercise 4 Let T be a symmetric bounded operator on a real or complex Hilbert space and let $V(T)$ be the closure of $\{(Tx, x);\, x \in H,\, ||x|| = 1\}$. Show that $\lambda - T$ is invertible if $\lambda \notin V(T)$ and give an estimate for the norm of the inverse. Hint: $|\lambda - (Tx, x)| \leq ||\lambda x - Tx||$ gives that $\lambda - T : H \to R(\lambda - T)$ is a linear homeomorphism and $R(\lambda - T) = H$ because otherwise $\bar{\lambda}$ is an eigenvalue. (In fact you can do this without the symmetry assumption.)

Exercise 5 Let T be a symmetric bounded operator on a real or complex Hilbert space and let $m(T)$ and $M(T)$ be defined by Let

$$m(T) = \inf_{x \in H,\, ||x||=1} (Tx, x) \quad \text{and} \quad M(T) = \sup_{x \in H,\, ||x||=1} (Tx, x). \tag{1.1}$$

Then

$$\sigma(T) = \{\lambda \in \mathbb{C} : (\lambda - T)^{-1} \text{ is not a bijection}\}$$

is contained in $[m(T), M(T)]$ and both endpoints of this interval are contained in $\sigma(T)$. Hint: for the last statement, assume $m(T) = 0$ and show that T cannot have a bounded inverse (and therefore not be a bijection).

Exercise 6 State and prove Theorem 1.1 for complex Hilbert spaces.

Exercise 7 In the context of Theorem 1.1, show that

$$Tx = \sum_k \lambda_k (x, x_k) x_k. \tag{1.2}$$

Give a similar representation of $(\mu - T)^{-1}$ if $\mu \notin \sigma(T) = \{0, \lambda_1, \lambda_2, \ldots\}$.

Exercise 8 In the context of Theorem 1.1, let

$$E_\lambda x = \sum_{k, \lambda_k \leq \lambda} (x, x_k) x_k \quad \text{for} \quad \lambda < 0,$$

and

$$E_\lambda x = x - \sum_{k, \lambda_k > \lambda} (x, x_k) x_k \quad \text{for} \quad \lambda \geq 0.$$

Show that the orthogornal projections E_λ have

$$\lambda \geq \mu \;\Rightarrow\; E_\lambda E_\mu = E_\mu E_\lambda = E_\mu,$$

and

$$\lambda \downarrow \mu \;\Rightarrow\; E_\lambda x \downarrow E_\mu x.$$

Explain the formula

$$Tx = \int \lambda d E_\lambda x, \tag{1.3}$$

and give similar formulas for $(\mu - T)^{-1}x$ if $\mu \notin \sigma(T)$ and for $p(T)x$ where p is a polynomial. From this exercise you can guess a general theorem for bounded symmetric operators, see Theorem VI-6.1 in [5].

The spectral theory of symmetric operators may be formulated in both the complex and the real setting. The more general theory for normal operators needs a complex formulation because the spectrum is no longer real. A treatment of normal operators employing representation theory of C^*-algebra's may be found in [4] or in [5]. In many applications however the operators are not normal. Then there is no reason to restrict to Hilbert spaces (which exclude most of the standard function spaces). Therefore we will leave the Hilbert space setting and see what can be done in the (complex) Banach space setting. The main reference here is [5], which is also the source of the exercises above.

2 Banach spaces: spectra of bounded operators

Unless stated otherwise from here on X is always a complex Banach space, $X \neq \{0\}$. The dual (Banach) space of bounded linear functionals $f : X \to \mathbb{C}$ is denoted by X^*, the norm on X^* being given by

$$||f|| = \sup_{0 \neq x \in X} \frac{|f(x)|}{||x||}.$$

Also, $T : X \to X$ is always a bounded linear operator, and the Banach space of bounded linear operators on X is denoted by $B(X)$, with norm

$$||T|| = \sup_{0 \neq x \in X} \frac{||Tx||}{||x||}.$$

Finally, Ω is always an open subset of $\mathbb{C}$.

In [5] the concepts below are discussed for general unbounded operators.

