Elliptic and Parabolic Equations

by

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1. Harmonic functions

Throughout this section, $\Omega \subset \mathbb{R}^n$ is a bounded domain.

1.1 Definition A function $u \in C^2(\Omega)$ is called *subharmonic* if $\Delta u \geq 0$ in Ω, *harmonic* if $\Delta u \equiv 0$ in Ω, and *superharmonic* if $\Delta u \leq 0$ in Ω.

1.2 Notation The measure of the unit ball in $\mathbb{R}^n$ is

$$\omega_n = |B_1| = |\{x \in \mathbb{R}^n : x_1^2 + ... + x_n^2 \leq 1\}| = \int_{B_1} dx = \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

The $(n-1)$-dimensional measure of the boundary ∂B_1 of B_1 is equal to $n\omega_n$.

1.3 Mean Value Theorem Let $u \in C^2(\Omega)$ be subharmonic, and

$$\overline{B_R(y)} = \{x \in \mathbb{R}^n : |x-y| \leq R\} \subset \Omega.$$

Then

$$u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x) dS(x),$$

where dS is the $(n-1)$-dimensional surface element on $\partial B_R(y)$. Also

$$u(y) \leq \frac{1}{\omega_n R^n} \int_{B_R(y)} u(x) dx.$$

Equalities hold if u is harmonic.

Proof We may assume $y = 0$. Let $\rho \in (0, R)$. Then

$$0 \leq \int_{B_\rho} \Delta u(x) dx = \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}(x) dS(x) = \int_{\partial B_\rho} \frac{\partial u}{\partial r}(x) dS(x) =$$

(substituting $x = \rho\omega$)

$$\int_{\partial B_1} \frac{\partial u}{\partial r}(\rho\omega)\rho^{n-1} dS(\omega) = \rho^{n-1} \int_{\partial B_1} (\frac{\partial}{\partial \rho} u(\rho\omega)) dS(\omega) = \rho^{n-1} \frac{d}{d\rho} \int_{\partial B_1} u(\rho\omega) dS(\omega)$$

(substituting $\omega = x/\rho$)

$$= \rho^{n-1} \frac{d}{d\rho} \frac{1}{\rho^{n-1}} \int_{\partial B_\rho} u(x) dS(x),$$

which implies, writing

$$f(\rho) = \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho} u(x)dS(x),$$

that $f'(\rho) \geq 0$. Hence

$$u(0) = \lim_{\rho \downarrow 0} f(\rho) \leq f(R) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u(x)dS(x),$$

which proves the first inequality. The second one follows from

$$\int_{B_R} u(x)dx = \int_0^R \{ \int_{\partial B_\rho} u(x)dS(x) \} d\rho \geq \int_0^R n\omega_n \rho^{n-1} u(0)d\rho = \omega_n R^n u(0).$$

This completes the proof. ∎

1.4 Corollary (Strong maximum principle for subharmonic functions) Let $u \in C^2(\Omega)$ be bounded and subharmonic. If for some $y \in \Omega$, $u(y) = \sup_\Omega u$, then $u \equiv u(y)$.

Proof Exercise (hint: apply the mean value theorem to the function $\tilde{u}(x) = u(x) - u(y)$, and show that the set $\{x \in \Omega : \tilde{u}(x) = 0\}$ is open). ∎

1.5 Corollary (weak maximum principle for subharmonic functions) Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is subharmonic. Then

$$\sup_\Omega u = \max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof Exercise. ∎

1.6 Corollary Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be subharmonic. If $u \equiv 0$ on $\partial\Omega$, then $u < 0$ on Ω, unless $u \equiv 0$ on $\overline{\Omega}$.

Proof Exercise. ∎

1.7 Corollary Let $\varphi \in C(\partial\Omega)$. Then there exists at most one function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $u = \varphi$ on $\partial\Omega$.

Proof Exercise. ∎

1.8 Theorem (Harnack inequality) Let $\Omega' \subset\subset \Omega$ (i.e. $\Omega' \subset \overline{\Omega'} \subset \Omega$) be a subdomain. Then there exists a constant C which only depends on Ω' and Ω, such that for all harmonic nonnegative functions $u \in C^2(\Omega)$,

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

Proof Suppose that $\overline{B_{4R}(y)} \subset \Omega$. Then for any $x_1, x_2 \in B_R(y)$ we have

$$B_R(x_1) \subset B_{3R}(x_2) \subset B_{4R}(y) \subset \Omega,$$

so that by the mean value theorem,

$$u(x_1) = \frac{1}{\omega_n R^n} \int_{B_R(x_1)} u(x)dx \leq \frac{3^n}{\omega_n (3R)^n} \int_{B_{3R}(x_2)} u(x)dx = 3^n u(x_2).$$

Hence, x_1, x_2 being arbitrary, we conclude that

$$\sup_{B_R(y)} u \leq 3^n \inf_{B_R(y)} u.$$

Thus we have shown that for $\Omega' = B_R(y)$, with $B_{4R}(y) \subset \Omega$, the constant in the inequality can be taken to be 3^n. Since any $\Omega' \subset\subset \Omega$ can be covered with finitely many of such balls, say

$$\Omega' \subset B_{R_1}(y_1) \cup B_{R_2}(y_2) \cup \ldots \cup B_{R_N}(y_N),$$

we obtain for Ω' that $C = 3^{nN}$. ∎

Next we turn our attention to radially symmetric harmonic functions. Let $u(x)$ be a function of $r = |x|$ alone, i.e. $u(x) = U(r)$. Then u is harmonic if and only if

$$0 = \Delta u(x) = \sum_{i=1}^{n} (\frac{\partial}{\partial x_i})^2 u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(U'(r)\frac{\partial r}{\partial x_i}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\frac{x_i}{r}U'(r))$$

$$= \sum_{i=i}^{n} (\frac{1}{r}U'(r) + \frac{x_i}{r}U''(r)\frac{\partial r}{\partial x_i} - x_i U'(r)\frac{\partial}{\partial x_i}(\frac{1}{r}))$$

$$= \frac{n}{r}U'(r) + \sum_{i=1}^{n} \frac{x_i^2}{r}U''(r) - \sum_{i=1}^{n} x_i U'(r)\frac{x_i}{r^3} =$$

$$U''(r) + \frac{n-1}{r}U'(r) = \frac{1}{r^{n-1}}(r^{n-1}U'(r))',$$

implying

$$r^{n-1}U'(r) = C_1,$$

so that

$$U(r) = \begin{cases} C_1 r + C_2 & n = 1; \\ C_1 \log r + C_2 & n = 2; \\ \dfrac{C_1}{2-n}\dfrac{1}{r^{n-2}} + C_2 & n > 2. \end{cases} \tag{1.1}$$

We define the *fundamental solution* by

$$\Gamma(x) = \begin{cases} \dfrac{1}{2}|x| & n = 1 \\ \dfrac{1}{2\pi}\log|x| & n = 2 \\ \dfrac{1}{n\omega_n(2-n)}\dfrac{1}{|x|^{n-2}} & n > 2, \end{cases} \tag{1.2}$$

i.e. $C_1 = 1/n\omega_n$ and $C_2 = 0$ in (1.1). Whenever convenient we write $\Gamma(x) = \Gamma(|x|) = \Gamma(r)$.

1.9 Theorem The fundamental solution Γ is a solution of the equation $\Delta\Gamma = \delta$ in the sense of distributions, i.e.

$$\int_{\mathbb{R}^n} \Gamma(x)\Delta\psi(x)dx = \psi(0) \qquad \forall\psi \in D(\mathbb{R}^n).$$

Proof First observe that for all $R > 0$, we have $\Gamma \in L^\infty(B_R)$ if $n = 1$, $\Gamma \in L^P(B_R)$ for all $1 \le p < \infty$ if $n = 2$, and $\Gamma \in L^P(B_R)$ for all $1 \le p < \frac{n}{n-2}$ if $n > 2$, so for all ψ in $D(\mathbb{R}^n)$, choosing R large enough, we can compute

$$\int_{\mathbb{R}^n} \Gamma(x)\Delta\psi(x)dx = \int_{B_R} \Gamma(x)\Delta\psi(x)dx = \lim_{\rho\downarrow 0}\int_{A_{R,\rho}} \Gamma(x)\Delta\psi(x)dx =$$

(here $A_{R,\rho} = \{x \in B_R : |x| > \rho\}$)

$$\lim_{\rho\downarrow 0}\{\int_{\partial A_{R,\rho}} \Gamma\frac{\partial\psi}{\partial\nu} - \int_{A_{R,\rho}} \nabla\Gamma\nabla\psi\} = \lim_{\rho\downarrow 0}\{\int_{\partial A_{R,\rho}} (\Gamma\frac{\partial\psi}{\partial\nu} - \frac{\partial\Gamma}{\partial\nu}\psi) + \int_{A_{R,\rho}} \psi\Delta\Gamma\} =$$

$$\lim_{\rho\downarrow 0}\int_{\partial B_\rho}\{\frac{-\partial\psi/\partial\nu}{n\omega_n(2-n)\rho^{n-2}} + \frac{\psi}{n\omega_n\rho^{n-1}}\} = \psi(0).$$

For $n = 1, 2$ the proof is similar. ∎

Closely related to this theorem we have

1.10 Theorem (Green's representation formula) Let $u \in C^2(\overline{\Omega})$ and suppose $\partial\Omega \in C^1$. Then, if ν is the outward normal on $\partial\Omega$, we have

$$u(y) = \int_{\partial\Omega} \{u(x)\frac{\partial}{\partial\nu}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial\nu}(x)\}dS(x) + \int_{\Omega} \Gamma(x-y)\Delta u(x)dx.$$

Here the derivatives are taken with respect to the x-variable.

Proof Exercise (Hint: take $y = 0$, let $\Omega_\rho = \{x \in \Omega : |x| > \rho\}$, and imitate the previous proof). ∎

If we want to solve $\Delta u = f$ on Ω for a given function f, this representation formula strongly suggests to consider the convolution

$$\int_{\Omega} \Gamma(x-y)f(x)dx$$

as a function of y, or equivalently,

$$(\Gamma * f)(x) = \int_{\Omega} \Gamma(x-y)f(y)dy \tag{1.3}$$

as a function of x. This convolution is called the *Newton potential* of f.

For any harmonic function $h \in C^2(\overline{\Omega})$ we have

$$\int_{\Omega} h\Delta u = \int_{\partial\Omega} (h\frac{\partial u}{\partial\nu} - u\frac{\partial h}{\partial\nu}),$$

so that, combining with Green's representation formula,

$$u(y) = \int_{\partial\Omega} \{u\frac{\partial G}{\partial\nu} - G\frac{\partial u}{\partial\nu}\} + \int_{\Omega} G\Delta u, \tag{1.4}$$

where $G = \Gamma(x-y) + h(x)$. The trick is now to take instead of a function $h(x)$ a function $h(x, y)$ of two variables $x, y \in \overline{\Omega}$, such that h is harmonic in x, and for every $y \in \Omega$,

$$G(x, y) = \Gamma(x-y) + h(x, y) = 0 \qquad \forall x \in \partial\Omega.$$

This will then give us the solution formula

$$u(y) = \int_{\partial\Omega} u\frac{\partial G}{\partial\nu} + \int_{\Omega} G\Delta u.$$

In particular, if $u \in C^2(\overline{\Omega})$ is a solution of

$$(D) \begin{cases} \Delta u = f & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

then

$$u(y) = \int_{\partial\Omega} \varphi(x) \frac{\partial G(x,y)}{\partial \nu} + \int_{\Omega} G(x,y) f(x) dx. \tag{1.5}$$

The function $G(x, y) = \Gamma(x - y) + h(x, y)$ is called the *Green's function* for the Dirichletproblem. Of course $h(x, y)$ is by no means trivial to find. The function h is called the *regular part* of the Green's function. If we want to solve

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

(1.5) reduces to

$$u(y) = \int_{\partial\Omega} \varphi(x) \frac{\partial G(x,y)}{\partial \nu} dS(x). \tag{1.6}$$

We shall evaluate (1.6) in the case that $\Omega = B = B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$. Define the *reflection* in ∂B by

$$S(y) = \frac{y}{|y|^2} \quad \text{if} \quad y \neq 0; \quad S(0) = \infty; \quad S(\infty) = 0. \tag{1.7}$$

Here ∞ is the point that has to be added to $\mathbb{R}^n$ in order to construct the one-point compactification of $\mathbb{R}^n$. If $0 \neq y \in B$, then $\overline{y} = S(y)$ is uniquely determined by asking that 0, y and $\overline{y}$ lie (in that order) on one line l, and that the boundary ∂B of B is tangent to the cone C with top $\overline{y}$ spanned by the circle obtained from intersecting the ball B with the plane perpundicular to l going through y (you'd better draw a picture here). Indeed if $\overline{x}$ lies on this circle, then the triangles $0y\overline{x}$ and $0\overline{xy}$ are congruent and

$$|y| = \frac{|y - 0|}{|\overline{x} - 0|} = \frac{|\overline{x} - 0|}{|\overline{y} - 0|} = \frac{1}{|\overline{y}|},$$

so that $\overline{y} = S(y)$. It is also easily checked that

$$\partial B = \{x \in \mathbb{R}^n;\ |x - y| = |y||x - \overline{y}|\}.$$

But then the construction of $h(x, y)$ is obvious. We simply take

$$h(x, y) = -\Gamma(|y|(x - \overline{y})),$$

so that

$$G(x, y) = \Gamma(x - y) - \Gamma(|y|(x - \overline{y})). \tag{1.8}$$

Note that since $|y||\overline{y}| = 1$, and since $y \to 0$ implies $\overline{y} \to \infty$, we have, with a slight abuse of notation, that $G(x, 0) = \Gamma(x) - \Gamma(1)$. It is convenient to rewrite $G(x, y)$ as

$$G(x, y) = \Gamma(\sqrt{|x|^2 + |y|^2 - 2xy}) - \Gamma(\sqrt{|x|^2|y|^2 + |y|^2|\overline{y}|^2 - 2|y|^2 x\overline{y}})$$

$$= \Gamma(\sqrt{|x|^2+|y|^2-2xy}) - \Gamma(\sqrt{|x|^2|y|^2+1-2xy}),$$

which shows that G is symmetric in x and y. In particular G is also harmonic in the y variables.

