

‘I like functions of one variable’

Xavier Cabré addressing Abel prize winner Louis Nirenberg and a small analysis group at Tor Vergata in June 2015.

1 Introduction

These are lecture notes for a first course in mathematical analysis and what can follow later. In the days of corona I made videos for most of the topics covered. The video <https://youtu.be/taKW6cNomJk> should get you started. Click MEER WEERGEVEN for an overview of the video playlists that I listed on the front page of these notes. To go the next video click

go to https://youtu.be/4vj5LX_okSA next

and likewise in every second line of the text under every next video. The link in every first line takes you back to the previous video.

Read the text with every video before you watch it, read the text with every playlist before you play it. I will still be editing them. Every video has a link to the playlist it belongs to. First course topics covered are

1. Cauchy sequences, convergence, limits;
2. Completeness of the real numbers; theorem of Bolzano-Weierstrass;
3. Continuity and uniform continuity;
4. The concept of differentiability;
(including differentiability of power series);
5. The concept of Riemann integrability (including Riemann integrability of monotone and uniformly continuous functions);
6. The language of metric topology;
7. Completeness of the space $C([a, b])$, uniform convergence;
8. The Banach Fixed Point Theorem (with applications to integral and differential equations, and after all not the implicit function theorem).

Some of these terms may mean nothing to you yet. This introduction is meant to give you a flavour of how and what we do in analysis, with some historical perspective. We introduce some of the notation along the way, as well as a few basic principles. Some familiarity with what once was *highschool calculus* is assumed: limits, continuity, differentiability and integration, in the context of real valued functions $f(x)$ of a real variable x . In particular you must have seen the *integration formula*

$$\int_a^b f(x) dx = F(b) - F(a),$$

in which F is a primitive function of f , meaning that the derivative of $F(x)$ is given by $F'(x) = f(x)$.

Perhaps you have also seen Newton's method, the *scheme*

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = F(x_{n-1}) \quad (1.1)$$

for solving the equation $f(x) = 0$ numerically. Starting with some x_0 and $n = 1$ this scheme produces a sequence $x_1, x_2, \dots$, which typically converges to a solution of $f(x) = 0$ very fast, see Chapter [12.2](#)

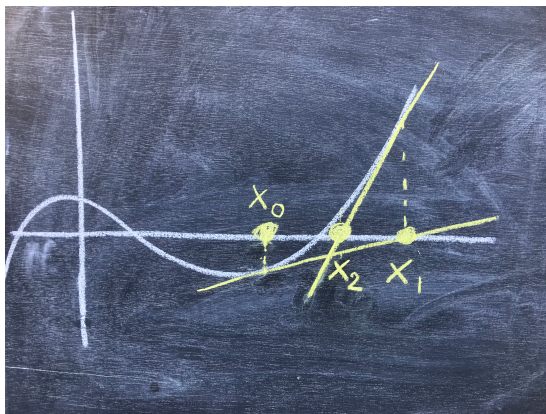


Figure 1: Newton's iterative method for solving $f(x) = 0$ pictured with the graph of f . To find the next iterate intersect the line tangent to the graph in the previous iterate with the horizontal axis.

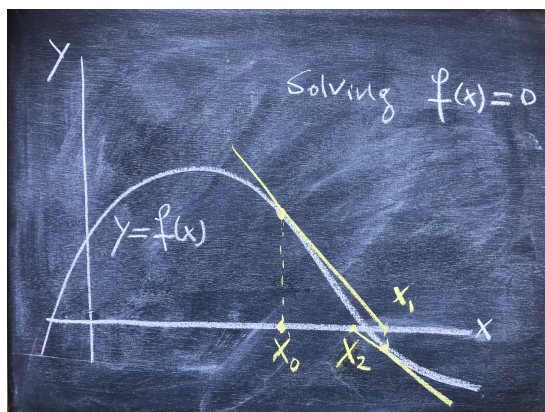
Exercise 1.1. Consider the graph defined by $y = f(x)$. Use your highschool maths to write down a formula for the line tangent to the graph of f in the point $(x, y) = (x_{n-1}, f(x_{n-1}))$. Intersect this line with the x -axis and denote the x -value in the intersection point by x_n . Show that it is given by [\(1.1\)](#).

Hint: make a picture first, for instance if f is given by $f(x) = x^2 - 2$.

Exercise 1.2. Show for $f(x) = x^2 - 2$ that [\(1.1\)](#) reduces to

$$x_n = F(x_{n-1}) := \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}} \quad (1.2)$$

and experiment, with $x_0 = 1$ as starting value for instance.



1.1 The square root of two

Have a look at <https://youtu.be/DyG1B3WYh-k>.