Definition 2.1 The resolvent set $\rho(T)$ is the set of all complex λ such that $\lambda - T = \lambda I - T : X \to X$ is a bijection. By the bounded inverse theorem $(\lambda - T)^{-1}$ is bounded if it exists. The operator $R_\lambda = (\lambda - T)^{-1}$ is called the resolvent of T. The complement of $\rho(T)$ in $\mathbb{C}$ is called the spectrum of T, notation $\sigma(T)$.

Theorem 2.2 *Since*

$$\lambda, \mu \in \rho(T) \;\Rightarrow\; R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu, \tag{2.1}$$

R_λ and R_μ commute. In view of

$$|\lambda - \mu| < \frac{1}{||R_\mu||} \;\Rightarrow\; R_\lambda = \sum_{n=0}^{\infty} (\mu - \lambda)^n R_\mu^{n+1},$$

$\rho(T)$ is open and the resolvent $\lambda \to R_\lambda$ is analytic on $\rho(T)$. Finally,

$$|\lambda| > ||T|| \;\Rightarrow\; \lambda \in \rho(T), \quad R_\lambda = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1} \;\text{ and }\; ||R_\lambda|| \leq \frac{1}{|\lambda| - ||T||}.$$

so the spectrum $\sigma(T)$ is compact.

This theorem is easily proved manipulating the "geometric series"

$$(I - T)^{-1} = I + T + T^2 + T^3 + \cdots \quad (||T|| < 1). \tag{2.2}$$

We recall that a function $F : \Omega \to X$ is called analytic if

$$F'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0}$$

exists for every $\lambda_0 \in \Omega$. This is equivalent to $\lambda \to f(F(\lambda)) \in \mathbb{C}$ being analytic on Ω for every $f \in X^*$. A function $F : \Omega \to X$ which is analytic has the same nice properties as an ordinary $\mathbb{C}$-valued analytic function: Coursat's theorem, Cauchy formula's for F and its derivatives, local powerseries representation, maximum modulus theorem, Liouville's theorem, etc. Liouville gives:

Theorem 2.3 $\sigma(T) \neq \emptyset$.

Proof. Otherwise $\rho(T) = \mathbb{C}$. By Theorem 2.2 the resolvent R_λ goes to zero at infinity. For any f in $B(X)^*$, the dual of the Banach space of bounded operators, $f(R_\lambda) \equiv 0$ by an application Liouville's theorem. Thus $R_\lambda \equiv 0 \in B(X)$, a contradiction.

Theorem 2.4 $\sigma(p(T)) = p(\sigma(T))$ *for every polynomial p.*

Proof. If $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$, then, for fixed μ in $\mathbb{C}$,

$$p(\lambda) - \mu = (\lambda - \beta_1) \cdots (\lambda - \beta_n),$$

which also holds if λ is replaced by T. Thus $p(T) - \mu I$ is a bijection if and only if all $T - \beta_j I$ are bijections. Since $p(\beta_j) = \mu$ the theorem follows.

Theorem 2.5 *The representation of R_λ as a power series in $\frac{1}{\lambda}$ in Theorem 2.2 is valid for all $|\lambda| > r(T)$, where*

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}} \tag{2.3}$$

is the spectral radius of T.

Proof. By the formula for the radius of convergence (2.3) holds with lim sup. Theorem 2.4 implies $||T^n|| \geq r(T^n) = r(T)^n$, whence $r(T) \leq \liminf_{n\to\infty} ||T^n||^{\frac{1}{n}}$ and (2.3) follows.

3 Banach spaces: compact operators

Definition 3.1 A linear operator $T : X \to X$ is called compact if the closure of the image of the unit ball is compact.

Proposition 3.2 *The compact linear operators on X form a closed two-sided ideal $K(X)$ in the Banach algebra $B(X)$ of bounded linear operators.*

Theorem 3.3 *If T is a compact bounded operator then $\sigma(T)$ is a countable set, including $\lambda = 0$, having no accumulation points except possibly $\lambda = 0$. Every nonzero $\lambda \in \sigma(T)$ is an eigenvalue with a finite dimensional space of generalized eigenvectors.*

We need some preparations for the proof.