Next we compute $\partial G/\partial\nu$ on ∂B. We write

$$r = |x-y|; \;\; \overline{r} = |x-\overline{y}|; \;\; \frac{\partial}{\partial \nu} = \nu\cdot\nabla = \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}; \;\; \frac{\partial r}{\partial x_i} = \frac{x_i - y_i}{r}; \;\; \frac{\partial \overline{r}}{\partial x_i} = \frac{x_i - \overline{y}_i}{\overline{r}},$$

so that since $G = \Gamma(r) - \Gamma(|y|\overline{r})$,

$$\frac{\partial \Gamma(r)}{\partial \nu} = \Gamma'(r)\frac{\partial r}{\partial \nu} = \frac{1}{n\omega_n r^{n-1}} \sum_{i=1}^n x_i \frac{x_i - y_i}{r} = \frac{1-xy}{n\omega_n r^n},$$

and

$$\frac{\partial \Gamma(|y|\overline{r}|)}{\partial \nu} = \Gamma'(|y|\overline{r})|y|\frac{\partial \overline{r}}{\partial \nu} = \frac{1}{n\omega_n |y|^{n-2}\overline{r}^{n-1}}\frac{\partial \overline{r}}{\partial \nu} = \frac{1}{n\omega_n |y|^{n-2}\overline{r}^{n-1}} \sum_{i=1}^n x_i \frac{x_i - \overline{y}_i}{\overline{r}}$$

$$= \frac{1}{n\omega_n}|y|^{2-n}\overline{r}^{-n} \sum_{i=1}^n x_i(x_i - \overline{y}_i) = \quad \text{(substituting } \; r = |y|\overline{r})$$

$$\frac{1}{n\omega_n}|y|^{2-n}(\frac{|y|}{r})^n \sum_{i=1}^n (x_i^2 - x_i\overline{y}_i) = \frac{1}{n\omega_n r^n}\{|y|^2 - xy\},$$

whence

$$\frac{\partial G(x,y)}{\partial \nu(x)} = \frac{1}{n\omega_n r^n}(1-|y|^2) = \frac{1-|y|^2}{n\omega_n |x-y|^n}. \tag{1.9}$$

1.11 Theorem (Poisson integration formula) Let $\varphi \in C(\partial B)$. Define $u(y)$ for $y \in B$ by

$$u(y) = \frac{1-|y|^2}{n\omega_n}\int_{\partial B} \frac{\varphi(x)}{|x-y|^n} dS(x),$$

and for $y \in \partial B$, by $u(y) = \varphi(y)$. Then $u \in C^2(B) \cup C(\overline{B})$, and $\Delta u = 0$ in B.

Proof First we show that $u \in C^\infty(B)$ and that $\Delta u = 0$ in B. We have

$$u(y) = \frac{1-|y|^2}{n\omega_n}\int_{\partial B} \frac{\varphi(x)}{|x-y|^n} dS(x) = \int_{\partial B} K(x,y)\varphi(x) dS(x),$$

where the integrand is smooth in $y \in B$, and $K(x,y)$ is positive, and can be written as

$$K(x,y) = \frac{\partial G(x,y)}{\partial \nu(x)} = \sum_{i=1}^n x_i \frac{\partial G(x,y)}{\partial x_i}.$$

Thus $u \in C^\infty(B)$ and

$$\Delta u(y) = \sum_{j=1}^n \frac{\partial^2 u}{\partial y_j} = \sum_{j=1}^n (\frac{\partial}{\partial y_j})^2 \int_{\partial B} \sum_{i=1}^n x_i \frac{\partial G(x,y)}{\partial x_i} \varphi(x) dS(x)$$

$$= \int_{\partial B} \{\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \sum_{j=1}^n \frac{\partial^2 G(x,y)}{\partial y_j^2}\} \varphi(x) dS(x) = 0.$$

Next we show that $u \in C(\overline{B})$. Observe that

$$\int_{\partial B} K(x,y) dS(x) = 1,$$

because $\tilde{u} \equiv 1$ is the unique harmonic function with $\tilde{u} \equiv 1$ on the boundary. We have to show that for all $x_0 \in \delta B$

$$\lim_{\substack{y \to x_0 \\ y \in B}} u(y) = \varphi(x_0) = u(x_0),$$

so we look at

$$u(y) - u(x_0) = \int_{\partial B} K(x,y)(\varphi(x) - \varphi(x_0)) dS(x).$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(x_0)| < \varepsilon \quad \text{for all } x \in \delta B \text{ with } \ |x - x_0| < \delta.$$

Thus we have, with $M = \max_{\partial B} |\varphi|$, that

$$|u(y) - u(x_0)| \leq \int_{x \in \partial B,\ |x - x_0| < \delta} K(x,y) |\varphi(x) - \varphi(x_0)| dS(x)$$

$$+ \int_{x \in \partial B,\ |x - x_0| \geq \delta} K(x,y) |\varphi(x) - \varphi(x_0)| dS(x) \leq$$

$$\int_{\partial B} K(x,y) \varepsilon dS(x) + \int_{x \in \partial B,\ |x - x_0| \geq \delta} K(x,y) 2M dS(x) =$$

$$\varepsilon + 2M \int_{x \in \partial B,\ |x - x_0| \geq \delta} K(x,y) dS(x) \leq \quad (\text{choosing } \ y \in B \text{ with } \ |y - x_0| < \frac{\delta}{2}\ )$$

$$\varepsilon + 2M \int_{x \in \partial B,\ |x-y| \geq \frac{\delta}{2}} \frac{1 - |y|^2}{n\omega_n |x-y|^n} dS(x) \leq \varepsilon + 2M \frac{1 - |y|^2}{n\omega_n} \int_{\partial B} (\frac{2}{\delta})^n dS(x) =$$

$$\varepsilon + 2M (\frac{2}{\delta})^n (1 - |y|^2) \to \varepsilon \quad \text{as} \quad y \to x_0.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. ■

1.12 Remark On the ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ the Poisson formula reads

$$u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{\varphi(x)}{|x-y|^n} dS(x).$$

1.13 Corollary A function $u \in C(\Omega)$ is harmonic if and only if

$$u(y) = \frac{1}{\omega_n R^n} \int_{B_R(y)} u(x)dx$$

for all $B_R(y) \subset\subset \Omega$.

Proof Exercise (hint: use Poisson's formula in combination with the weak maximum principle which was proved using the mean value (in-)equalities). ■

1.14 Corollary Uniform limits of harmonic functions are harmonic.

Proof Exercise. ■

1.15 Corollary (Harnack convergence theorem) For a nondecreasing sequence of harmonic functions $u_n : \Omega \to \mathbb{R}$ to converge to a harmonic limit function u, uniformly on compact subsets, it is sufficient that the sequence $(u_n(y))_{n=1}^{\infty}$ is bounded for just one point $y \in \Omega$.

Proof Exercise (hint: use Harnack's inequality to establish convergence). ■

1.16 Corollary If $u : \Omega \to \mathbb{R}$ is harmonic, and $\Omega' \subset\subset \Omega$, $d =$ distance $(\Omega', \partial\Omega)$, then

$$\sup_{\Omega'} |\nabla u| \leq \frac{n}{d} \sup_{\Omega} |u|.$$

For higher order derivatives the factor $\frac{n}{d}$ has to be replaced by $(\frac{ns}{d})^s$, where s is the order of the derivative.

Proof Since $\Delta \nabla u = \nabla \Delta u = 0$, we have by the mean value theorem for $y \in \Omega'$

$$\big|\nabla u(y)\big| = \big|\frac{1}{\omega_n d^n} \int_{B_d(y)} \nabla u(x)dx\big| =$$

(by the vector valued version of Gauss' Theorem)

$$\big|\frac{1}{\omega_n d^n} \int_{\partial B_d(y)} u(x)\nu(x)dS(x)\big| \leq \frac{1}{\omega_n d^n} n\omega_n d^{n-1} \sup_{B_d(y)} |u(x)||\nu(x)| = \frac{n}{d} \sup_{B_d(y)} |u(x)|,$$

since ν is the unit normal. ■

1.17 Corollary (Liouville) If $u : \mathbb{R}^n \to \mathbb{R}^+$ is harmonic, then $u \equiv$ constant.

Proof We have

$$|\nabla u(y)| = \Big|\frac{1}{w_n R^n}\int_{\partial B_R(y)} u(x)\nu(x)dS(x)\Big| \leq \frac{1}{w_n R^n}\int_{\partial B_R(y)} |u(x)\nu(x)|dS(x)$$

$$= \frac{n}{R}\frac{1}{n\omega_n R^{n-1}}\int_{\partial B_R(y)} u(x)dS(x) = \frac{n}{R}u(y)$$

for all $R > 0$. Thus $\nabla u(y) = 0$ for all $y \in \mathbb{R}^n$, so that $u \equiv$ constant. ■

We have generalized a number of properties of harmonic functions on domains in $\mathbb{R}^2$, which follow from the following theorem for harmonic functions of two real variables.

1.18 Theorem Let $\Omega \subset \mathbb{R}^2$ be simply connnected. Suppose $u \in C(\Omega)$ is harmonic. Then there exists $v : \Omega \to \mathbb{R}$ such that

$$F(x+iy) = u(x,y) + iv(x,y)$$

is an analytic function on Ω. In particular $u, v \in C^\infty(\Omega)$ and $\Delta u = \Delta v = 0$ in Ω.

1.19 Exercise Let $u : \Omega \to \mathbb{R}$ be harmonic. Show that the function $v = |\nabla u|^2$ is subharmonic in Ω.

1.20 Exercise Adapt the proof of the mean value theorem to show that under the same assumptions, for dimension $n > 2$,

$$u(y) = \frac{1}{\omega_n R^n}\int_{B_R(y)} u(x)dx - a_n \int_{B_R(y)} G(|x-y|;R)\Delta u(x)dx,$$

where

$$G(r,R) = \frac{1}{r^{n-2}} - \frac{1}{R^{n-2}} + \frac{n-2}{2}\frac{r^2 - R^2}{R^n},$$

and $a_n > 0$ can be computed explicitly. Derive a similar formula for $n = 2$.

1.21 Exercise Show directly that the function

$$K(x,y) = \frac{R^2 - |y|^2}{n\omega_n R|x-y|^n}$$

is positive and harmonic in $y \in B = B_R$, and, without to much computation, that $\int_B K(x,y)dx = 1$. (These are the three essential ingredients in the proof of Theorem 1.11 and Remark 1.13).

1.22 Exercise Use Corollary 1.16 to prove that, if u is harmonic in Ω and $B_{2R}(x_0) \subset\subset \Omega$, then

$$|u(x) - u(y)| \leq |x-y|^\alpha (2R)^{1-\alpha} \frac{n}{R} \sup_{B_{2R}(x_0)} |u| \quad \forall x, y \in B_R(x_0).$$

1.23 Exercise (Schwarz reflection principle) Let Ω be a domain which is symmetric with respect to $x_n = 0$, and let $\Omega^+ = \Omega \cap \{x_n > 0\}$. Show that a function u which is harmonic in Ω^+, and continuous at $\Omega \cap \{x_n = 0\}$, has a unique harmonic extension to Ω, provided $u = 0$ on $\Omega \cap \{x_n = 0\}$.

2. Perron's method

2.1 Theorem Let Ω be bounded and suppose that the exterior ball condition is satisfied at every point of $\partial\Omega$, i.e. for every point $x_0 \in \partial\Omega$ there exists a ball B such that $\overline{B} \cap \overline{\Omega} = \{x_0\}$. Then there exists for every $\varphi \in C(\partial\Omega)$ exactly one harmonic function $u \in C(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$.

For the proof of Theorem 2.1 we need to extend the definition of sub- and superharmonic to continuous functions.

2.2 Definition A function $u \in C(\Omega)$ is called *subharmonic* if $u \leq h$ on B for every ball $B \subset\subset \Omega$ and every $h \in C(\overline{B})$ harmonic with $u \leq h$ on ∂B. The definition of superharmonic is likewise.

Clearly this is an extension of Definition 1.1, that is, every $u \in C^2(\Omega)$ with $\Delta u \geq 0$ is subharmonic in the sense of Definition 2.2. See also the exercises at the end of this section.

2.3 Theorem Suppose $u \in C(\overline{\Omega})$ is subharmonic, and $v \in C(\overline{\Omega})$ is superharmonic. If $u \leq v$ on $\partial\Omega$, then $u < v$ on Ω, unless $u \equiv v$.