The example in Exercises [1.1](#)[1.2](#) takes us way back to Babylonian times, and the origins of differential calculus. It concerns $\sqrt{2}$, a *geometric number* which appears as the length of the diagonal in the unit square. The first recorded attempt¹ to compute the positive number r defined by $r^2 = 2$ can be found on the *Babylonian clay tablet* YBC7289. Dating back around 37 centuries, it contains the picture of a square with its diagonals, and several number sequences written in cuneiform.

In decimal notation one of these number sequences is

$$1 \quad 24 \quad 51 \quad 10$$

and stands for²

$$1 + \frac{24}{60} + \frac{51}{3600} + \frac{10}{216000} = 1.41421\underline{296},$$

which is a remarkably good *hexagesimal approximation* of

$$\sqrt{2} = 1.4142135\dots,$$

the irrational square root of 2.

In our notation this approximation is believed to have resulted from rather clever calculations employing the approximation

$$\sqrt{1+x} \approx 1 + \frac{x}{2}$$

¹That I know of.

²The repeating part of the decimal expansion is underlined.

for small x . The clarifying formula would be that

$$\sqrt{2} \approx \frac{577}{408} \approx 1 + \frac{24}{60} + \frac{51}{3600} + \frac{10}{216000},$$

in which the Babylonian approximation is a truncated hexigesimal expansion for $\frac{577}{408}$. This works as follows.

Let $r > 0$ be a possibly not so very good approximation of $\sqrt{2}$. Then

$$\sqrt{2} = \sqrt{r^2 + 2 - r^2} = r \sqrt{1 + \frac{2 - r^2}{r^2}} \approx r \left(1 + \frac{1}{2} \frac{2 - r^2}{r^2} \right) = \frac{r}{2} + \frac{1}{r},$$

which is possibly a better approximation of $\sqrt{2}$. You should recognise the example of Newton's method in Exercises [1.1](#)[1.2](#). Starting with the bad approximation $r = 1$ the new approximation of $\sqrt{2}$ is $\frac{3}{2}$, which is not that bad really. Redoing the approximation with $r = \frac{3}{2}$ gives $\frac{17}{12}$, much better, and $r = \frac{17}{12}$ in turn gives

$$\frac{17}{24} + \frac{12}{17} = \frac{289 + 288}{24 \times 17} = \frac{577}{408} \approx 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3},$$

the approximation on YBC7289, which is where the Babylonians apparently stopped.

This method for approximating $\sqrt{2}$ is also known as *Heron's method*. In this course we will take these methods to the limit. In Chapter 2 we will give a proper formulation and proof of the statement that the sequence x_n defined in Exercise [1.2](#) has the property that

$$x_n \rightarrow \sqrt{2} \quad \text{as } n \rightarrow \infty, \tag{1.3}$$

to be pronounced as “ x_n goes to $\sqrt{2}$ as n goes to infinity”. We shall show that it does so extremely fast.

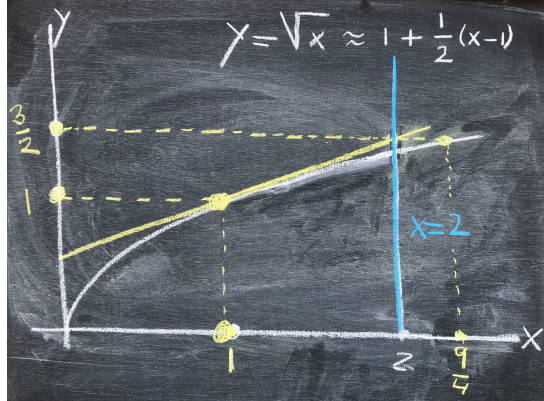


Figure 2: The first step in the Babylonian scheme. The line tangent to $y = \sqrt{x}$ in $(1, 1)$ intersects $x = 2$ in $y = \frac{3}{2}$. To find the next iterate intersect the line tangent in $(\frac{9}{4}, \frac{3}{2})$ with $x = 2$. And so on.

1.2 One third of what?

https://www.youtube.com/playlist?list=PLQgy2W8pIli91L_Yp_4Tr_CG0grGUBdZa

The playlist linked to in the previous line starts with <https://youtu.be/taKW6cNomJk>, and concerns another *geometric number*: the number $\frac{1}{3}$ that appears as the *volume* V of a *pyramid* with unit base area and unit height. To see how $\frac{1}{3}$ appears we divide the pyramid into say 6 horizontal layers of height $\frac{1}{6}$ and write n for 6. The maximal width of each layer varies from 1 at the bottom to $\frac{1}{6} = \frac{1}{n}$ at the top.