Lemma 3.4 *Let $T \in K(X)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Then $N(T-\lambda)^n$ is finite dimensional.*

Proof. Let $n = 1$. It is no restriction to assume $\lambda = 1$. If $N(T - I)$ has infinite dimension, its unit ball is not compact (this easily follows from Riesz' Lemma below), so there exists a sequence $x_n = Tx_n \in N(T - I)$, $||x_n|| \leq 1$, having no convergent subsequence, contradicting $T \in K(X)$. For $n > 1$ we write $(T - I)^n = T^n - nT^{n-1} + \cdots - I = S - I$, then S is compact (why?) and the conclusion follows from the case $n = 1$.

Lemma 3.5 *Let $T \in K(X)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. If M is a closed subspace of X such that $M \cap N(T-\lambda) = \{0\}$, then the restriction $T - \lambda : M \to (T - \lambda)(M)$ has a bounded inverse and $(T - \lambda)(M)$ is closed.*

Proof. Again it is no restriction to assume $\lambda = 1$. Clearly the inverse exists. Suppose it is not bounded, then there exists a sequence $x_n \in M$, $||x_n|| = 1$, such that $||Tx_n - x_n|| \to 0$. But T being compact we may extract a subsequence, again denoted by x_n, such that Tx_n converges to a limit y. Then also $x_n \to y$, $y \in M$ because M is closed, $||y|| = 1$ and $(T - I)y = \lim(Tx_n - x_n) = 0$, contradicting $M \cap N(T - I) = \{0\}$. Thus $T - I : M \to (T - I)(M)$ has a bounded inverse. Hence it is a linear homeomorphism. Since M is closed it is Banach, so $(T - I)(M)$ is Banach and therefore closed in X.

Corollary 3.6 *Let $T \in K(X)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Then $R((T-\lambda)^n)$ is closed.*

Proof. Take $n = 1$ and $\lambda = 1$. Since $N(T-I)$ is finite dimensional there is a closed subspace M such that X is the direct product of $N(T-I)$ and M. Thus $M \cap N(T-I) = \{0\}$ so by Lemma 3.5, $R(T-I)$ is closed. The case $n > 1$ follows as above writing $T^n - I = S - I$ with $S \in K(X)$.

In the next definition and theorem we only need the vector space structure.

Definition 3.7 Let $T : X \to X$ be linear. Observing that always $N(T^n) \subset N(T^{n+1})$ and $R(T^n) \supset R(T^{n+1})$, the ascent $\alpha(T)$ is the smallest integer n such that $N(T^n) = N(T^{n+1})$ (if no such n exists $\alpha(T) = \infty$) and the descent $\delta(T)$ is the smallest integer n such that $R(T^n) = R(T^{n+1})$ (if no such n exists $\delta(T) = \infty$).

Theorem 3.8 *Let $T : X \to X$ be linear. If $\alpha(T)$ and $\delta(T)$ are finite then $\alpha(T) = \delta(T) = p$ and X is the direct vector space product of $R(T^p)$ and $N(T^p)$, both of which are invariant under T.*

Proof. The proof follows from two observations. First, consider, for any positive integers i, j,

$$T^i : N(T^{i+j}) \to R(T^i) \cap N(T^j).$$

This map is surjective and has kernel $N(T^i)$, so

$$N(T^{i+j})/N(T^i) \cong R(T^i) \cap N(T^j). \tag{3.1}$$

Second, if

$$Q : R(T^j) \to R(T^j)/R(T^{i+j})$$

is the quotient map, then

$$QT^j : X \to R(T^j)/R(T^{i+j})$$

is surjective and $N(QT^j) = R(T^i) + N(T^j)$, so

$$R(T^j)/R(T^{i+j}) \cong X/(R(T^i) + N(T^j)). \tag{3.2}$$

If $i \geq \alpha(T)$ the spaces in (3.1) are trivial and the same holds for (3.2) if $j \geq \delta(T)$. Consequently

$$X = R(T^i) \oplus N(T^j) \quad \text{if} \quad i \geq \alpha(T) \quad \text{and} \quad j \geq \delta(T).$$

It follows that $\alpha(T) = \delta(T)$ (why?) and the proof is complete.