Proof First we prove that $u \leq v$ in Ω. If not, then the function $u - v$ must have a maximum $M > 0$ achieved in some interior point x_0 in Ω. Since $u \leq v$ on $\partial\Omega$ and $M > 0$, we can choose a ball $B \subset\subset \Omega$ centered in x_0, such that $u - v$ is not identical to M on ∂B. Because of the Poisson Integral Formula, there exist harmonic functions $\overline{u}, \overline{v} \in C(\overline{B})$ with $\overline{u} = u$ and $\overline{v} = v$ on ∂B. By definition, $\overline{u} \geq u$ and $\overline{v} \leq v$. Hence $\overline{u}(x_0) - \overline{v}(x_0) \geq M$, while on ∂B we have $\overline{u} - \overline{v} = u - v \leq M$. Because $\overline{u}$ and $\overline{v}$ are harmonic it follows that $\overline{u} - \overline{v} \equiv M$ on B, and therefore the same holds for $u - v$ on ∂B, a contradiction.

Next we show that also $u < v$ on Ω, unless $u \equiv v$. If not, then the function $u - v$ must have a zero maximum achieved in some interior point x_0 in Ω, and, unless $u \equiv v$, we can choose x_0 and B exactly as above, reading zero for M. Again this gives a contradiction. ∎

Using again the Poisson Integral Formula we now introduce

2.4 Definition Let $u \in C(\Omega)$ be subharmonic, and let $B \subset\subset \Omega$ be a ball. The unique function $U \in C(\Omega)$ defined by

(i) $U = u$ for $\Omega \backslash B$;

(ii) U is harmonic on B,

is called the *harmonic lifting* of u in B.

2.5 Proposition The harmonic lifting U on B in Definition 2.4 is also subharmonic in Ω.

Proof Let $B' \subset\subset \Omega$ be an arbitrary closed ball, and suppose that $h \in C(\overline{B}')$ is harmonic in B', and $U \leq h$ on $\partial B'$. We have to show that also $U \leq h$ on B'. First observe that since u is subharmonic $U \geq u$ so that certainly $u \leq h$ on $\partial B'$, and hence $u \leq h$ on B'. Thus $U \leq h$ on $B' \backslash B$, and also on the boundary $\partial \Omega'$ of $\Omega' = B' \cap B$. But both U and h are harmonic in $\Omega' = B' \cap B$, so by the maximum principle for harmonic functions, $U \leq h$ on $\Omega' = B' \cap B$, and hence on the whole of B'. ∎

2.6 Proposition If $u_1, u_2 \in C(\Omega)$ are subharmonic, then $u = \max(u_1, u_2) \in C(\Omega)$ is also subharmonic.

Proof Exercise. ∎

2.7 Definition A function $u \in C(\overline{\Omega})$ is called a *subsolution* for $\varphi : \partial\Omega \to \mathbb{R}$ if u is subharmonic in Ω and $u \leq \varphi$ in $\partial\Omega$. The definition of a supersolution is likewise.

2.8 Theorem For $\varphi : \partial\Omega \to \mathbb{R}$ bounded let S_φ be the collection of all subsolutions, and let

$$u(x) = \sup_{v \in S_\varphi} v(x), \quad x \in \Omega.$$

Then $u \in C(\Omega)$ is harmonic in Ω.

Proof Every subsolution is smaller then or equal to every supersolution. Since $\sup_{\partial\Omega}\varphi$ is a supersolution, it follows that u is well defined. Now fix $y \in \Omega$ and choose a sequence of functions $v_1, v_2, v_3, \ldots \in S_\varphi$ such that $v_n(y) \to u(y)$ as $n \to \infty$. Because of Proposition 2.6 we may take this sequence to be nondecreasing in $C(\Omega)$, and larger then or equal to $\inf_{\partial\Omega}\varphi$. Let $B \subset\subset \Omega$ be a ball with center y, and let V_n be the harmonic lifting of v_n on B. Then $v_n \leq V_n \leq u$ in Ω, and V_n is also nondecreasing in $C(\Omega)$. By the Harnack Convergence Theorem, the sequence V_n converges on every ball $B' \subset\subset B$ uniformly to a harmonic function $v \in C(B)$. Clearly $v(y) = u(y)$ and $v \leq u$ in B. The proof will be complete if we show that $v \equiv u$ on B for then it follows that u is harmonic in a neighbourhood of every point y in Ω. So suppose $v \not\equiv u$ on B. Then there exists $z \in B$ such that $u(z) > v(z)$, and hence we can find $\overline{u} \in S_\varphi$ such that $v(z) < \overline{u}(z) \leq u(z)$. Define $w_n = \max(v_n, \overline{u})$ and let W_n be the harmonic lifting of w_n on B. Again it follows that the sequence W_n converges on every ball $B' \subset\subset B$ uniformly to a harmonic function $w \in C(B)$, and clearly $v \leq w \leq u$ in B, so $v(y) = w(y) = u(y)$. But v and w are both harmonic, so by the strong maximum principle for harmonic functions they have to coincide. However, the construction above implies that $v(z) < \overline{u}(z) \leq w(z)$, a contradiction. ∎

Next we look at the behaviour of the harmonic function u in Theorem 2.8 near the boundary.

2.9 Definition Let $x_0 \in \partial\Omega$. A function $w \in C(\overline{\Omega})$ with $w(x_0) = 0$ is called a barrier function in x_0 if w is superharmonic in Ω and $w > 0$ in $\overline{\Omega}\backslash\{x_0\}$.

2.10 Proposition Let u be as in Theorem 2.8, and let $x_0 \in \partial\Omega$, and suppose there exists a barrier function w in x_0. If φ is continuous in x_0, then $u(x) \to \varphi(x_0)$ if $x \to x_0$.

Proof The idea is to find a sub- and a supersolution of the form $u^\pm = \varphi(x_0) \pm \epsilon \pm kw(x)$. Fix $\epsilon > 0$ and let $M = \sup_{\partial\Omega}|\varphi|$. We first choose $\delta > 0$ such that $|\varphi(x) - \varphi(x_0)| < \epsilon$ for all $x \in \partial\Omega$ with $|x - x_0| < \delta$, and then $k > 0$ such that $kw > 2M$ on $\overline{\Omega}\backslash B_\delta(x_0)$. Clearly then u^- is a sub- and u^+ is a supersolution, so that $\varphi(x_0) - \epsilon - kw(x) \leq u(x) \leq \varphi(x_0) + \epsilon + kw(x)$ for all $x \in \Omega$. Since $\epsilon > 0$ was arbitrary, this completes the proof. ∎

2.11 Exercise Finish the proof of Theorem 2.1, and prove that the map

$$\varphi \in C(\partial\Omega) \to u \in C(\overline{\Omega})$$

is continuous with respect to the supremum norms in $C(\partial\Omega)$ and $C(\overline{\Omega})$.

2.12 Exercise Show that for a function $u \in C(\Omega)$ the following three statements are equivalent:

(i) u is subharmonic in the sense of Definition 2.2;

(ii) for every nonnegative compactly supported function $\phi \in C^2(\Omega)$ the inequality

$$\int_\Omega u \Delta \phi \geq 0$$

holds;

(iii) u satisfies the conclusion of the Mean Value Theorem, i.e. for every $B_R(y) \subset\subset \Omega$ the inequality

$$u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x) dS(x)$$

holds.

Hint: In order to deal with (ii) show that it is equivalent to the existence of a sequence $(\Omega_n)_{n=1}^\infty$ of strictly increasing domains, and a corresponding sequence of subharmonic functions $(u_n)_{n=1}^\infty \in C^\infty(\Omega_n)$, with the property that for every compact $K \subset \Omega$ there exists an integer N such that $K \subset \Omega_N$, and moreover, the sequence $(u_n)_{n=N}^\infty$ converges uniformly to u on K.

Finally we formulate an optimal version of Theorem 2.1.

2.13 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Then there exists for every $\varphi \in C(\partial\Omega)$ exactly one harmonic function $u \in C(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$. The map $\varphi \in C(\partial\Omega) \to u \in C(\overline{\Omega})$ is continuous with respect to the supremum norm.

Proof Exercise.

3. Potential theory

We recall that the fundamental solution of Laplace's equation is given by

$$\Gamma(x) = \Gamma(|x|) = \begin{cases} \dfrac{1}{2\pi} \log(|x|) & \text{if } n = 2; \\ \dfrac{1}{n(2-n)\omega_n} |x|^{2-n} & \text{if } n > 2, \end{cases}$$

and that the Newton potential of a bounded function $f : \Omega \to \mathbb{R}$ is defined by

$$w(x) = \int_\Omega \Gamma(x-y) f(y) dy.$$

Note that we have interchanged the role x and y in the previous section.

When $n = 3$, one can view $w(x)$ as the gravitational potential of a body Ω with density function f, that is, the gravitational field is proportional to $-\nabla w(x)$. This gradient is well defined because of the following theorem.

3.1 Theorem Let $f \in L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open and bounded, and let $w(x)$ be the Newton potential of f. Then $w \in C^1(\mathbb{R}^n)$ and

$$\frac{\partial w(x)}{\partial x_i} = \int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i} f(y)dy.$$

Proof First observe that

$$\frac{\partial \Gamma(x-y)}{\partial x_i} = \frac{x_i - y_i}{n\omega_n |x-y|^n} \quad \text{so that} \quad \left|\frac{\partial \Gamma(x-y)}{\partial x_i}\right| \leq \frac{1}{n\omega_n |x-y|^{n-1}}.$$

Hence

$$\int_{B_R(y)} \left|\frac{\partial \Gamma(x-y)}{\partial x_i}\right| dx \leq \int_{B_R(y)} \frac{dx}{n\omega_n |x-y|^{n-1}} = \int_{B_R(0)} \frac{dx}{n\omega_n |x|^{n-1}}$$

$$= \int_0^R \frac{1}{r^{n-1}} r^{n-1} dr = R < \infty,$$

and

$$\frac{\partial \Gamma(x-y)}{\partial x_i} \in L^1(B_R(y)) \quad \text{for all} \quad R > 0.$$

Thus the function

$$v_i(x) = \int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i} f(y)dy$$

is well defined for all $x \in \mathbb{R}^n$.

Now let $\eta \in C^\infty([0,\infty))$ satisfy

$$\begin{cases} \eta(s) = 0 \quad \text{for} \quad 0 \leq s \leq 1; \\ 0 \leq \eta'(s) \leq 2 \quad \text{for} \quad 1 \leq s \leq 2; \\ \eta(s) = 1 \quad \text{for} \quad s \geq 2, \end{cases}$$

and define

$$w_\varepsilon(x) = \int_\Omega \Gamma(x-y)\eta\left(\frac{|x-y|}{\varepsilon}\right) f(y)dy.$$

Then the integrand is smooth in x, and its partial derivates of any order with respect to x are also in $L^\infty(\Omega)$. Thus $w_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$\frac{\partial w_\varepsilon(x)}{\partial x_i} = \int_\Omega \frac{\partial}{\partial x_i}\big(\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon})f(y)\big)dy =$$

$$\int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i}\eta(\frac{|x-y|}{\varepsilon})f(y)dy + \int_\Omega \Gamma(x-y)\eta'(\frac{|x-y|}{\varepsilon})\frac{|x_i-y_i|}{\varepsilon|x-y|}f(y)dy.$$

We have for $n>2$, and for all $x\in\mathbb{R}^n$, that

$$\Big|\frac{\partial w_\varepsilon(x)}{\partial x_i} - v_i(x)\Big| = \Big|\int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i}\big(\eta(\frac{|x-y|}{\varepsilon})-1\big)f(y)dy$$

$$+\int_\Omega \Gamma(x-y)\eta'(\frac{|x-y|}{\varepsilon})\frac{x_i-y_i}{\varepsilon|x_i-y_i|}f(y)dy\Big| \le$$

$$\|f\|_\infty\{\int_{|x-y|\le 2\varepsilon}\frac{1}{n\omega_n|x-y|^{n-1}}dy + \int_{|x-y|\le 2\varepsilon}\frac{1}{n(n-2)\omega_n|x-y|^{n-2}}\frac{2}{\varepsilon}dy\}$$

$$= \|f\|_\infty\{\int_0^{2\varepsilon}\frac{1}{r^{n-1}}r^{n-1}dr + \int_0^{2\varepsilon}\frac{1}{(n-2)r^{n-2}}\frac{2}{\varepsilon}r^{n-1}dr\}$$

$$= \|f\|_\infty\{2\varepsilon + \frac{1}{n-2}\frac{1}{\varepsilon}(2\varepsilon)^2\} = \|f\|_\infty\big(2+\frac{4}{n-2}\big)\varepsilon,$$

so that

$$\frac{\partial w_\varepsilon}{\partial x_i} \to v_i \quad \text{uniformly in } \mathbb{R}^n \quad \text{as} \quad \varepsilon \downarrow 0.$$

Similarly one has

$$|w_\varepsilon(x) - w(x)| = \Big|\int_\Omega \Gamma(x-y)\big(\eta(\frac{|x-y|}{\epsilon})-1\big)f(y)dy\Big|$$

$$\le \|f\|_\infty \int_0^{2\varepsilon}\frac{1}{(n-2)r^{n-2}}r^{n-1}dr = \|f\|_\infty\frac{\varepsilon^2}{2(n-2)},$$

so that also $w_\varepsilon \to w$ uniformly on $\mathbb{R}^n$ as $\varepsilon\downarrow 0$. This proves that $\omega, v_i \in C(\mathbb{R}^n)$, and that $v_i = \partial w/\partial x_i$. The proof for $n=2$ is left as an exercise. ∎

The next step would be to show that for $f\in C(\Omega)$, $w\in C^2(\Omega)$ and $\Delta w = f$. Unfortunately this is not quite true in general. For a counterexample see Exercise 4.9 in [GT]. To establish $w\in C^2(\Omega)$ we introduce the concept of *Dini continuity*.