Exercise 1.3. Draw a picture and convince yourself that from top to bottom these maximal widths are

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, 1.$$

Thus the total volume V of the “unit” pyramid is certainly less than

$$\frac{1}{n} \left(\frac{1}{n^2} + \frac{4}{n^2} + \frac{9}{n^2} + \dots + 1 \right) = \frac{1}{n^3} \sum_{k=1}^n k^2.$$

Likewise the minimal widths of the layers are

$$\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-1}{n},$$

so V is larger than

$$\frac{1}{n^3} \sum_{k=0}^{n-1} k^2.$$

Combining the two bounds we have³

$$\underline{S}_n := \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 < V < \frac{1}{n^3} \sum_{k=1}^n k^2 =: \bar{S}_n, \quad \text{while} \quad \bar{S}_n - \underline{S}_n = \frac{n^2}{n^3} - \frac{0}{n^3} = \frac{1}{n},$$

in which we don't really have to exhaust ourselves to understand that this is also true for values of n different from 10 as large as we like.

How many numbers V can satisfy this inequality for all n ? At most one according to Archimedes. Because for two such numbers, say $V < W$, we would have

$$0 < W - V < \bar{S}_n - \underline{S}_n = \frac{1}{n} \quad \text{for all } n \in \mathbb{N}. \quad (1.4)$$

Archimedes took it for granted⁴ that therefore the difference of V and W must be zero, and who are we to dispute? As a consequence of what we now call the *Archimedean Principle* there is indeed at most one number that qualifies as the volume of the pyramid.

By the way, Archimedes also knew the identity

$$(C_n) \quad \sum_{k=1}^n k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6},$$

so the inequalities become

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} < V < \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

and we see that $V = \frac{1}{3}$ fits. If we agree that the unit pyramid has a volume, then its volume must be $\frac{1}{3}$ because it is the *only* value that fits⁵. It's quite amusing that we actually found this value as the coefficient of n^3 in (C_n) .

In modern language we say that V is the *integral*

$$\int_0^1 (1-z)^2 dz = \frac{1}{3},$$

³This is the first time we use the symbol $<$ but you know what it means.

⁴https://youtu.be/4vj5LX_okSA

⁵There is no obvious way to think of this volume as one third of the unit cube!

in which $(1 - z)^2$ is the area of the intersection of the pyramid with a horizontal plane at height z . The integration variable z ranges from $z = 0$ at the bottom to $z = 1$ at the top of the pyramid.

Having guessed (C_n) one way or another you can *prove it by induction*: starting with $n = 1$ and (C_1) being a statement that is trivially true, the implication

$$(C_n) \implies (C_{n+1})$$

is easy to verify. Indeed, using (C_n) we have that

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^n k^2 + (n+1)^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + (n+1)^2,$$

which happens to be equal to

$$\frac{(n+1)^3}{3} + \frac{(n+1)^2}{2} + \frac{n+1}{6}.$$

So (C_{n+1}) holds if (C_n) holds. This is called the *induction step*, which here is valid for every $n \geq 1$. Verifying (C_1) via

$$\sum_{k=1}^1 k^2 = 1^2 = 1 = \frac{1^3}{3} + \frac{1^2}{2} + \frac{1}{6}$$

we then conclude that for every *natural number* n the identity (C_n) holds because

$$C_1 \implies C_2 \implies C_3 \implies C_4 \implies \dots$$

This trick to prove (C_n) for all positive integers n is also called proof by induction, or *domino principle*. Think of the n^{th} statement (C_n) as being written on the n^{th} domino. Put all dominos in a never ending queue. Kick the first domino ($n = 1$) over and watch. The statements still to be checked are on the dominos still standing.

You may have noted that⁶

$$\int_0^1 (1 - z)^2 dz = \int_0^1 x^2 dx.$$

This integral belongs to a family

$$J_1 = \int_0^1 x dx = \frac{1}{2}, \quad J_2 = \int_0^1 x^2 dx = \frac{1}{3}, \quad J_3 = \int_0^1 x^3 dx = \frac{1}{4}, \dots,$$

⁶Via the substitution $z = 1 - x$.

expressions that you must have seen before for the area J_p of the set

$$A_p = \{(x, y) : 0 \leq y \leq x^p \leq 1\}$$

in the xy -plane.

Archimedean type expressions for *sums of powers* can be used to show directly that the sequence $J_1, J_2, J_3, \dots$ continues as suggested. Unfortunately the sum formulas for exponents p larger than 3 become a bit cumbersome.

The inequalities⁷

$$\sum_{k=0}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^n k^p$$

do a quicker job. They hold for all positive integers p, n and dividing by n^{p+1} it follows that

$$\frac{1}{n^{p+1}} \sum_{k=0}^{n-1} k^p < \frac{1}{p+1} < \frac{1}{n^{p+1}} \sum_{k=1}^n k^p$$

for lower and upper approximations of J_p . Since these approximations differ by $\frac{1}{n}$, Archimedes tells us that

$$J_p = \int_0^1 x^p dx = \frac{1}{p+1}. \quad (1.5)$$

holds for every positive integer p . An important goal in this course will be to give a rigorous meaning to integrals such as (1.5). The above reasoning will guide us in Chapter 6.