Theorem 3.9 *Let $T \in K(X)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Then $\alpha(T-\lambda)$ and $\delta(T-\lambda)$ are finite. Thus Lemma 3.8 applies with T replaced by $T-\lambda$, and both factors of the product are closed and invariant under T:*

$$X = R((T-\lambda)^p) \oplus N((T-\lambda)^p), \quad p = \alpha(T-\lambda) = \delta(T-\lambda). \tag{3.3}$$

Proof. Restricting again to $\lambda = 1$ we only have to show that $\alpha(T-I)$ and $\delta(T-I)$ are finite. Hereto we need

Lemma 3.10 *(Riesz' Lemma) Let X be a normed space and $M \subset X$, $M \neq X$, a closed subspace. For every $\theta \in (0,1)$ there exists $x_\theta \in X$ with $||x_\theta|| = 1$ such that $d(x_\theta, M) = \inf\{||x - x_\theta|| : x \in M\} > \theta$.*

Now suppose $\alpha(T-I)$ is not finite, then, for every positive integer n, using Riesz, we may choose $x_n \in N((T-I)^n)$ with $||x_n|| = 1$ and such that $d(x_n, N((T-I)^{n-1})) \geq \theta$. Then $Tx_n - x_n \in N((T-I)^{n-1})$, so, with $1 \leq m < n$,

$$||Tx_n - Tx_m|| = ||x_n - x_m + (x_m - Tx_m) + (Tx_n - x_n)|| \geq \theta.$$

Thus Tx_n cannot have a convergent subsequence, contradicting $T \in K(X)$. Hence $\alpha(T-I)$ is finite.

Exercise 9 Prove that also $\delta(T-I)$ is finite.

Proof of Theorem 3.3. Let $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Let $p = \alpha(T-\lambda) = \delta(T-\lambda)$. If $p = 0$ then $N(T-\lambda) = N(I) = \{0\}$ and $R(T-\lambda) = R(I) = X$. Thus $T - \lambda : X \to X$ is bijective and $\lambda \in \rho(T)$.

If $p > 0$, then

$$X = N((T-\lambda)^p) \oplus R((T-\lambda)^p) = X_1 \oplus X_2,$$

with X_1 finite dimensional. Clearly X_1 and X_2 are invariant under T. Denoting the restriction of T to X_i by T_i, we have $\sigma(T_1) = \{\lambda\}$ (why?). Moreover, $T_2 - \lambda : X_2 \to X_2$ is bijective. This implies that λ belongs to $\rho(T_2)$, and hence the same holds for a neighbourhood $\mathcal{N}$ of λ. Finally, for every $\mu \neq \lambda$, $T_1 - \mu : X_1 \to X_1$ is a bijection, so for $\mu \in \mathcal{N}\backslash\{\lambda\}$ both $T_1 - \mu$ and $T_2 - \mu$ are bijective and hence so is $T - \mu$, i.e. $\mu \in \rho(T)$.

This completes the proof of the spectral theorem for compact linear operators. We note that the essential result is really (3.3). In Sections 5 and 6 we develop machinery which allows to decide whether or not (3.3) holds for $\lambda \in \sigma(T)$ when $T \in B(X)$ is not assumed to be compact.

Exercise 10 Let $T \in K(X)$ and consider, for given $y \in X$, the equation $x - Tx = y$. Deduce the Fredholm alternative: either there exists a unique solution x for all y, or the homogeneous equation $x - Tx = 0$ has nontrival solutions.

Exercise 11 In the context of Theorem 3.8, show that $N(T^p)$ and $R(T^p)$ are closed if X is a Banach space and T is bounded. Hint: take $p = 1$ and consider $\tilde{T} : X \to X/N(T)$ defined by $\tilde{T}x = [Tx] = Tx + N(T)$ and show that the Open Mapping Theorem applies to $\tilde{T}$.