3.2 Definition $f:\Omega\to\mathbb{R}$ is called (locally) Dini continuous in Ω, if for every $\Omega' \subset\subset \Omega$ there exists a measurable function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ with

$$\int_0^R \frac{\varphi(r)}{r}dr < \infty \qquad \text{for all } R>0,$$

such that

$$|f(x) - f(y)| \leq \varphi(|x-y|)$$

for all x, y in Ω'. If the function φ can be chosen independent of Ω', then f is called uniformly Dini continuous in Ω.

3.3 Definition $f : \Omega \to \mathbb{R}$ is called (uniformly) Hölder continuous with exponent $\alpha \in (0,1]$ if f is (uniformly) Dini continuous with $\varphi(r) = r^{\alpha}$.

3.4 Theorem Let Ω be open and bounded, and let $f \in L^{\infty}(\Omega)$ be Dini continuous. Then $w \in C^2(\Omega)$, $\Delta w = f$ in Ω, and for every bounded open set $\Omega_0 \supset \Omega$ with smooth boundary $\partial\Omega_0$,

$$\frac{\partial^2 w(x)}{\partial x_i \partial x_j} = \int_{\Omega_0} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}(f(y) - f(x))dy - f(x)\int_{\partial\Omega_0} \frac{\partial \Gamma(x-y)}{\partial x_i}\nu_j(y)dS(y),$$

where $\nu = (\nu_1, ..., \nu_n)$ is the outward normal on $\partial\Omega_0$, and f is assumed to be zero on the complement of Ω.

Proof We give the proof for $n \geq 3$. Note that

$$\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j} = \frac{1}{n\omega_n} \frac{|x-y|^2 \partial_{ij} - n(x_i - y_i)(x_j - y_j)}{|x-y|^{n+2}},$$

so that

$$\left|\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}\right| \leq \frac{1}{\omega_n} \frac{1}{|x-y|^n},$$

which is insufficient to establish integrability near the singularity $y = x$. Let

$$u_{ij}(x) = \int_{\Omega_0} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}(f(y) - f(x))dy - f(x)\int_{\partial\Omega_0} \frac{\partial \Gamma(x-y)}{\partial x_i}\nu_j(y)dS(y).$$

Since f is Dini continuous, it is easy to see that $u_{ij}(x)$ is well defined for every $x \in \Omega$, because the first integrand is dominated by

$$\frac{1}{\omega_n} \frac{\varphi(r)}{r^n}$$

and the second integrand is smooth. Now let

$$v_{i\varepsilon}(x) = \int_{\Omega} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta\left(\frac{|x-y|}{\varepsilon}\right) f(y)dy.$$

Then

$$\left|v_{i\varepsilon}(x) - \frac{\partial w(x)}{\partial x_i}\right| = \left|\int_{\Omega} \frac{\partial \Gamma(x-y)}{\partial x_i}\left\{\eta\left(\frac{|x-y|}{\varepsilon}\right) - 1\right\} f(y)dy\right|$$

$$\leq \|f\|_\infty n\omega_n \int_0^{2\varepsilon} \frac{1}{n\omega_n r^{n-1}} r^{n-1} dr = 2\|f\|_\infty \varepsilon,$$

so that $v_{i\varepsilon} \to \partial w/\partial x_i$ uniformly in $\mathbb{R}^n$ as $\varepsilon \downarrow 0$. Extending f to Ω_0 by $f \equiv 0$ in Ω^c, we find for $x \in \Omega$, using the smoothness of $(\partial\Gamma/\partial x_i)\eta f$, that

$$\frac{\partial v_{i\varepsilon}(x)}{\partial x_j} = \int_{\Omega_0} \frac{\partial}{\partial x_j} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta(\frac{|x-y|}{\varepsilon}) f(y) dy =$$

$$\int_{\Omega_0} \{f(y) - f(x)\} \frac{\partial}{\partial x_j} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta(\frac{|x-y|}{\varepsilon}) dy +$$

$$f(x) \int_{\Omega_0} \frac{\partial}{\partial x_j} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta(\frac{|x-y|}{\varepsilon}) dy =$$

$$\int_{\Omega_0} \{f(y) - f(x)\} \frac{\partial}{\partial x_j} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta(\frac{|x-y|}{\varepsilon}) dx - f(x) \int_{\partial\Omega_0} \frac{\partial \Gamma(x-y)}{\partial x_i} \nu_j(y) dS(y),$$

provided $2\varepsilon < d(x, \partial\Omega)$, so that

$$\left|u_{ij}(x) - \frac{\partial v_{i\varepsilon}(x)}{\partial x_j}\right| = \left|\int_{\Omega_0} \{f(y) - f(x)\} \frac{\partial}{\partial x_j} \big(1 - \eta(\frac{|x-y|}{\varepsilon})\big) \frac{\partial \Gamma(x-y)}{\partial x_i} dy\right| =$$

$$\left|\int_{\Omega_0} \{\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j} \big(1 - \eta(\frac{|x-y|}{\varepsilon})\big) - \eta'(\frac{|x-y|}{\varepsilon}) \frac{x_j - y_j}{\varepsilon|x-y|} \frac{\partial \Gamma(x-y)}{\partial x_i}\} \times\right.$$

$$\left.\{f(y) - f(x)\} dy\right| \leq \int_{|x-y| \leq 2\varepsilon} \{\frac{1}{\omega_n |x-y|^n} + \frac{2}{\varepsilon n\omega_n |x-y|^{n-1}}\} \varphi(|x-y|) dy \leq$$

$$\int_0^{2\varepsilon} (\frac{n}{r^n} + \frac{2}{\varepsilon r^{n-1}}) \varphi(r) r^{n-1} dr \leq$$

$$n \int_0^{2\varepsilon} \frac{\varphi(r)}{r} dr + 2 \int_0^{2\varepsilon} \frac{r}{\varepsilon} \frac{\varphi(r)}{r} dr \leq (n+2) \int_0^{2\varepsilon} \frac{\varphi(r)}{r} dr,$$

implying

$$\frac{\partial v_{i\varepsilon}}{\partial x_j} \to u_{ij} \quad \text{as} \quad \varepsilon \downarrow 0,$$

uniformly on compact subsets of Ω. This gives $v_i \in C^1(\Omega)$ and

$$u_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} = \frac{\partial^2 w(x)}{\partial x_i \partial x_j}.$$

It remains to show that $\Delta w = f$. Fix $x \in \Omega$ and let $\Omega_0 = B_R(x) \supset \Omega$. Then

$$\Delta w(x) = \sum_{i=1}^n u_{ii}(x) = \sum_{i=1}^n \frac{\partial^2 w(x)}{\partial x_i^2} = -f(x) \sum_{i=1}^n \int_{\partial B_R(x)} \frac{\partial \Gamma(x-y)}{\partial x_i} \nu_i(y) dS(y) =$$

$$f(x)\int_{\partial B_R(x)}\sum_{i=1}^{n}\frac{\partial\Gamma(x-y)}{\partial y_i}\nu_i(y)dS(y)=f(x)\int_{\partial B_R(0)}\frac{\partial\Gamma}{\partial\nu}dS=$$

$$f(x)n\omega_nR^{n-1}\frac{1}{n\omega_nR^{n-1}}=f(x),$$

and this completes the proof. ∎

3.5 Definition Let f be locally integrable on Ω. A function $u\in C(\Omega)$ is called a weak C_0-solution of $\Delta u=f$ in Ω if, for every compactly supported $\psi\in C^2(\Omega)$, the equality

$$\int_\Omega u\Delta\psi=\int_\Omega\psi f$$

holds.

3.6 Exercise Let $f\in C(\overline{\Omega})$, and let $w\in C^1(\mathbb{R}^n)$ be the Newton potential of f. Show that w is a weak C_0-solution of $\Delta u=f$ in Ω, and that the map

$$f\in C(\overline{\Omega})\to w\in C(\overline{\Omega})$$

is compact with respect to the supremum norm in $C(\overline{\Omega})$.

4. Existence results; the method of sub- and supersolutions

We begin with some existence results which follow from the previous results. The first one combines the results of Perron's method (Theorem 2.1 and Exercise 2.13) with the continuity of the second derivatives of the Newton potential of a Dini continuous function (Theorem 3.4).

4.1 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Then the problem

$$\begin{cases}\Delta u=f & \text{in } \Omega;\\ u=\varphi & \text{on } \partial\Omega,\end{cases}$$

has a unique classical solution $u\in C^2(\Omega)\cap C(\overline{\Omega})$ for every bounded Dini continuous $f\in C(\overline{\Omega})$ and for every $\varphi\in C(\partial\Omega)$.

Proof Exercise (hint: write $u=\tilde{u}+w$, where w is the Newton potential of f). ∎

The previous theorem gives a classical solution $u\in C^2(\Omega)\cap C(\overline{\Omega})$. We recall that for f locally integrable on Ω, the function $u\in C(\overline{\Omega})$ is called a weak C_0-solution of $\Delta u=f$ in Ω if, for every compactly supported $\psi\in C^2(\Omega)$, the equality

$\int u\Delta\psi = \int \psi f$ holds. The next theorem combines Perron's method with Exercise 3.5.

4.2 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Then the problem

$$\begin{cases} \Delta u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak C_0-solution for every $f \in C(\overline{\Omega})$. The map

$$f \in C(\overline{\Omega}) \to u \in C(\overline{\Omega})$$

is compact with respect to the supremum norm in $C(\overline{\Omega})$.

Proof Exercise.

4.3 Exercise Compute the solution of

$$\begin{cases} \Delta u = -1 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the case that

$$\Omega = \Omega_{\epsilon,R} = \{x \in \mathbb{R}^n : \epsilon < |x| < R\}.$$

4.4 Exercise Let Ω be bounded and suppose that for some $\epsilon > 0$ the exterior ball condition is satisfied at every point of $\partial\Omega$ by means of a ball with radius $r \geq \epsilon$. For $f \in C(\overline{\Omega})$ let u be the unique weak C_0-solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that

$$|u(x)| \leq C||f||_\infty \text{dist}(x, \partial\Omega),$$

where C is a constant which depends only on ϵ and the diameter of Ω.

The concept of weak solutions allows one to obtain existence results for semilinear problems without going into the details of linear regularity theory, which we shall discuss later on in this course. We consider the problem

$$(D) \quad \begin{cases} \Delta u = f(x,u) & \text{in } \Omega; \\ u = 0 \quad \text{on} & \partial\Omega, \end{cases}$$

where $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is continuous.

4.5 Definition A function $\underline{u} \in C(\overline{\Omega})$ is called a weak C_0-subsolution of (D), if $\underline{u} \leq 0$ on $\partial\Omega$, and if for every compactly supported nonnegative $\psi \in C^2(\Omega)$, the equality

$$\int_\Omega \underline{u}\Delta\psi \geq \int_\Omega \psi f(x, \underline{u}(x))dx$$

holds. A C_0-supersolution $\overline{u}$ is defined likewise, but with reversed inequalities. A function u which is both a C_0-subsolution and a C_0-supersolution, is called a C_0-solution of (D).

4.6 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Let $f : \mathbb{R} \times \Omega \to \mathbb{R}$ be continuous. Suppose that Problem (D) admits a C_0-subsolution $\underline{u}$ and a C_0-supersolution $\overline{u}$, satisfying $\underline{u} \leq \overline{u}$ in Ω. Then Problem (D) has at least one C_0-solution u with the property that $\underline{u} \leq u \leq \overline{u}$.

Sketch of the proof The proof is due to Clement and Sweers and relies on an application of Schauder's fixed point theorem. Let

$$[\underline{u}, \overline{u}] = \{u \in C(\overline{\Omega}) : \ \underline{u} \leq u \leq \overline{u}\}.$$

In order to define the map T we first replace f by f^* defined by $f^*(x, s) = f(x, s)$ for $\underline{u}(x) \leq s \leq \overline{u}(x)$, $f^*(x, s) = f(x, \overline{u}(x))$ for $s \geq \overline{u}(x)$, and $f^*(x, s) = f(x, \underline{u}(x))$ for $s \leq \underline{u}(x)$. It then follows from the maximum principle that every solution for the problem with f^* must belong to $[\underline{u}, \overline{u}]$. Writing f for f^* again, the map T is now defined by $T(v) = u$, where u is the weak C_0-solution of the problem

$$\begin{cases} \Delta u = f(x, v(x)) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $e \in C^2(\overline{\Omega})$ be the (positive) solution of

$$\begin{cases} \Delta e = -1 & \text{in } \Omega; \\ e = 0 & \text{on } \partial\Omega. \end{cases}$$

Then e is bounded in Ω by some constant M. We introduce the set

$$A_k = \{u \in C(\overline{\Omega}) : \ |u(x)| \leq ke(x) \ \forall x \in \Omega\}.$$

From the maximum principle it follows again that $T : A_k \to A_k$ is well defined, provided k is larger then the supremum of $f = f^*$. The compactness of T follows from Theorem 4.2. Hence there exists a fixed point, which is the solution we seek.
∎

4.7 Exercise Fill in the details of the proof.

4.8 Exercise Prove the existence of a positive weak solution in the case that $f(u) = -u^\beta(1-u)$ with $0 < \beta < 1$.