⁷Frits Beukers showed me this neat trick.

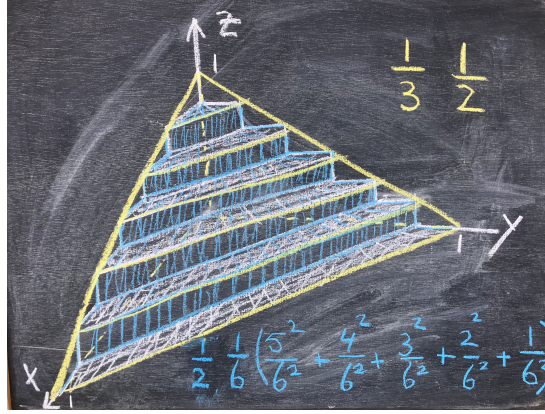


Figure 3: The volume of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$ in xyz -space is equal to the product of $\frac{1}{3}$ and $\frac{1}{2}$. The latter factor is the area of the triangular base, but how does $\frac{1}{3}$ appear? And how would you measure the object bounded by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and $x_1 + x_2 + x_3 + x_4 = 1$ in 4-space?

1.3 Number sets and the Archimedean Principle

We continue this introduction with an overview of the different number sets that we use in analysis, tied up with Archimedes' principle. You are of course familiar with

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\} \subset \mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\},$$

the set of all *integers* and the set of all *rational*s. We think of $\mathbb{Z}$ as a bi-infinite sequence of marked points on a number line with no endpoints. The other numbers of $\mathbb{Q}$ lie in the intervals between. If $r \in \mathbb{Q}$ is not in $\mathbb{Z}$ then $r = m + q$ with $m \in \mathbb{Z}$, $q \in \mathbb{Q}$ and $0 < q < 1$.

Many geometrically defined numbers such as π and $\sqrt{2}$ are not rational and correspond to other points on the number line, which we think of as corresponding to the set $\mathbb{R}$ of all *real numbers*. Thus

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Beginning with $\mathbb{N}$ all of these are sets with infinitely many elements, as they all contain the infinite set $\mathbb{N}$ *enumerated* by $1, 2, 3, \dots$. It is also easy to enumerate $\mathbb{Q}$, but you really should convince yourself that such a one-to-one

correspondence between $\mathbb{N}$ and the set of all points on the real number line cannot exist.

To wit, assume

$$x_1, x_2, x_3, \dots$$

is an enumeration of $\mathbb{R}$. Then $\mathbb{R}$ is completely covered by the intervals⁸

$$\left(x_1 - \frac{1}{4}, x_1 + \frac{1}{4}\right), \left(x_2 - \frac{1}{8}, x_2 + \frac{1}{8}\right), \left(x_3 - \frac{1}{16}, x_3 + \frac{1}{16}\right), \\ \left(x_4 - \frac{1}{32}, x_4 + \frac{1}{32}\right), \left(x_5 - \frac{1}{64}, x_5 + \frac{1}{64}\right), \left(x_6 - \frac{1}{128}, x_6 + \frac{1}{128}\right),$$

etcetera. The total length of these covering intervals is at most

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \dots,$$

which I hope you agree is 1. Similar reasoning would bound the total length by $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, and so on. This is an absurdity that we are not willing to accept: the total length⁹ of the real number line should be larger than any positive number. Have we proved the following theorem?

Theorem 1.4. *The set $\mathbb{R}$ of real numbers is not enumerable. In other words, $\mathbb{R}$ is not a sequence of numbers.*

A more direct proof of Theorem 1.4 is via never ending *decimal expansions*. Indeed: one possible and very natural definition of the set $\mathbb{R}$ of real numbers is by means of such expansions. Assume that the real numbers between 0 and 1 are enumerated by

$$x_n = \sum_{j=1}^{\infty} \frac{d_{nj}}{10^j} \quad \text{for } n = 1, 2, 3, \dots,$$

and put the digits¹⁰ d_{nj} in a block

⁸For numbers $a < b$ we denote by (a, b) the set of all real numbers x with $a < x < b$.