4 Quasi-nilpotent or not quasi-nilpotent?

It may happen that $T \in K(X)$ is quasi-nilpotent, i.e.

$$r(T) = \lim_{n\to\infty} ||T^n|| = 0.$$

In that case there is no information from Theorem 3.3. This raises two questions.

(i) can we still say something in the quasi-nilpotent case?

(ii) how can we conclude that $T \in K(X)$ is not quasi-nilpotent?

We state two theorems in this context.

Theorem 4.1 *(Lomonosov) If $0 \neq T \in K(X)$ is quasi-nilpotent then T has a nontrivial closed invariant subspace.*

Theorem 4.2 *(de Pagter, [2], Thm 4.2.2) If X is a Banach lattice and $0 \neq T \in K(X)$ is quasi-nilpotent and positive then T has a nontrivial closed invariant order ideal.*

To explain the last theorem we first restrict to the case of real spaces. A real Banach lattice X is a Banach space which is also a vector lattice. A vector lattice or Riesz space is a real vector space endowed with a partial order $\geq$, such that, in addition to the partial order axioms,

$$x \geq x; \quad x \geq y \text{ and } y \geq x \Rightarrow x = y; \qquad x \geq y \text{ and } y \geq z \Rightarrow x \geq z,$$

for all $x, y, z \in X$, also

$$x \geq y \Rightarrow x + z \geq y + z \text{ and } \alpha x \geq \alpha y,$$

for all $x, y, z \in X$ and $0 < \alpha \in \mathbb{R}$, and, finally, for every pair $x, y \in X$ there exist a lowest upper bound and a largest lower bound of x and y. The positive cone in X is $X^+ = \{x \in X : x \geq 0\}$. $T \in B(X)$ is called positive, notation $T \geq 0$, if $T(X^+) \subset X^+$. An order ideal in X is a subspace I with the additional property that

$$x \in I,\ y \in X,\ |x| \geq |y| \quad \Rightarrow \quad y \in I.$$

Here $|x|$ is the lowest upper bound of x and $-x$.

Theorem 4.2 is really a result for real Banach lattices. It can be used if the operator has a stronger positivity property which we will not formulate in the abstract setting. In applications to elliptic boundary value problems one typically has results of the form

$$-Lu = f \geq 0 \quad \Rightarrow \quad u \geq 0,$$

and $u > 0$ (everywhere or almost everywhere, depending on the choice of function spaces, see [1], chapter 5, for classical maximum principles). This prevents the (usually compact) solution operator $T : f \to u$ from having a nontrivial closed invariant order ideal. We note that in $X = C([0,1])$, or, more generally, in $X = C(Y)$, where Y is a compact normal topological space, the closed ideals are of the form $\{f \in X : \ f \equiv 0 \text{ on } M\}$, where M is closed in Y. A similar characterisation holds in L^p-spaces in the a.e. sense with M measurable, provided $1 \leq p < \infty$. The L^p setting is discussed in [3].

The complex version follows by complexification and puts us back into our framework for the analysis of $\sigma(T)$. In particular we have

Theorem 4.3 *(Krein-Rutman, [2], Thm 4.1.4) If X is a Banach lattice and $T \in K(X)$ is positive and has $r(T) > 0$, then $r(T)$ is an eigenvalue corresponding to an eigenvector in X^+. Moreover, $r(T)$ is a pole of the resolvent of maximal order on $\{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$.*

Note that the assumption $r(T) > 0$ is essential (it is not stated in [2]).

Exercise 12 Consider $X = C([0,1])$ and the operators K and G defined by $(Kf)(x) = \int_0^x f(y)\,dy$ and $(Gf)(x) = \int_0^1 g(x,y)f(y)\,dy$, where $g(x,y)$ is the Green's function for the problem

$$-u''(x) = f(x) \quad -1 \leq x \leq 1, \quad u(-1) = u(1) = 0.$$

Compute g and evaluate the theorems above for K and G.

Exercise 13 (Hilden's proof of Thm 4.1) Let $\mathcal{A} = \{A \in B(X) : \ AK = KA\}$.
(i) Show that $\mathcal{A}y = \{Ay : \ A \in \mathcal{A}\}$ is invariant under $\mathcal{A}$ and thus also under K for each $y \in H$.