4.9 Exercise Prove the existence of a positive weak solution in the case that $f(u) = -u^\beta$ with $0 < \beta < 1$.

5. Classical maximum principles for elliptic equations

In this section we replace the Laplacian Δ by the operator L, defined by

$$Lu = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu. \tag{5.1}$$

Here the coefficients $a(x), b(x)$ and $c(x)$ are continuous functions of $x \in \Omega$, and u is taken in $C^2(\Omega)$. It is no restriction to assume that $a_{ij}(x) = a_{ji}(x)$ for all $x \in \Omega$. The matrix

$$A(x) = \big(a_{ij}(x)\big)_{i,j=1,\ldots,N} = \begin{pmatrix} a_{11}(x) & \cdots & a_{1N}(x) \\ \vdots & & \vdots \\ a_{N1}(x) & \cdots & a_{NN}(x) \end{pmatrix}$$

is symmetric and defines a quadratic form on $\mathbb{R}^n$ for every $x \in \Omega$. Denoting the elements of $\mathbb{R}^n$ by $\xi = (\xi_1, ..., \xi_n)$, this form is given by

$$(A(x)\xi, \xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j. \tag{5.2}$$

Note that $A(x)$, being a symmetric matrix, has exactly N real eigenvalues $\lambda_1(x) \leq ... \leq \lambda_N(x)$ (counted with multiplicity), corresponding to an orthonormal basis of eigenvectors.

5.1 Definition The operator L is called *elliptic* in $x_0 \in \Omega$ if

$$(A(x_0)\xi, \xi) > 0 \quad \forall\, \xi \in \mathbb{R}^n, \xi \neq 0,$$

i.e. if the quadratic form is positive definite in x_0.

If L is elliptic at x_0, there exist numbers $0 < \lambda(x_0) \leq \Lambda(x_0)$ such that $\lambda(x_0)|\xi|^2 \leq (A(x_0))\xi, \xi) \leq \Lambda(x_0)|\xi|^2$ for all $\xi \in \mathbb{R}^n$, and it is easy to see that $0 < \lambda(x_0) = \lambda_1(x_0) \leq \lambda_N(x_0) = \Lambda(x_0)$.

5.2 Definition The operator L is called *uniformly elliptic* in Ω if there exist numbers $0 < \lambda \le \Lambda < \infty$, independent of $x \in \Omega$, such that

$$\lambda|\xi|^2 \le (A(x)\xi, \xi) \le \Lambda|\xi|^2 \quad \forall x \in \Omega \quad \forall \xi \in \mathbb{R}^n.$$

To check the uniform elipticity of a given operator L of the form (5.1), it is sufficient to check that all the eigenvalues of the matrix $A(x)$ are positive, and bounded away from zero and infinity uniformly for $x \in \Omega$. Throughout this section we shall assume that this is always so, and that for some fixed number $b_0 > 0$,

$$|b_i(x)| \le b_0 \quad \forall x \in \Omega, \quad \forall i = 1, ..., N. \tag{5.3}$$

In many ways uniformly elliptic operators resemble the Laplacian.

5.3 Theorem (weak maximum principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let L be uniformly elliptic with bounded continuous coefficients, and $c \equiv 0$ on Ω. Suppose that for some $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$Lu \ge 0 \quad \text{in} \quad \Omega.$$

Then

$$\sup_{\Omega} u = \max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof First we assume that $Lu > 0$ in Ω and that u achieves a maximum in $x_0 \in \Omega$. Then $\nabla u(x_0) = 0$, and the Hessian of u in x_0,

$$(Hu)(x_0) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)\right)_{i,j=1,\dots,N},$$

is negative semi-definite, i.e.

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)\xi_i\xi_j \le 0 \quad \forall \xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^n.$$

We claim that consequently

$$(Lu)(x_0) = \sum_{i,j=1}^{n} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \le 0,$$

contradicting the assumption. This claim follows from a lemma from linear algebra which we state without proof.

5.4 Lemma Let $A = (a_{ij})$ and $B = (b_{ij})$ be two positive semi-definite matrices, i.e.

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq 0 \quad \text{and} \quad \sum_{i,j=1}^{n} b_{ij}\xi_i\xi_j \geq 0,$$

for all $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^n$. Then

$$\sum_{i,j=1}^{n} a_{ij}\ b_{ij} \geq 0.$$

We continue with the proof of Theorem 5.3. Suppose $Lu \geq 0$ in Ω, and let

$$v(x) = e^{\gamma x_1}, \qquad \gamma > 0.$$

Then

$$(Lv)(x) = (a_{11}\gamma^2 + \gamma b_1)e^{\gamma x_1} \geq \gamma(\lambda\gamma - b_0)e^{\gamma x_1} > 0,$$

if $\gamma > b_0/\lambda$. Hence, by the first part of the proof, we have for all $\varepsilon > 0$ that

$$\sup_{\Omega}(u + \varepsilon v) = \max_{\partial\Omega}(u + \varepsilon v).$$

Letting $\varepsilon \downarrow 0$ completes the proof. ∎

5.5 Theorem Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let L be uniformly elliptic with bounded continuous coefficients, and $c \leq 0$ on Ω. Suppose that for some $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$Lu \geq 0 \quad \text{in} \quad \Omega.$$

Then

$$\sup_{\Omega} u = \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+,$$

where $u^+ = \max(u, 0)$ denotes the positive part of u.

Proof Let $\Omega^+ = \{x \in \Omega : u(x) > 0\}$. If $\Omega^+ = \emptyset$ there is nothing to prove. Assume $\Omega^+ \neq \emptyset$. For every component Ω_0^+ of Ω^+ we have

$$u = 0 \qquad \text{on} \qquad \partial\Omega_0^+ \setminus \partial\Omega,$$

so

$$\max_{\partial\Omega_0^+} u \leq \max_{\partial\Omega} u^+.$$

Define the operator L_0 by

$$L_0 u = Lu - cu.$$

Then $L_0 u \geq Lu$ in Ω_0^+ and by Theorem 5.3

$$\sup_{\Omega_0^+} u = \max_{\overline{\Omega_0^+}} u = \max_{\partial\Omega_0^+} u \leq \max_{\partial\Omega} u^+,$$

and this holds for every component of Ω^+. ■

5.6 Corollary Let Ω, u and L be as in Theorem 5.5. If

$$Lu = 0 \quad \text{in} \quad \Omega,$$

then

$$\sup_{\Omega} |u| = \max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Proof Exercise. ■

5.7 Definition $u \in C^2(\Omega)$ is called a *subsolution* of the equation $Lu = 0$ if $Lu \geq 0$ in Ω, and a *supersolution* if $Lu \leq 0$.

5.8 Corollary Let Ω and L be as in Theorem 5.5, and assume that $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subsolution and $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ a supersolution. Then $\underline{u} \leq \overline{u}$ on $\partial\Omega$ implies $\underline{u} \leq \overline{u}$ on Ω. (*Comparison principle*)

Proof Exercise. ■

5.9 Corollary For $f \in C(\Omega), \varphi \in C(\partial\Omega), \Omega$ and L as in Theorem 5.5, the problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases}$$

has atmost one solution in $C^2(\Omega) \cap C(\overline{\Omega})$.

Proof Exercise. ■

What we have done so far is based on the weak maximum principle. As in the case of the Laplacian, we shall also prove a strong maximum principle. We recall that Ω is said to satisfy a interior ball condition at $x_0 \in \partial\Omega$ if there exists a ball $B \subset \Omega$ such that $\overline{B} \cap \partial\Omega = \{x_0\}$.

5.10 Theorem (*Boundary Point Lemma*) Let Ω and L be as in Theorem 5.5, let $x_0 \in \partial\Omega$ be a point where the interior ball condition is satisfied by means of a ball $B = B_R(y)$, and $u \in C^2(\Omega) \cap C(\Omega \cup \{x_0\})$. Suppose that

$$Lu \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u(x) < u(x_0) \quad \forall x \in \Omega,$$

Then, if $u(x_0) \geq 0$, we have

$$\lim \inf_{\substack{x \to x_0 \\ x \in S_\delta}} \frac{u(x_0) - u(x)}{|x - x_0|} > 0 \quad \text{for all} \quad \delta > 0,$$

where

$$S_\delta = \{x \in \Omega : (y - x_0, x - x_0) \geq \delta R |x - x_0|\}.$$

For $c \equiv 0$ in Ω the same conclusion holds if $u(x_0) < 0$, and if $u(x_0) = 0$ the sign condition on c may be omitted. N.B. If the outward normal ν on $\partial\Omega$ and the normal derivative $\frac{\partial u}{\partial \nu}$ exist in x_0, then $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Proof Choose $\rho \in (0, R)$ and let $A = B_R(y) \backslash \overline{B_\rho(y)}$. For $x \in A$ we define

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}, \quad r = |x - y|,$$

where $\alpha > 0$ is to be specified later on. Then

$$\frac{\partial v}{\partial x_i}(x) = -2\alpha e^{-\alpha r^2}(x_i - y_i),$$

$$\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = 4\alpha^2 e^{-\alpha r^2}(x_i - y_i)(x_j - y_j) - 2\alpha e^{-\alpha r^2}\delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$, so

$$(Lv)(x) = e^{-\alpha r^2}\Big\{\sum_{i,j=1}^N 4\alpha^2(x_i - y_i)(x_j - y_j)a_{ij}(x)$$

$$-\sum_{i=1}^N 2\alpha(a_{ii}(x) + b_i(x)(x_i - y_i)) + c(x)\Big\} - c(x)e^{-\alpha R^2}.$$

Hence, if c is nonpositive, by definition 5.2 and (5.3),

$$(Lv)(x) \geq e^{-\alpha r^2}\Big\{4a^2\lambda r^2 - 2\alpha(N\Lambda + Nb_0 r) + c\Big\} \geq 0 \quad \text{in} \quad A,$$

provided α is chosen sufficiently large.

Now let

$$w_\varepsilon(x) = u(x_0) - \varepsilon v(x) \quad x \in \overline{A}, \quad \varepsilon > 0.$$

Then

$$(Lw_\varepsilon)(x) = -\varepsilon(Lv)(x) + c(x)u(x_0) \leq -\varepsilon(Lv)(x) \leq 0 \leq Lu(x) \quad \forall x \in A,$$

if $c(x)u(x_0) \leq 0$ for all $x \in \Omega$. Because $u(x) < u(x_0) \quad \forall x \in \partial B_\rho(y)$ we can choose $\varepsilon > 0$ such that

$$w_\varepsilon(x) = u(x_0) - \varepsilon v(x) \geq u(x) \quad \forall x \in \partial B_\rho(y),$$

while for $x \in \partial B_R(y)$

$$w_\varepsilon(x) = u(x_0) - \varepsilon v(x) = u(x_0) \geq u(x).$$

Hence

$$\begin{cases} Lw_\varepsilon \leq Lu & \text{in} \quad A \\ w_\varepsilon \geq u & \text{on} \quad \partial A, \end{cases}$$

so that by the comparison principle (Corollary 5.8) $u \leq w_\varepsilon$ in A, whence

$$u(x_0) - u(x) \geq \varepsilon v(x) \quad \forall x \in A.$$

Since

$$\nabla v(x_0) = 2\alpha e^{-\alpha r^2}(y - x_0),$$

this completes the proof for the case that $c \leq 0$ and $u(x_0) \geq 0$. Clearly the case $c \equiv 0$ and $u(x_0)$ arbitrary is also covered by this proof. Finally, if c is allowed to change sign, and $u(x_0) = 0$, we replace L by $\hat{L}u = Lu - c_+u$. ∎

5.11 Theorem (Strong Maximum Principle, Hopf) Let Ω and L be as in Theorem 5.5 and let $u \in C^2(\Omega)$ satisfy

$$Lu \geq 0 \quad \text{in} \quad \Omega.$$

(i) If $c \equiv 0$ in Ω then u cannot have a global maximum in Ω, unless u is constant.

(ii) If $c \leq 0$ in Ω then u cannot have a global nonnegative maximum in Ω, unless u is constant.

(iii) If u has a global maximum zero in Ω, then u is identically equal to zero in Ω.

Proof Suppose $u(y_0) = M$ and $u(x) \leq M$ for all $x \in \Omega$. Let

$$\Omega_- = \{x \in \Omega; u(x) < M\},$$

and assume that $\Omega_- \neq \emptyset$. Then $\partial\Omega_- \cap \Omega \neq \emptyset$, so there exists $y \in \Omega_-$ with $d(y, \partial\Omega_-) < d(y, \partial\Omega)$. Hence we may choose a maximal $R > 0$ such that $B_R(y) \subset \Omega_-$, and on $\partial B_R(y) \subset \Omega$ there must be a point x_0 where $u(x_0) = M$. In view of the boundary point lemma we have $\nabla u(x_0) \neq 0$, contradicting the assumption that M is a global maximum. ∎

5.12 Corollary Let Ω and L be as in Theorem 5.5 and let $u \in C^2(\Omega)$ satisfy

$$Lu = 0 \quad \text{in} \quad \Omega.$$

(i) If $c \equiv 0$ in Ω then u cannot have a global extremum in Ω, unless u is constant.

(ii) If $c \leq 0$ in Ω then $|u|$ cannot have a global maximum in Ω, unless u is constant.

5.13 Theorem (a priori estimate) Let L and Ω be as in Theorem 5.5, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $f \in C(\Omega)$. If

$$Lu \geq f \quad \text{in} \quad \Omega,$$

then

$$\sup_{\Omega} u = \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C \sup_{\Omega} |f_-|,$$

where C is a constant only depending on λ, b_0, the diameter of Ω.