⁹Here we touch upon measure theory, see cartoon before Chapter 8 and Section 8.4

¹⁰Which can be any of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

$$\begin{array}{cccccccc}
d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} & d_{17} & d_{18} & \dots \\
d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} & d_{27} & d_{28} & \dots \\
d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} & d_{37} & d_{38} & \dots \\
d_{41} & d_{42} & d_{43} & d_{44} & d_{45} & d_{46} & d_{47} & d_{48} & \dots \\
d_{51} & d_{52} & d_{53} & d_{54} & d_{55} & d_{56} & d_{57} & d_{58} & \dots \\
d_{61} & d_{62} & d_{63} & d_{64} & d_{65} & d_{66} & d_{67} & d_{68} & \dots \\
d_{71} & d_{72} & d_{73} & d_{74} & d_{75} & d_{76} & d_{77} & d_{78} & \dots \\
d_{81} & d_{82} & d_{83} & d_{84} & d_{85} & d_{86} & d_{87} & d_{88} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Now choose d_n with $d_n - d_{nn} = 2$ or $d_{nn} - d_n = 2$. Then the real number

$$\sum_{j=1}^{\infty} \frac{d_j}{10^j}$$

does not appear as any x_n in our enumeration, a contradiction.

To make decimal representations unique, we may choose to exclude expansions which only have finitely many nonzero digits. The number $1 \in \mathbb{R}$ is then represented in $\mathbb{R}$ as

$$1 = 0.9999999 \dots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots, \quad (1.6)$$

whence for example

$$\frac{1}{9} = 0.1111111 \dots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots = \sum_{n=1}^{\infty} \frac{1}{10^n}.$$

This is just like

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^n}, \quad (1.7)$$

which relates to *binary representations* of the real numbers.

The equalities in the above expressions relate to the Archimedean principle again. For instance, the absolute value of the difference between 1 and

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

is clearly smaller than every power of $\frac{1}{2}$, and thus smaller than every $\frac{1}{n}$. According to Archimedes it must thus be zero. We shall honour Archimedes by stating his principle as a theorem in which we use the modern symbols $\forall$ and $\exists$, as well as the proverbial epsilon. Enjoy <https://youtu.be/CFGzPqAadEU>

1.4 Enter the epsilons

The theorem below says that there is no positive real number smaller than every $\frac{1}{n}$, which is what we used to conclude from (1.4) that the only candidate for the volume of the pyramid is $\frac{1}{3}$. Our task will be to understand the mathematical proof of what was obvious to Archimedes¹¹.

Theorem 1.5. *The Archimedean Principle:*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \frac{1}{N} < \varepsilon.$$

Exercise 1.6. Maybe Archimedes thought of his principle as¹²

$$\forall \varepsilon > 0 \underbrace{\exists N \in \mathbb{N} : \frac{1}{N} \leq \varepsilon}_{\text{whatever}}$$

Looks weaker as a statement but it's not¹³ why?

Exercise 1.7. Explain why the Archimedean Principle with $\varepsilon = \frac{1}{x}$ and $x \in \mathbb{R}$ positive leads to the equivalent statement

$$\forall x \in \mathbb{R} \exists N \in \mathbb{N} : N > x.$$

Note that for $x \leq 0$ the inequality holds for all $N \in \mathbb{N}$.

Exercise 1.8. We used the symbol N to exhibit that the statements in Theorem 1.5 and Exercise 1.7 concern the existence of a single N . For which n other than $n = N$ do these Archimedean statements also hold?

1.5 The geometric series

See

https://en.wikipedia.org/wiki/Geometric_series

¹¹Don't we do *important work*?

¹²If he ever did. Note the first use of the symbol $\leq$ and that you know what it means.

¹³We could choose to write all future $\forall \varepsilon > 0 \dots < \varepsilon$ statements with $\leq \varepsilon$, but we won't.

for the title of this subsection. We have seen in Section 1.3 that in the set $\mathbb{R}$ it must hold that

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \frac{1}{10^5} + \cdots = \frac{1}{10-1}.$$

Substituting $10 = n$ we “discover” that

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \frac{1}{n^5} + \cdots = \frac{1}{n-1}. \quad (1.8)$$

It’s easy to convince yourself why (1.8) should be true for every integer $n > 1$: order one *pizza* for $n-1$ persons, slice it in n pieces, eat, slice, eat, and so. If you have been born with n fingers ($n > 1$) you are likely to discover (1.8) as a fact of every day arithmetic life, long before you eat pizza’s. Have a look at

https://en.wikipedia.org/wiki/Zeno_of_Elea

before we continue but don’t spend too long there.

For $x \in \mathbb{R}$ the more general expression

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad (1.9)$$

is called a *geometric series*. The formula

$$\sum_{n=0}^N x^n = 1 + x + x^2 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x} \quad (1.10)$$

for the finite sums leads to a remarkable conclusion.

Theorem 1.9. *For $x \in \mathbb{R}$ it holds that¹⁴*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x| < 1. \quad (1.11)$$

Exercise 1.10. Sketch the graphs defined by

$$y = \frac{1}{1-x}, \quad y = 1+x, \quad y = 1+x+x^2, \quad y = 1+x+x^2+x^3, \dots$$

to see what this actually means.

¹⁴We use the *absolute value* $|x|$ of x for the first time here.