Arguing by contradiction assume that $\overline{\mathcal{A}y} = X$ for all $0 \neq y \in H$.

(ii) Show that for any $0 \neq x_0 \in X$ with $Kx_0 \neq 0$ there exists an open ball B centered at x_0 with $0 \notin \bar{B}$ and $0 \notin \overline{KB}$.

(iii) Show that for every $y \in KB$ there exists an $A \in \mathcal{A}$ and an open neighbourhood U of y such that $AU \subset B$.

(iv) Use compactness to conclude that finitely many $U_1, \ldots, U_n$ cover $\overline{KB}$.

Let $A_1, \ldots, A_n$ be the corresponding elements of $\mathcal{A}$.

(v) (ping-pong) Jump back and forth between B and KB to construct a sequence

$$x_n = A_{i_n} K \cdots A_{i_0} K x_0$$

in B and derive a contradiction using $||K^n|| \to 0$.

5 Functional calculus

We recall that X is always a complex Banach space, $X \neq \{0\}$, $T : X \to X$ a bounded linear operator and Ω an open subset of $\mathbb{C}$.

The formula (2.2) is a first example of $f(T)$ making sense for a fixed complex analytic function f and a bounded operator T. Clearly, if

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad \lambda \in \mathbb{C},\ |\lambda| < R,$$

then we may define

$$f(T) = \sum_{n=0}^{\infty} a_n T^n, \quad T \in B(X),\ ||T|| < R.$$

The following theorem defines $f(T)$ for fixed T and f analytic on a neighbourhood of $\sigma(T)$.

Theorem 5.1 *Let X be a Banach space, $T \in B(X)$, $\sigma(T) \subset \Omega$, $f : \Omega \to \mathbb{C}$ analytic and γ a contour in $\Omega \setminus \sigma(T)$ winding once (counterclockwise) around $\sigma(T)$ (and containing no holes of Ω). Define*

$$f(T) = \frac{1}{2\pi i} \oint_\gamma f(\lambda) R_\lambda \, d\lambda = \frac{1}{2\pi i} \oint_\gamma f(\lambda)(\lambda - T)^{-1} \, d\lambda.$$

Then $f(T)$ is a bounded linear operator on X and $\sigma(f(T)) = f(\sigma(T))$. The definition is independent of the particular choice of γ and thus only depends on the values of f in a neighbourhood of $\sigma(T)$. If $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ has radius of convergence larger than $r(T)$, then $f(T) = \sum_{n=0}^{\infty} a_n T^n$. Any bounded linear operator on X which commutes with T also commutes with $f(T)$. Moreover, if $g : \Omega \to \mathbb{C}$ is another analytic function, then

$$f(T) + g(T) = (f+g)(T) \quad \text{and} \quad f(T)g(T) = (fg)(T).$$

If $fg \equiv 1$ in a neighbourhood of $\sigma(T)$ in Ω then $g(T) = f(T)^{-1}$. Finally, the choice $f(\lambda) = \lambda^n$ $(n = 0, 1, 2, \dots)$ gives

$$T^n = \frac{1}{2\pi i} \oint_\gamma \lambda^n R_\lambda \, d\lambda = \frac{1}{2\pi i} \oint_\gamma \lambda^n (\lambda - T)^{-1} \, d\lambda.$$

Proof. We will prove the product formula for $f(T)$ and $g(T)$. Choose contours γ for f and Γ for g such that $\gamma = \delta D_\gamma$, $\Gamma = \delta D_\Gamma$, $\overline{D}_\gamma \subset D_\Gamma \subset \overline{D}_\Gamma \subset \Omega$. Then

$$f(T)g(T) = \frac{1}{2\pi i} \oint_\gamma f(\lambda) R_\lambda \, d\lambda \, \frac{1}{2\pi i} \oint_\Gamma g(\mu) R_\mu \, d\mu \ =$$

$$\frac{1}{2\pi i} \oint_\gamma f(\lambda) \frac{1}{2\pi i} \oint_\Gamma g(\mu) R_\lambda R_\mu \, d\mu d\lambda \ =$$