Proof We may assume that

$$\Omega \subset \{x \in \mathbb{R}^n; 0 < x_1 < d\}.$$

Let L_0 be defined by $L_0 u = Lu - cu$. For $x \in \Omega$ and α a positive parameter we have

$$L_0 e^{\alpha x_1} = (\alpha^2 a_{11} + \alpha b_1) e^{\alpha x_1} \geq (\lambda\alpha^2 - b_0\alpha) e^{\alpha x_1} = \lambda\alpha(\alpha - \frac{b_0}{\lambda}) e^{\alpha x_1} \geq \lambda,$$

if $\alpha = \frac{b_0}{\lambda} + 1$. Now define $v(x)$ by

$$v = \max_{\partial\Omega} u_+ + \frac{1}{\lambda}(e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} |f_-|.$$

Then

$$Lv = L_0 v + cv \leq -\sup_{\Omega} |f_-|,$$

so

$$L(v - u) \leq -(\sup_{\Omega} |f_-| + f) \leq 0 \quad \text{in} \quad \Omega.$$

Clearly $v - u \geq 0$ on $\partial\Omega$, whence, by the weak maximum principle, $v - u \geq 0$, and

$$u \leq \max_{\partial\Omega} u^+ + \frac{1}{\lambda} e^{\alpha d} \sup_{\Omega} |f^-| \quad \text{in} \quad \Omega.$$

■

5.14 Corollary In the same situation, if $Lu = f$, then

$$\sup_{\Omega} |u| = \max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u| + C \sup_{\Omega} |f|.$$

Another important consequence of the strong maximum principle is what is widely known as Serrin's sweeping principle, which is very useful in the study of semilinear elliptic equations of the form

$$Lu + f(x, u) = 0.$$

5.15 Theorem (Serrin's sweeping principle) Let L and Ω be as in Theorem 5.5, and suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous, and has a continuous partial derivative f_u, which is locally bounded in $u \in \mathbb{R}$, uniformly in $x \in \Omega$. Suppose that there exists a family (of supersolutions) $\{\overline{u}_\lambda,\ \lambda \in [0,1]\} \subset C^2(\Omega) \cap C(\overline{\Omega})$, with $\overline{u}_\lambda$ varying continuously with λ in the sense that the map $\lambda \in [0,1] \to \overline{u}_\lambda \in C(\overline{\Omega})$ is continuous. Suppose also that, for all $\lambda \in [0,1]$,

$$L\overline{u}_\lambda + f(x, \overline{u}_\lambda) \leq 0 \ \text{ in } \Omega, \text{and } \ u_\lambda > 0 \text{ on } \partial\Omega,$$

and that there exists a (subsolution) $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$, with

$$L\underline{u} + f(x, \underline{u}) \geq 0 \ \text{ in } \Omega, \text{and } \ \underline{u} \leq 0 \text{ on } \partial\Omega.$$

Then, if $\underline{u} \leq \overline{u}_\lambda$ in Ω for some $\lambda = \lambda_0 \in [0,1]$, it follows that $\underline{u} < \overline{u}_\lambda$ in Ω for all $\lambda \in [0,1]$.

Proof We first prove the statement for $\lambda = \lambda_0$. Assume it is false. In view of the assumptions this means that the function $w = \overline{u}_\lambda - \underline{u}$ has a global maximum zero in some interior point of Ω. But w is easily seen to satisfy the equation

$$Lw + c_\lambda(x)w \leq 0 \ \text{ in } \Omega,$$

with

$$c_\lambda(x) = \int_0^1 f_u(x, s\overline{u}_\lambda(x) + (1-s)\underline{u}(x))ds,$$

which is a bounded continuous function in Ω. By Theorem 5.11(iii), it follows that w is identically equal to zero, a contradiction. Thus the statement is proved for $\lambda = \lambda_0$.

Next vary λ starting at $\lambda = \lambda_0$. As long as $\underline{u} < \overline{u}_\lambda$ there is nothing to prove, the only thing that can go wrong is, that for some $\lambda = \lambda_1$, with $|\lambda_1 - \lambda_0|$ chosen minimal, $\overline{u}_\lambda$ touches $\underline{u}$ again from below. But this is ruled out by the same argument as above. ∎

6. More regularity, Schauder's theory for general elliptic operators

We have seen that in order to obtain solutions with continuous second order derivatives, continuity of the right hand side was not sufficient. This goes back to Theorem 3.4, which asks for a modulus of continuity without returning one. In order to extend existence results for the Laplacian to more general operators, it is necessary to fill this gap. We shall prove in detail that the Newton potential of a Hölder continuous function has Hölder continuous second order derivatives. The proofs of the consequences of this result are only sketched.

Here and from now on, unless stated otherwise, Ω is a bounded domain. For $f : \Omega \to \mathbb{R}$ and $0 < \alpha < 1$ define the dimensionless seminorm

$$[f]_\alpha = [f]_{\alpha,\Omega} = d^\alpha \sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$

where d is the diameter of Ω, and let

$$C^\alpha(\overline{\Omega}) = \{f : \Omega \to \mathbb{R}, [f]_\alpha < \infty\}.$$

This space is a Banach space with respect to the norm

$$||f||_\alpha = ||f||_{\alpha,\Omega} = ||f||_{0,\Omega} + [f]_\alpha.$$

where

$$||f||_{0,\Omega} = \sup_{x\in\Omega} |f(x)|.$$

The set of functions which are in $C^\alpha(\overline{\Omega}')$ for every subdomain $\Omega' \subset\subset \Omega$ is denoted by $C^\alpha(\Omega)$. Note that this is only a vector space. We also define the seminorms

$$[f]_{1,\Omega} = d \sup_{x\in\Omega, i=1,\ldots,n} |\frac{\partial f(x)}{\partial x_i}|; \quad [f]_{2,\Omega} = d^2 \sup_{x\in\Omega, i,j=1,\ldots,n} |\frac{\partial^2 f(x)}{\partial x_i \partial x_j}|;$$

$$[f]_{2,\alpha,\Omega} = d^2 \sup_{i,j=1,\ldots,n} [\frac{\partial^2 f}{\partial x_i \partial x_j}]_{\alpha,\Omega}.$$

and the norm

$$||f||_{2,\alpha,\Omega} = ||f||_{0,\Omega} + [f]_{1,\Omega} + [f]_{2,\Omega} + [f]_{2,\alpha,\Omega}.$$

The space of all functions with second derivatives in $C^\alpha(\Omega)$ for which the latter norm is finite is the Banach space $C^{2,\alpha}(\overline{\Omega})$. Again the vector space of functions for which this is only valid on subdomains $\Omega' \subset\subset \Omega$ is denoted by $C^{2,\alpha}(\Omega)$.

6.1 Theorem Let $f \in C^\alpha(\overline{B}_{2R})$. Then the Newton potential w of f belongs to $C^\alpha(\overline{B}_R)$, and there exists a constant $C(n,\alpha)$, such that

$$||\frac{\partial^2 w}{\partial x_i \partial x_j}||_{\alpha,B_R} \leq C(n,\alpha)||f||_{\alpha,B_{2R}}.$$

Proof We recall (Theorem 3.4) that

$$\frac{\partial^2 w(x)}{\partial x_i \partial x_j} = \int_{B_{2R}} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}(f(y)-f(x))dy - f(x)\int_{\partial B_{2R}} \frac{\partial \Gamma(x-y)}{\partial x_i}\nu_j(y)dS(y), \tag{6.1}$$

and that the derivatives of Γ satisfy

$$\left|\frac{\partial \Gamma(x-y)}{\partial x_i}\right| \leq \frac{1}{n\omega_n |x-y|^{n-1}}, \quad \left|\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}\right| \leq \frac{1}{\omega_n}\frac{1}{|x-y|^n}.$$

For $x \in B_R$ we now first estimate

$$\left|\frac{\partial^2 w(x)}{\partial x_i \partial x_j}\right| \leq \frac{[f]_{\alpha,B_{2R}}}{(4R)^\alpha}\int_{B_{2R}} \frac{1}{\omega_n}|x-y|^{\alpha-n}dy + |f(x)|\int_{\partial B_{2R}} \frac{1}{n\omega_n}|x-y|^{1-n}dS(y)$$

$$\leq \frac{n}{\alpha}(\frac{3}{4})^\alpha [f]_{\alpha,B_{2R}} + 2^{n-1}|f(x)| \leq \max(\frac{n}{\alpha}(\frac{3}{4})^\alpha, 2^{n-1})||f||_{\alpha,B_{2R}}. \tag{6.2}$$

To get a similar estimate for the Hölder seminorm is a bit more complicated. We begin with formula (6.1) for two values of x in B_R, say x and $\bar{x}$. Subtraction yields

$$\frac{\partial^2 w(x)}{\partial x_i \partial x_j} - \frac{\partial^2 w(\bar{x})}{\partial x_i \partial x_j} = J_1 + J_2 + J_3 + J_4 + J_5 + J_6,$$

where, writing

$$\delta = |x-\bar{x}|, \quad \xi = \frac{x+\bar{x}}{2},$$

$$J_1 = f(x)\int_{\partial B_{2R}} (\frac{\partial \Gamma(x-y)}{\partial x_i} - \frac{\partial \Gamma(\bar{x}-y)}{\partial x_i})\nu_j(y)dS(y),$$

$$J_2 = (f(x)-f(\bar{x}))\int_{\partial B_{2R}} \frac{\partial \Gamma(\bar{x}-y)}{\partial x_i}\nu_j(y)dS(y),$$

$$J_3 = \int_{B_\delta(\xi)} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}(f(x)-f(y))dy,$$

$$J_4 = \int_{B_\delta(\xi)} \frac{\partial^2 \Gamma(\bar{x}-y)}{\partial x_i \partial x_j}(f(y)-f(\bar{x}))dy,$$

$$J_5 = (f(x)-f(\bar{x}))\int_{B_{2R}-B_\delta(\xi)} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}dy,$$

and

$$J_6 = \int_{B_{2R}-B_\delta(\xi)} (\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j} - \frac{\partial^2 \Gamma(\bar{x}-y)}{\partial x_i \partial x_j})(f(\bar{x})-f(y))dy.$$

Next we estimate each of these six integrals by a constant times

$$[f]_{\alpha,B_{2R}}\big(\frac{\delta}{2R}\big)^{\alpha}.$$

This will complete the proof.

Observe that by the mean value theorem there exists a $\hat{x}$ between x and $\bar{x}$ such that

$$J_1 = f(x)\int_{\partial B_{2R}} \big(\nabla\frac{\partial\Gamma(\hat{x}-y)}{\partial x_i}, x-\bar{x}\big)\nu_j(y)dS(y),$$

whence

$$|J_1| \le |f(x)||x-\bar{x}|\int_{\partial B_{2R}} \frac{n}{\omega_n}|\hat{x}-y|^{-N}dS(y)$$

$$\le |f(x)|\frac{|x-\bar{x}|}{2R}n^2 2^n \le n^2 2^n [f]_{\alpha,B_{2R}}\big(\frac{\delta}{2R}\big)^{\alpha}.$$

For J_2 we have, as in the first part of the proof, that

$$|J_2| \le |f(x)-f(\bar{x})|2^{n-1} \le 2^{n-1}[f]_{\alpha}\big(\frac{\delta}{2R}\big)^{\alpha}.$$

Next for J_3,

$$|J_3| \le [f]_{\alpha,B_{2R}}(4R)^{-\alpha}\int_{B_\delta(\xi)} \frac{1}{\omega_n}|x-y|^{\alpha-n}dy \le$$

$$[f]_{\alpha,B_{2R}}(4R)^{-\alpha}\int_{B_{\frac{3\delta}{2}}(x)} \frac{1}{\omega_n}|x-y|^{\alpha-n}dy = [f]_{\alpha,B_{2R}}\big(\frac{3}{4}\big)^{\alpha}\frac{n}{\alpha}\big(\frac{\delta}{2R}\big)^{\alpha},$$

and likewise, for J_4,

$$|J_3| \le [f]_{\alpha,B_{2R}}\big(\frac{3}{4}\big)^{\alpha}\frac{n}{\alpha}\big(\frac{\delta}{2R}\big)^{\alpha}.$$

For J_5 we have, using Green's formula,

$$J_5 = (f(x)-f(\bar{x}))\Big(\int_{\partial B_{2R}} \frac{\partial\Gamma(x-y)}{\partial x_i}\nu_j(y)dS(y) - \int_{\partial B_\delta(\xi)} \frac{\partial\Gamma(x-y)}{\partial x_i}\nu_j(y)dS(y)\Big),$$

so that, exactly as in the estimate for J_2,

$$|J_5| \le 2|f(x)-f(\bar{x})|2^{n-1} \le 2^n[f]_{\alpha}\big(\frac{\delta}{2R}\big)^{\alpha}.$$

Finally, for J_6, applying the mean value theorem again, for some $\hat{x}$ between x and $\bar{x}$,

$$J_6 = \int_{B_{2R}-B_\delta(\xi)} \big(\nabla\frac{\partial^2\Gamma(\hat{x}-y)}{\partial x_i\partial x_j}, x-\bar{x}\big)(f(\bar{x})-f(y))dy,$$

so that

$$|J_6| \leq c(n,\alpha)|x-\bar{x}| \int_{B_{2R}-B_\delta(\xi)} \frac{|f(\bar{x})-f(y)|}{|\hat{x}-y|^{n+1}} dy \leq$$

$$c(n,\alpha)|x-\bar{x}|[f]_{\alpha,B_{2R}}(4R)^{-\alpha} \int_{|y-\xi|>\delta} \frac{|\bar{x}-y|^\alpha}{|\hat{x}-y|^{n+1}} dy \leq$$

$$c(n,\alpha)\delta[f]_{\alpha,B_{2R}}(4R)^{-\alpha}(\frac{3}{2})^\alpha 2^{n+1} \int_{|y-\xi|>\delta} |\xi-y|^{\alpha-n-1} dy =$$

$$\frac{c(n,\alpha)}{1-\alpha} n\omega_n \delta [f]_{\alpha,B_{2R}}(\frac{3}{4})^\alpha 2^{n+1} \big(\frac{\delta}{2R}\big)^\alpha.$$