In particular it follows for all x with $|x| < 1$ that¹⁵

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = \frac{x}{1-x},$$

which reduces to (1.8) for $x = \frac{1}{n}$, but is a far more general statement¹⁶

A mathematical proof of Theorem 1.9 first of all requires an algebraic proof of (1.10), i.e. that

$$\sum_{n=0}^N x^n = \underbrace{\frac{1-x^{N+1}}{1-x}}_{\text{LaTeX sucks}} = \frac{1-x^{N+1}}{1-x},$$

and then a limit argument for $N \rightarrow \infty$, which boils down to the statement that

$$x^N \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad \text{if } |x| < 1. \quad (1.12)$$

You should contrast this with¹⁷

$$\sqrt[n]{x} \rightarrow 1 \quad \text{as } n \rightarrow \infty \quad \text{if } x > 0. \quad (1.13)$$

Making such and other limits statements mathematically sound is another important task for this course, but don't ignore the algebraic beauty in (1.10). The temptation to for now leave Archimedes and the epsilons for what they are and jump to Chapter 9 is hard to resist. Very hard.

1.6 Exercises

Exercise 1.11. Write

$$\frac{x}{1+x}$$

as a power series for x with $|x| < 1$, and as a power series in $\frac{1}{x}$ for x with $|x| > 1$. As in Exercise 1.10, draw graphs to examine how well the partial sums do as approximations.

Exercise 1.12. We turn (1.10) around. Show that for every $x \neq 1$ it holds that

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1},$$

¹⁵Subtracting 1 on both sides, or multiplying by x .

¹⁶Play with this formula, for instance, replace x by $-x$ and draw some graphs.

¹⁷How would you use (1.12) to prove (1.13)?

and observe that the right hand side is equal to n if $x = 1$. Generalise to

$$\frac{x^n - a^n}{x - a}$$

with a and x in $\mathbb{R}$. What does this tell¹⁸ you about the line tangent to the graph defined by $y = x^n$ in $x = a$?

Exercise 1.13. Look at (1.1). Verify that

$$f(x) = \frac{x}{(1 - x^7)^{\frac{1}{7}}} \quad \text{gives} \quad F(x) = x^8.$$

What does the scheme $x_n = F(x_{n-1})$ do in relation to f ? Play with the obvious similar examples.

Exercise 1.14. Use long division to find the expansion

$$\frac{1}{7} = \sum_{j=1}^{\infty} \frac{d_j}{10^j}.$$

What's the periodic part in the expansion? Divide the sum by that periodic part to obtain (1.8) with n a power of 10 and check that your answer was right.

Exercise 1.15. Find the complete hexigesimal expansion¹⁹ of

$$\frac{17}{24} + \frac{12}{17} = \frac{289 + 288}{24 \times 17} = \frac{577}{408}.$$

Hint: use hexigesimal *long division* to write the hexigesimal expansion of $\frac{12}{17}$, which should come out periodic. Then add the finite hexigesimal expansion of $\frac{17}{24}$.

Exercise 1.16. Find a formula for

$$\sum_{k=1}^n k^3.$$

Hint: try $an^4 + bn^3 + cn^2 + dn$, find a, b, c, d from $n = 1, 2, 3, 4$, then use dominos.

¹⁸In Chapter 9 this starts an approach to *differentiation* that *avoids the usual limits*.

¹⁹You need the multiplicative tables in base 60.

Exercise 1.17. Let $p \in \mathbb{N}$. Complete the expression

$$a^{p+1} - b^{p+1} = (a - b) \sum_{j=0}^p \dots$$

and show that

$$(p+1)b^p < \frac{a^{p+1} - b^{p+1}}{a - b} < (p+1)a^p$$

for $a > b > 0$. Then put $a = k+1$ and $b = k$ and take the sum over $k = 0, 1, \dots, n-1$ to show that

$$\sum_{k=0}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=0}^n k^p$$

for $p, n \in \mathbb{N}$. NB In Chapter 6 these inequalities lead to

$$\int_0^1 x^p dx = \frac{1}{p+1}.$$

Exercise 1.18. Referring to Exercise 1.12 take $n = 7$. Use *long division* to show that

$$\frac{x^7 - a^7}{x - a} = x^6 + ax^5 + a^2x^4 + a^3x^3 + a^4x^2 + a^5x + a^6,$$

and then whatever algebra you like to deduce that

$$x^7 = a^7 + 7a^6(x - a) + (x^5 + 2ax^4 + 3a^2x^3 + 4a^3x^2 + 5a^4x + 6a^5)(x - a)^2.$$

What's the formula for general $n \in \mathbb{N}$?