(using the resolvent formula (2.1) and exchanging the order of integration in the second term below))

$$\frac{1}{2\pi i} \oint_\gamma f(\lambda) \frac{1}{2\pi i} \oint_\Gamma \frac{g(\mu)}{\mu - \lambda} \, d\mu \, R_\lambda \, d\lambda \ + \frac{1}{2\pi i} \oint_\Gamma \frac{1}{2\pi i} \oint_\gamma \frac{f(\lambda)}{\mu - \lambda} \, d\lambda \, g(\mu) R_\mu \, d\mu \ =$$

$$\frac{1}{2\pi i}\oint_{\gamma} f(\lambda)g(\lambda)R_{\lambda}\,d\lambda + 0 = (fg)(T).$$

The rest of the theorem is left as an exercise. Note that $f(T)$ is invertible if and only if $f \neq 0$ on $\sigma(T)$. Since $f(T) - \mu I = (f - \mu)(T)$ this implies $\sigma(f(T)) = f(\sigma(T))$.

Thus $f(T)$ is defined for every $\mathbb{C}$-valued function f defined and analytic in a neighbourhood of $\sigma(T)$. See [5] for a generalisation to unbounded closed operators with $\sigma(T) \neq \mathbb{C}$ and f analytic in a neighbourhood of $\sigma(T) \cup \{\infty\}$ having a well defined limit $f(\infty)$.

Exercise 14 Suppose that in the context of Theorem 5.1 the $\mathbb{C}$-valued function is analytic on a neighbourhood of $f(\sigma(T)) = \sigma(f(T))$. Show that $g(f(T)) = (g \circ f)(T)$.

6 Spectral projections; poles of the resolvent

We recall that X is always a complex Banach space, $X \neq \{0\}$, $T : X \to X$ a bounded linear operator. Using Theorem 5.1 we wish to establish when (3.3) holds for $\lambda \in \sigma(T)$.

Suppose first that $\sigma(T)$ is disconnected. Then

$$\sigma(T) = \sigma_1 \cup \sigma_2 \quad \sigma_1 \cap \sigma_2 = \emptyset,$$

with σ_i nonempty and compact. Thus there exist disjoint open sets Ω_1, Ω_2 such that $\sigma_i \subset \Omega_i$ and we may take contours γ_i in Ω_i around σ_i to apply Theorem 5.1 with $\gamma = \gamma_1 \cup \gamma_2$:

$$f(T) = \frac{1}{2\pi i}\oint_{\gamma_1} f(\lambda)R_{\lambda}d\lambda + \frac{1}{2\pi i}\oint_{\gamma_2} f(\lambda)R_{\lambda}d\lambda.$$

If we choose χ_i to be the characteristic function of Ω_i and define $f_i(\lambda) = \lambda\chi_i(\lambda)$, we obtain, using the algebraic properties of the mapping $f \to f(T)$, a splitting

$$I = \chi_1(T) + \chi_2(T) = E_1 + E_2, \quad T = f_1(T) + f_2(T) = T_1 + T_2,$$

where

$$E_i = \chi_i(T), \quad T_i = f_i(T) = TE_i = E_iT, \quad E_1E_2 = E_2E_1 = 0, \quad E_i^2 = E_i,$$

and

$$N(E_1) = R(E_2), \quad N(E_2) = R(E_1), \quad X = R(E_1) \oplus R(E_2).$$

The projections E_1 and E_2 are called spectral projections and the sets σ_1 and σ_2 are called spectral sets.

It may happen that $\sigma_1 = \{\mu\}$ is a singleton (if the reader wishes he may set $\mu = 0$ in the reasoning below). The resolvent R_λ is then analytic in a punctured neighbourhood of μ and we may write its Laurent series as

$$R_{\lambda} = (\lambda - T)^{-1} = \sum_{n=-\infty}^{\infty} (\lambda - \mu)^n A_n, \quad A_n \in B(X). \tag{6.1}$$

If this series has only finitely many nonzero terms with n negative we say that R_λ has a pole in $\lambda = \mu$ of order p where A_{-p} is the first nonzero A_n. If this is not the case, R_λ has an essential singularity in $\lambda = \mu$. In what follows, whenever we say that R_λ has a pole in μ, it is implicitly understood that μ is an isolated point of $\sigma(T)$.