■

6.2. Corollary If $u \in C^2(\mathbb{R}^n)$ has compact support and $f = \Delta u$ then $u \in C^{2,\alpha}(\mathbb{R}^n)$, and, if the support of u is contained in a ball B with radius R,

$$||\frac{\partial^2 u}{\partial x_i \partial x_j}||_{\alpha,B} \leq C(n,\alpha)||f||_{\alpha,B}; \quad ||\frac{\partial u}{\partial x_i}||_{0,B} \leq C(n)R^2||f||_{0,B}.$$

Proof Exercise: use the fact that u in this case is the Newton potential of f. The first estimate follows from Theorem 6.1, the second from Theorem 3.1. ■

6.3 Corollary If $f \in C^\alpha(\Omega)$ and $u \in C^2(\Omega)$ is a solution of $\Delta u = f$ in Ω, then for every ball $B_{2R}(x_0) \subset\subset \Omega$ the following estimate holds true.

$$||u||_{2,\alpha,B_R(x_0)} \leq C(n,\alpha)\big(||u||_{0,B_{2R}(x_0)} + R^2||f||_{\alpha,B_{2R}(x_0)}\big).$$

Proof Write $u = v + w$ where w is the Newton potential of f on B_{2R}. For w the estimate follows from the previous results in this section. For (the harmonic function) v the estimates follow from Corollary 1.16 combined with Exercise 1.22.
■

So far the estimates give no information about the behaviour near the boundary. To improve on this we introduce new norms. Let

$$d_x = \inf_{z\in\partial\Omega} |x-z|; \quad d_{x,y} = \min(d_x, d_y),$$

write D for a general partial derivative, and $|D|$ for its order. Define

$$[f]^*_{k,\Omega} = [f]^*_{k,0,\Omega} = \sup_{x\in\Omega,|D|=k} d_x^k|Df(x)|; \quad ||f||^*_{k,\Omega} = ||f||^*_{k,0,\Omega} = \sum_{j=0}^{k}[f]^*_{j,\Omega},$$

and

$$[f]^*_{k,\alpha,\Omega} = \sup_{x\neq y\in\Omega, |D|=k} d^{k+\alpha}_{x,y} \frac{|Df(x) - Df(y)|}{|x-y|^\alpha}; \quad ||f||^*_{k,\alpha,\Omega} = ||f||^*_{k,\Omega} + [f]^*_{k,\alpha,\Omega}.$$

and finally

$$||f||^{(k)}_{\alpha,\Omega} = \sup_{x\in\Omega} d^k_x|f(x)| + \sup_{x\neq y\in\Omega} d^{k+\alpha}_{x,y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

We shall not introduce the corresponding spaces. Whenever we write these norms, it is understood that they may be infinite.

6.4. Theorem For $u \in C^2(\Omega)$ and $f \in C^\alpha(\Omega)$ satisfying $\Delta u = f$ we have $u \in C^{2,\alpha}(\Omega)$ and

$$||u||^*_{2,\alpha,\Omega} \leq C(n,\alpha)\big(||u||_{0,\Omega} + ||f||^{(2)}_{\alpha,\Omega}\big).$$

Proof Apply the previous estimates to balls with center x and radius $d_x/3$. ∎

The previous theorem gives information about the behaviour near the boundary. Next we look at the behaviour at the boundary. First we take very simple boundaries and formulate a variant of Theorem 6.1 for the intersection of a ball and a half space. We use the notation $\Omega^+ = \{x \in \Omega : \ x_n > 0\}$.

6.5 Theorem Let B_R be a ball with center lying in $(\mathbb{R}^n)^+$. Then Theorem 6.1 holds true with B_R and B_{2R} replaced by B^+_R and B^+_{2R} respectively.

Proof Identical to the proof of Theorem 6.1 if i and j are not both equal to n. Note that on the flat part of the boundary all components of ν are zero except for ν_n. Finally

$$\frac{\partial^2 w}{\partial x_n^2} = f - \sum_{j=1}^{n-1} \frac{\partial^2 w}{\partial x_j^2},$$

so the estimate for the left hand side follows from the estimates for the terms on the right hand side. This completes the proof. ∎

6.6 Theorem The estimate in Corollary 6.3 remains true if we replace B_R and B_{2R} by B^+_R and B^+_{2R} respectively, and assume that $u = 0$ at $x_n = 0$.

Proof Write $u = v + \tilde{w}$, where

$$\tilde{w}(x) = \int_{B_{2R}} (\Gamma(x-y) - \Gamma(\bar{x}-y)) f(y)dy.$$

Here $\bar{x}$ denotes the reflection of x in $\{x_n = 0\}$. Note that $G(x, y) = \Gamma(x - y) - \Gamma(\bar{x} - y)$ is really the Green's function for $\Delta u = f$ on the half space with Dirichlet boundary data on $\{x_n = 0\}$. Using also the Schwarz reflection principle for harmonic functions (Exercise 1.23), the proof is then similar to the proof of Corollary 6.3. ■

6.7 Theorem Replacing Ω by Ω^+ and assuming $u = 0$ and u and f continuous at $x_n = 0$, the estimate in Theorem 6.4 remains valid, if in the definition of d_x and $d_{x,y}$ the boundary $\partial\Omega$ is replaced by $(\partial\Omega)^+$.

In other words, this theorem gives Hölder continuity of the second derivatives up to flat parts of the boundary. Using local coordinate transformations, this estimate upto the boundary can be extended to boundaries which are locally the graph of a $C^{2,\alpha}$-function, and thereby obtaining the solvability of the Dirichlet problem for $\Delta u = f$ in $C^{2,\alpha}(\overline{\Omega})$ if $f \in C^{\alpha}(\overline{\Omega})$. Note however that the Laplacian is not invariant under such a transformation, so this is far from straightforward. For the ball however this result can be obtained directly.

6.8 Theorem Let $u \in C^2(B) \cap C(\overline{\Omega})$ be a solution of $\Delta u = f$ in B with $u = 0$ on ∂B. Then $u \in C^{2,\alpha}(\overline{B})$ if $f \in C^{\alpha}(\overline{B})$.

Proof We may assume that $0 \in \partial B$ and that $B \subset (\mathbb{R}^n)^+$. Then

$$x \to x^* = \frac{x}{|x|^2}$$

is a bijection between B and $B^* = \{x_n > \frac{1}{2R}\}$, which has a flat boundary. The function $v(x) = x^{2-n} v(x^*)$ satisfies

$$\Delta v(x) = |x|^{-(n+2)} f(\frac{x}{|x|^2}).$$

Thus the previous results apply. ■

The results so far are a starting point for obtaining existence results for more general elliptic boundary value problems. We consider

$$(D) \quad \begin{cases} Lu = f & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c$$

is an elliptic operator, and f and φ are given functions. We shall use the method of continuity to obtain an existence result for general L from the results in the case $L = \Delta$. From now on the domains under consideration are assumed to be bounded with

$$\partial\Omega \in C^{2,\alpha}.$$

We shall make the following assumptions on L.

H1. The functions a_{ij}, b_i, and c belong to $C^\alpha(\overline{\Omega})$. (Thus their norms in this space are majorized by a fixed number Λ.)

H2. L is uniformly elliptic, i.e.

$$\exists \lambda > 0 : \quad \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall x \in \Omega \ \forall \xi \in \mathbb{R}^n,$$

and $c \leq 0$ in Ω. Also $a_{ij} = a_{ji}$.

Until the end of this section we shall say that Ω is α-regular, if, in the case that $L = \Delta$, problem (D) with $\varphi = 0$ has a (unique) solution $u \in C^{2,\alpha}(\overline{\Omega})$ for every $f \in C^\alpha(\overline{\Omega})$. At the end of this section every bounded domain with $\partial\Omega \in C^{2,\alpha}$ will turn out to have this property.

Next we consider the operator L as a bounded linear operator

$$L: \ V = \{u \in C^{2,\alpha}(\overline{\Omega}) : \ u = 0 \text{ on } \partial\Omega\} \ \to \ W = C^\alpha(\overline{\Omega}).$$

Note that we restrict our attention to the case of zero boundary data. For boundary data given by a function $\varphi \in C^{2,\alpha}(\overline{\Omega})$ the general case reduces to the case $\varphi = 0$ by considering $u - \varphi$ as new unknown. If the estimate

$$||u||_V \leq C||tLu + (1-t)\Delta u||_W \quad \forall u \in V \ \forall t \in [0,1]$$

holds, the method of continuity (Theorem A.12) yields that $L : V \to W$ is invertible if and only if $\Delta : V \to W$ is invertible. We have already seen that for balls the latter is indeed true. By proving the estimate we then obtain the invertibility for L, and hence the solvability of the Dirichlet problem for L on balls in the class $C^{2,\alpha}(\overline{B})$. Finally we extend this result to more general domains by using Perron's method, with Δ replaced by L. Crucial however is the estimate $||u||_V \leq ||Lu||_W$, which follows from the Schauder estimates given below. These estimates are given for one fixed operator L, but applied to L_t it is easy to see that the constants can be taken independent of $t \in [0,1]$.

6.9. Theorem (Schauder estimate) Let Ω be a bounded domain with boundary $\partial\Omega \in C^{2,\alpha}$. Then the following estimate holds for solutions $u \in C^{2,\alpha}(\overline{\Omega})$ of (D).

$$||u||_{2,\alpha,\Omega} \leq C(n,\alpha,\lambda,\Lambda,\Omega)\big(||u||_{0,\Omega} + ||\varphi||_{2,\alpha,\Omega} + ||f||_{\alpha,\Omega}\big).$$

This estimate is again derived in two steps, first an interior estimate, and then an estimate upto the boundary. The Schauder interior estimate is given in the next lemma. From this estimate, estimates upto the boundary can also be obtained in much the same way as for the Laplacian in the first part of this section. For the details we refer to Gilbarg and Trudinger, Chapter 6.

6.10 Lemma Suppose $u \in C^{2,\alpha}(\Omega)$ is a bounded solution of $Lu = f$, and suppose that

$$\max_{i,j=1,\dots,n} ||a_{i,j}||^{(0)}_{0,\alpha,\Omega} + \max_{i=1,\dots,n} ||b_i||^{(1)}_{0,\alpha,\Omega} + ||c||^{(2)}_{0,\alpha,\Omega} \leq \Lambda^*.$$

Then, without any sign restriction on c, the following estimate holds.

$$||u||^*_{2,\alpha,\Omega} \leq C(n,\alpha,\lambda,\Lambda^*)\big(||u||_{0,\Omega} + ||f||^{(2)}_{0,\alpha,\Omega}\big).$$

Proof The proof is lengthy and technical, and will only be sketched here. It is based on freezing the coefficients. We first observe that if

$$L_0 = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where $(a_{i,j})$ is a symmetric matrix with only positive eigenvalues, then a linear transformation of coordinats transforms L_0 into Δ. Thus the estimates for solutions of $\Delta u = f$ also hold for solutions of $L_0 u = f$.

Also we shall use the following interpolation inequality.

$$||u||^*_{j,\beta,\Omega} \leq C(\epsilon, j, \beta)\big(||u||_{0,\Omega} + \epsilon[u]^*_{2,\alpha,\Omega}\big), \tag{6.3}$$

for

$$\epsilon > 0,\ j = 0,1,2,\ 0 \leq \beta < 1,\ j + \beta < 2 + \alpha.$$

Because of this inequality we only (sic!) have to estimate $[u]^*_{2,\alpha}$, and because of compactness arguments it suffices to prove the estimate for all $\Omega' \subset\subset \Omega$. For Ω' this seminorm is finite in view of the assumption that $u \in C^{2,\alpha}(\Omega)$.

We choose two points x_0 and y_0 in Ω with x_0 closer to the boundary then y_0. Let $d = \mu d_{x_0}$ where $\mu \in (0, \frac{1}{2}]$ is to be chosen later. On the ball $B = B_d(x_0)$ we rewrite $Lu = f$ as $L_0 u = F$, where L_0 is as above with coefficients $a_{i,j}(x_0)$, and

$$F(x) = \sum_{i,j=1}^{n} (a_{ij}(x_0) - a_{ij}(x)) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} - c(x)u(x) + f(x). \tag{6.4}$$

We apply Theorem 6.4 to the equation $L_0 u = F$, and obtain

$$d_{x_0,y_0}^{2+\alpha} \frac{|\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) - \frac{\partial^2 u}{\partial x_i \partial x_j}(y_0)|}{|x_0 - y_0|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \big(||u||_{0,B_d(x_0)} + ||F||_{\alpha,B_d(x_0)}^{(2)} \big) + \frac{4}{\mu^\alpha} [u]_{2,\Omega}^*. \tag{6.5}$$

The first term on the right hand side follows from Theorem 6.4 if $|x_0 - y_0| < \frac{d}{2}$. For $|x_0 - y_0| \geq \frac{d}{2}$ the triangle inequality gives that the second term on the right hand side majorizes the left hand side.