Exercise 1.19. Use the Archimedean Principle in Theorem 1.5 to show that

$$\forall \varepsilon > 0 \exists n \in \mathbb{N} : \frac{2019}{n} \leq \varepsilon.$$

Exercise 1.20. Show that

$$\forall \varepsilon > 0 \exists n \in \mathbb{N} : \frac{1}{n^2} < \varepsilon; \quad \forall \varepsilon > 0 \exists n \in \mathbb{N} : \frac{1}{\sqrt{n}} < \varepsilon; \quad \forall \varepsilon > 0 \exists n \in \mathbb{N} : \frac{n}{n^2 + 1} < \varepsilon.$$

Exercise 1.21. You may enjoy proving that

$$\frac{1}{n} \geq 2^{1-n}$$

for all $n \in \mathbb{N}$. This tells us that

$$\forall \varepsilon > 0 \exists n \in \mathbb{N} : \frac{1}{2^{n-1}} < \varepsilon.$$

Hint: domino principle. When you're done do this next one:

Exercise 1.22. This exercise and (1.14) will be crucial in (3.8). Recall that we accept (1.7) as the obvious inequality below, supplemented with an Archimedean argument that the inequality cannot be strict. Let's examine the inequality more closely and cut it up in pieces, for instance

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \underbrace{\frac{1}{2} + \frac{1}{4}}_{\frac{3}{4}} + \frac{1}{8} + \underbrace{\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}}_{\frac{15}{128} = \frac{15}{16} \cdot \frac{1}{8} < \frac{1}{8} \leq \frac{1}{4}} + \cdots \leq 1,$$

to draw additional conclusions such as for example

$$\sum_{k=4}^7 \frac{1}{2^k} < \frac{1}{2^2}.$$

Generalise and prove that

$$\forall_{m,n,N \in \mathbb{N}} : m \geq n \geq N \implies \sum_{k=n}^m \frac{1}{2^k} < \frac{1}{2^{N-1}}.$$

Then take N as in Exercise 1.21 to conclude that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall_{m,n \in \mathbb{N}} : m \geq n \geq N \implies \sum_{k=n}^m \frac{1}{2^k} < \varepsilon. \quad (1.14)$$

Exercise 1.23. Use (1.10) to show for $n \in \mathbb{N}$ that

$$nx^{n-1} < \frac{1}{1-x} \quad \text{if } 0 < x < 1.$$

Exercise 1.24. This exercise relates to (1.12). Suppose that $0 < x < 1$. From (1.10) it follows that²⁰

$$(N + 1)x^N < \frac{1}{1 - x}$$

for every $N \in \mathbb{N}$. Combine with Theorem 1.5 to show that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : x^n < \varepsilon.$$

Exercise 1.25. This exercise relates to (1.13). Suppose that $x > 1$. For each $n \in \mathbb{N}$ assume $y = \sqrt[n]{x}$ is well defined by $y^n = x$. This implies that $\sqrt[n]{x} < \sqrt[m]{x}$ if $n > m$. To prove that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : 0 < \sqrt[n]{x} - 1 < \varepsilon$$

it therefore suffices to prove that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : 0 < \sqrt[N]{x} - 1 < \varepsilon.$$

Prove this latter statement.

Exercise 1.26. This exercise also relates to (1.13). Suppose that $0 < x < 1$. Assume again that $y = \sqrt[n]{x}$ is well defined by $y^n = x$ for each $n \in \mathbb{N}$. Prove that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : 0 < 1 - \sqrt[n]{x} < \varepsilon.$$

Exercise 1.27. This exercise introduces a happy couple for later. Let $p > 1$ and $q > 1$ be real numbers. Show that

$$\frac{1}{p} + \frac{1}{q} = 1 \iff (p - 1)(q - 1) = 1 \iff q = \frac{p}{p - 1} \iff p = \frac{q}{q - 1}$$

²⁰Compare to Exercise 1.23.

1.7 Compound interest

Think of x in Exercise 1.29 as a fixed and positive annual interest rate on your savings account, n as the number of months in a year, and the bank multiplying your balance by $1 + \frac{x}{n}$ every month. Wouldn't you like to have $n \rightarrow \infty$? And what if x turns negative?

Exercise 1.28. Have a look at

https://en.wikipedia.org/wiki/Binomial_theorem

and then use the domino principle to show that

$$(1+a)^n = 1 + na + \frac{n(n-1)}{2}a^2 + \frac{n(n-1)(n-2)}{3 \cdot 2}a^3 + \dots$$

for every $n \in \mathbb{N}$ and every $a \in \mathbb{R}$.

Exercise 1.29. (continued) Show that

$$\left(1 + \frac{x}{n}\right)^n = 1 + x + \left(1 - \frac{1}{n}\right) \frac{x^2}{2!} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \frac{x^3}{3!} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) \frac{x^4}{4!} + \dots$$

for every $n \in \mathbb{N}$ and every $x \in \mathbb{R}$.