Theorem 6.1 *Let μ be an isolated point of $\sigma(T)$. Then R_λ is given by (6.1) in a punctured neighbourhood of μ and the coefficients satisfy*

$$n \neq 0 \;\Rightarrow\; (T-\mu)A_n = A_{n-1},$$

$$(T-\mu)A_0 = A_{-1} - I,$$

and A_{-1} is the spectral projection corresponding to $\{\mu\}$. In particular

$$n < 0 \;\Rightarrow\; A_n = (T-\mu)^{1-n}A_{-1}.$$

Proof. We retrieve the coefficients A_n from R_λ by

$$A_n = \frac{1}{2\pi i}\oint_{\gamma_1}(\lambda-\mu)^{-n-1}R_\lambda\,d\lambda.$$

Using Theorem 5.1 and the notation above we rewrite A_n as

$$A_n = F_n(T), \tag{6.2}$$

where

$$F_n(\lambda) = (\lambda-\mu)^{-n-1}\chi_1(\lambda), \quad n \leq -1, \tag{6.3}$$

and

$$F_n(\lambda) = -(\lambda-\mu)^{-n-1}\chi_2(\lambda), \quad n \geq 0. \tag{6.4}$$

To see this for $n \geq 0$ observe that

$$A_n - F_n(T) = \frac{1}{2\pi i}\oint_{\gamma_1}(\lambda-\mu)^{-n-1}R_\lambda\,d\lambda + \frac{1}{2\pi i}\oint_{\gamma_2}(\lambda-\mu)^{-n-1}R_\lambda\,d\lambda =$$

(for $r > r(T)$)

$$= \frac{1}{2\pi i}\oint_{|\lambda|=r}(\lambda-\mu)^{-n-1}R_\lambda\,d\lambda \to 0 \;\text{ as }\; r \to \infty.$$

We now have, using the algebra again,

$$n \neq 0 \;\Rightarrow\; F_{n-1}(\lambda) = (\lambda-\mu)F_n(\lambda) \;\Rightarrow\; A_{n-1} = (T-\mu)A_n, \tag{6.5}$$

$$F_{-1}(\lambda) = \chi_1(\lambda) \;\Rightarrow\; A_{-1} = E_1, \tag{6.6}$$

and

$$(\lambda-\mu)F_0(\lambda) = -\chi_2(\lambda) \;\Rightarrow\; (T-\mu)A_0 = -E_2 = E_1 - I = A_{-1} - I. \tag{6.7}$$

Proposition 6.2 *If, in Theorem 6.1, the range of the spectral projection A_{-1} is finite then μ is a pole of the resolvent (of finite rank).*

Theorem 6.3 *If $R_\lambda = (T-\lambda)^{-1}$ has a pole of order $p \geq 1$ in $\lambda = \mu$, then μ is an eigenvalue and*

$$X = R((T-\mu)^p) \oplus N((T-\mu)^p) \quad \text{and} \quad p = \alpha(T-\mu) = \delta(T-\mu). \tag{6.8}$$

Exercise 15 Prove Theorem 6.3.

Theorem 6.4 *If $\mu \in \sigma(T)$ is such that $\alpha(T-\mu)$ and $\delta(T-\mu)$ are both finite, then μ is a pole of R_λ.*

Exercise 16 Prove Theorem 6.4. Use Theorem 3.8 to conclude that (6.8) holds and show that both factors are closed. Then reason as in the proof of Theorem 3.3 to show that μ is an isolated point of $\sigma(T)$.

Exercise 17 Show that μ is a pole of finite rank of R_λ if $T \in K(X)$ and $0 \neq \mu \in \sigma(T)$.

Exercise 18 Show that μ is a pole of finite rank of R_λ if $T \in B(X)$, $0 \neq \mu \in \sigma(T)$ and $T^n \in K(X)$ for some positive integer.

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