Exercise 1.30. (continued) Write

$$s_n(x) = \left(1 + \frac{x}{n}\right)^n$$

and show that $s_{n+1}(x) > s_n(x)$ for $x > 0$. What would you guess for $-n < x < 0$? See if you were right by verifying and using that

$$\begin{aligned} s_{n+1}(x) - s_n(x) &= \left(1 + \frac{x}{n+1}\right)^{n+1} - \left(1 + \frac{x}{n}\right)^{n+1} + \left(1 + \frac{x}{n}\right)^{n+1} - \left(1 + \frac{x}{n}\right)^n \\ &= \left(\frac{x}{n+1} - \frac{x}{n}\right) \left(\left(1 + \frac{x}{n+1}\right)^n + \dots + \left(1 + \frac{x}{n}\right)^n \right) + \left(1 + \frac{x}{n}\right)^n \left(1 + \frac{x}{n} - 1\right) \\ &= \frac{-x}{n(n+1)} \left(\left(1 + \frac{x}{n+1}\right)^n + \dots + \left(1 + \frac{x}{n}\right)^n \right) + \frac{x}{n} \left(1 + \frac{x}{n}\right)^n \\ &= \frac{x}{n(n+1)} \left((n+1) \left(1 + \frac{x}{n}\right)^n - \left(1 + \frac{x}{n+1}\right)^n - \dots - \left(1 + \frac{x}{n}\right)^n \right). \end{aligned}$$

Exercise 1.31. Show that the supremum

$$S(x) = \sup_{n \geq -x} \left(1 + \frac{x}{n}\right)^n$$

exists as a positive number for every $x \in \mathbb{R}$ and think of a better name for S . Hint: how would $S(x+y)$, $S(x)$ and $S(y)$ be related?



1.8 Outlook: norms beyond the real numbers

A small detour: you will see elsewhere that (1.11) is true even in the form

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}, \quad (1.15)$$

where A is a square matrix, and in which A^n is a *matrix product*²¹ i.e.

$$A^2 = A A, A^3 = A A A, A^4 = A A A A,$$

and so on. To give a rigorous meaning to (1.15), we will in fact need a condition of the form $|A| < 1$. A possible “absolute value” of a matrix is

$$|A|_{\text{Frob}} = \sqrt{\sum_{i,j} A_{ij}^2},$$

²¹Matrix products are explained in Section 19.1

the square root of the sum of all the squared entries of A . This is called the *Frobenius*²² norm of A . The Frobenius norm has the remarkable properties that

$$|AB|_{\text{Frob}} \leq |A|_{\text{Frob}} |B|_{\text{Frob}} \quad \text{and} \quad |A+B|_{\text{Frob}} \leq |A|_{\text{Frob}} + |B|_{\text{Frob}} \quad (1.16)$$

for all square²³ matrices A and B of the same size.

In this course we will not so much study *matrices and matrix norms*. However, we will often work with the “absolute value” of functions $f : [a, b] \rightarrow \mathbb{R}$, many of which you have seen before. This absolute value or *maximum norm* is defined as

$$|f|_{\max} = \max_{a \leq x \leq b} |f(x)|, \quad (1.17)$$

if this maximum exists. For two functions²⁴ f and g we will have that

$$|fg|_{\max} \leq |f|_{\max} |g|_{\max} \quad \text{and} \quad |f+g|_{\max} \leq |f|_{\max} + |g|_{\max}, \quad (1.18)$$

where in general $|fg|_{\max} < |f|_{\max} |g|_{\max}$.

We will speak about $f_n \rightarrow f$ for sequences of such functions, just like we speak of *convergent sequences* of real numbers x_n . This concept of convergence of sequences of functions will be extremely useful for solving many problems in analysis, including integral and differential equations.

1.9 An exam question: inverting functions

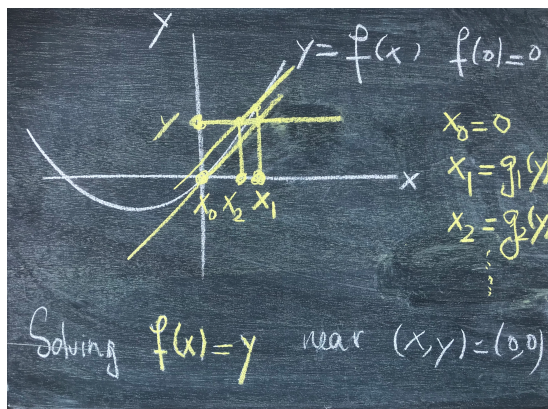


Figure 4: Using lines parallel to the tangent line in the starting point to define functions $g_1, g_2, \dots$ of y . Exam Problem 4 in Section 11.6

²²To some dismay of Euclides and Pythagoras perhaps.

²³In general $AB \neq BA$! In fact the estimates only require AB or $A+B$ to be defined.

²⁴With clearly $fg = gf$.