Before we estimate $||F||_{\alpha,B}^{(2)}$, observe that, for any $g \in C^\alpha(\Omega)$,

$$||g||_{\alpha,B}^{(2)} \leq 8\mu^2 ||g||_{\alpha,\Omega}^{(2)}, \tag{6.6}$$

while for any $f, g \in C^\alpha(\Omega)$ and $\sigma + \tau > 0$,

$$||fg||_{\alpha,\Omega}^{(\sigma+\tau)} \leq ||f||_{\alpha,\Omega}^{(\sigma)} ||g||_{\alpha,\Omega}^{(\tau)}. \tag{6.7}$$

We now begin with the second order terms in F. We have

$$||(a_{ij}(x_0) - a_{ij}(x)) \frac{\partial^2 u(x)}{\partial x_i \partial x_j}||_{\alpha,B}^{(2)} \leq ||a_{ij}(x_0) - a_{ij}(x)||_{\alpha,B}^{(0)} ||\frac{\partial^2 u(x)}{\partial x_i \partial x_j}||_{\alpha,B}^{(2)}. \tag{6.8}$$

The second factor on the right is majorized by

$$4\mu^2 [u]_{2,\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha,\Omega}^*,$$

and the first by

$$4\mu^\alpha \Lambda^*,$$

because of the assumptions on the coefficients. Thus all the second order terms together can be estimated by

$$\sum_{i,j=1,\ldots,n}^{n} ||(a_{ij}(x_0) - a_{ij}(x)) \frac{\partial^2 u(x)}{\partial x_i \partial x_j}||_{\alpha,B}^{(2)} \leq 32 n^2 \Lambda^* \mu^{2+\alpha} \big([u]_{2,\Omega}^* + [u]_{2,\alpha,\Omega}^* \big) \leq$$

(using the interpolation inequality with $\epsilon = \mu^\alpha$)

$$32 n^2 \Lambda^* \mu^{2+\alpha} \big(C(\mu) ||u||_{0,\Omega} + 2\mu^\alpha [u]_{2,\alpha,\Omega}^* \big). \tag{6.9}$$

By similar manipulations we obtain for the first order terms in F

$$\sum_{i,=1,\ldots,n}^{n} ||b_i(x) \frac{\partial u(x)}{\partial x_i}||_{\alpha,B}^{(2)} \leq 8 n \Lambda^* \mu^2 \big(C(\mu) ||u||_{0,\Omega} + 2\mu^\alpha [u]_{2,\alpha,\Omega}^* \big), \tag{6.10}$$

and for the two zero order terms

$$||cu||^{(2)}_{\alpha,B} \leq 8\Lambda^*\mu^2\big(C(\mu)||u||_{0,\Omega} + 2\mu^\alpha[u]^*_{2,\alpha,\Omega}\big), \tag{6.11}$$

and

$$||f||^{(2)}_{\alpha,B} \leq 8\mu^2||f||_{\alpha,\Omega}. \tag{6.12}$$

Combining (6.9-12) we have

$$||F||^{(2)}_{\alpha,B} \leq (n^2+8n+8)\Lambda^*\mu^{2+2\alpha}[u]^*_{2,\alpha,\Omega}+C(n)\Lambda^*C(\mu)||u||_{0,\Omega}+8\mu^2||f||_{\alpha,\Omega}, \tag{6.13}$$

which we can plug in (6.5) to yield an estimate in which d no longer appears:

$$d^{2+\alpha}_{x_0,y_0}\frac{|\frac{\partial^2 u}{\partial x_i\partial x_j}(x_0) - \frac{\partial^2 u}{\partial x_i\partial x_j}(y_0)|}{|x_0-y_0|^\alpha} \leq$$

$$C(n,\Lambda^*)\mu^\alpha[u]^*_{2,\alpha,\Omega} + C(n,\Lambda^*,\mu)||u||_{0,\Omega} + 8\mu^2||f||^{(2)}_{\alpha,\Omega}. \tag{6.14}$$

Here we have used $\epsilon = \mu^{2\alpha}$ in the interpolation inequality to get rid of $||u||_{2,\alpha,\Omega}$.

Now choose μ so small that $C(n,\Lambda^*)\mu^\alpha < \frac{1}{2}$. Taking the supremum over x_0 and y_0 and absorbing the first term on the right in the left hand side, it follows that

$$[u]^*_{2,\alpha,\Omega} \leq 2C(n,\Lambda^*,\mu)||u||_{0,\Omega} + 16\mu^2||f||^{(2)}_{\alpha,\Omega}, \tag{6.15}$$

which completes the proof. ∎

We now finish our application of the method of continuity. Theorem 6.9 does not yet give the precise estimate needed in this method, because of the appearance of $|u|_{0,\Omega}$ on the right hand side. This is where the sign of c comes into play. We have seen in the previous section on maximum principles, that solutions are a priori bounded, provided $c \leq 0$ on Ω (Theorem 5.13 and Corollary 5.14), by a constant times the suprema of the right hand side f and the boundary conditions φ. Combining the Schauder estimate with this apriori bound, the estimate for L_t follows.

6.11 Conclusion If Ω is a α-regular domain, and L satisfies assumptions H1 and H2, then problem (D) has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ for every $f \in C^\alpha(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$.

6.12 Remark In particular, since we have already seen that balls are α-regular (Theorem 6.8), problem (D) is solvable in $C^{2,\alpha}(\overline{B})$ for every ball B. A variant of Perron's method then gives the solvability in $C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ for every $f \in C^\alpha(\Omega)$ and $\varphi \in C(\partial\Omega)$, provided one can construct barrier functions in every point of the boundary. For $C^{2,\alpha}$ boundaries this is certainly the case. Using local transformations, it can then be shown that the solution u is in $C^{2,\alpha}(\overline{\Omega})$, whenever $f \in C^{\alpha}(\overline{\Omega})$. Thus every bounded domain with $C^{2,\alpha}$ boundary is α-regular. We summarize this result in the following theorem.

6.13 Theorem Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{2,\alpha}$, let $f \in C^{\alpha}(\overline{\Omega})$, let $\varphi \in C^{2,\alpha}(\overline{\Omega})$, and let L satisfy the assumptions H1 and H2. Then there exists a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ of the problem

$$\begin{cases} Lu = f & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Moreove, the map

$$(f, \varphi) \in C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}) \ \rightarrow \ u \in C^{2,\alpha}(\overline{\Omega})$$

is continuous.

We return to the problem

$$(D) \quad \begin{cases} \Delta u + f(x,u) = 0 & \text{in } \Omega; \\ u = 0 \quad \text{on} & \partial\Omega. \end{cases}$$

We shall deal now with classical solutions. We assume that f is uniformly Hölder continuous and $\partial\Omega \in C^{2,\alpha}$. A *classical subsolution* is a function $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $\Delta\varphi(x) + f(x, \varphi(x)) \geq 0$ is Ω and $\varphi \leq 0$ on $\partial\Omega$. For the defintion of a *classical supersolution* the inequalities have to be reversed. The basic result is that if a classical subsolution is below a classical supersolution, then there exists at least one classical solution in between. In Section 4 this was shown for weak C_0-solutions applying Schauder's theorem. However this did not give any additional information about the structure of the set of solutions. Here we shall show that one always has a *maximal* and a *minimal* solution by changing the iteration procedure as to make it monotone.

Suppose that the nonlinearity satisfies the assumption:

$$\text{There exists } c > 0 \text{ such that the function } f(x,u) + cu \text{ is increasing in } u. \tag{6.16}$$

Consider the problem

$$(L_c(v)) \quad \begin{cases} -\Delta u + cu = f(x, v(x)) + cv(x) & \text{in } \Omega; \\ u = 0 \quad \text{on} & \partial\Omega. \end{cases}$$

6.14 Exercise (i) Prove that for $\alpha > 0$ sufficiently small this problem has a unique solution $u \in C^{2,\alpha}(\Omega) \cap C^{\alpha}(\overline{\Omega})$ for every $v \in C^{\alpha}(\overline{\Omega})$.

(ii) Denoting the solution u as $T_\omega v$ show that the map $T = T_\omega : C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ has the following property:

$$u_1 \leq u_2 \text{ in } \Omega \;\Rightarrow Tu_1 < Tu_2 \text{ in } \Omega \text{ unless } u_1 \equiv u_2. \tag{6.17}$$

(iii) Show that if $\varphi \in C^\alpha(\overline{\Omega})$ is a classical subsolution then $T\varphi$ is also a classical subsolution.

(iv) Show that $T\varphi > \varphi$ in Ω, unless $T\varphi \equiv \varphi$ is a classical solution.

It follows that if $\varphi \in C^\alpha(\overline{\Omega})$ and $\psi \in C^\alpha(\overline{\Omega})$ are respectively a classical sub- and supersolution, with φ and ψ not themselves a solution and $\varphi \leq \psi$, then, defining the sequences u_n and v_n in $C^\alpha(\overline{\Omega})$ by

$$u_0 = \varphi, \quad u_n = Tu_{n-1}, \quad v_0 = \psi, \quad v_n = Tv_{n-1}, \tag{6.18}$$

we have

$$\varphi = u_0 < u_1 < u_2 < u_3 < \ldots < v_3 < v_2 < v_1 < v_0 = \psi \quad \text{in } \Omega. \tag{6.19}$$

6.15 Theorem Let $u(x) = \lim u_n(x)$ and $v = \lim v_n(x)$. Then u and v are classical solutions in $C^\alpha(\overline{\Omega})$. If $u \equiv v$, then u is the only classical solution between φ and ψ. If not then $u < v$ in Ω, and every other classical solution between φ and ψ is necessarily strictly between u and v.

Proof In this proof we only consider solutions in $C^\alpha(\overline{\Omega})$. It follows from the Schauder estimates that the sequences are bounded in $C^\alpha(\overline{\Omega})$. Obviously they converge uniformly to limit functions u and v in $C^\alpha(\overline{\Omega})$. Choosing a smaller α we have from the compactness of the embedding

$$C^\alpha(\overline{\Omega}) \to C^\beta(\overline{\Omega}) \quad \text{for} \quad 0 < \beta < \alpha,$$

that the convergence is in fact with respect to the $C^\alpha(\overline{\Omega})$-norm. Clearly then both u and v are classical solutions. Exercise: prove the additional statements in the theorem. ∎

This proof is not complete because it does not say anything about classical solutions which are not in $C^\alpha(\overline{\Omega})$. This is reflected in the assumptions on φ and ψ, which are also taken in $C^\alpha(\overline{\Omega})$. We would like to relax these assumptions. Let us see what the complications are for for classical solutions which are only in $C(\overline{\Omega})$. We would like to conclude that these are automatically in $C^\alpha(\overline{\Omega})$, but this is not as clear as one might expect, because why should $f(x, u(x))$ be Hölder continuous in x? In fact this relies on the L^p-estimates for weak solutions (Gilbarg and Trudinger, Chapter 9). The main result is the following: if u is a weak solution of

$$(D) \quad \begin{cases} Lu = f(x) & \text{in } \Omega; \\ u = 0 & \text{on } \quad \partial\Omega, \end{cases}$$

where L is a uniformly elliptic operator with bounded coefficients and $\partial\Omega \in C^{1,1}$, then there exists a constant such that for all $1 < p < \infty$

$$||u||_{2,p,\Omega} \leq C(||u||_p + ||f||_p).$$

Here $||u||_{2,p}$ is the sum of the L^p-norms of all derivatives upto order 2 of u. The corresponding Banach space is denoted by $W^{2,p}(\Omega)$ (see also the next sections). By Sobolev's embedding theorem we have that $W^{2,p}(\Omega) \to C^{1,\alpha}(\overline{\Omega})$ is a bounded injection, provided $p > n$ and $0 < \alpha < 1 - n/p$. Thus, when f is bounded, we certainly have $u \in C^\alpha(\overline{\Omega})$.

We conclude this section with a uniqueness result for positive solutions of

$$(D) \quad \begin{cases} \Delta u + f(x, u) = 0 & \text{in } \Omega; \\ u = 0 & \text{on } \quad \partial\Omega. \end{cases}$$

6.16 Theorem Suppose the function f has the additional property that

$$\frac{f(x, s)}{s} \quad \text{is decreasing in} \quad s.$$

Then (D) has at most one positive solution.

Proof Suppose we have two positive solutions u_1 and u_2. Note that in view of the discussion above both these solutions belong to $C^\alpha(\overline{\Omega})$, and hence to $C^{2,\alpha}(\overline{\Omega})$. Multiplying the equation for respectively u_1 by u_2 and for u_2 by u_1, we obtain, integrating by parts and subtracting, that

$$0 = \int_\Omega (f(x, u_1)u_2 - f(x, u_2)u_1)dx = \int_\Omega u_1 u_2 (\frac{f(x, u_1)}{u_1} - \frac{f(x, u_2)}{u_2})dx.$$

In the case that $u_1 < u_2$ this yields an immediate contradiction with the assumptionsls ls on f. We conclude that u_1 and u_2 have to intersect. Now let

$$\varphi(x) = \max(u_1(x), u_2(x)),$$

and replace f by a new function f which coincides with the old one on $[0, \max\varphi]$, but with the additional property that $f(x, M) \equiv 0$ for some large $M > 0$. In other words, $\psi(x) \equiv M$ is then a supersolution. Exercise: show that the iteration procedure starting with φ and ψ leads again to the existence of a maximal and a minimal solution between φ and ψ. Consequently we have again at least two solutions whch are strictly ordered. In the first part we have seen that this is impossible. ■

6.17 Exercise Let $\lambda_1 > 0$ be the first eigenvalue of minus the Laplacian with zero boundary conditions, and let $\varphi_1 > 0$ be the corresponding eigenfunction. Thus

$$\begin{cases} \Delta\varphi_1 + \lambda_1\varphi_1 = 0 & \text{in } \Omega; \\ \varphi_1 = 0 & \text{on } \quad \partial\Omega. \end{cases}$$

(For the existence of λ_1 and φ_1 see the next sections, in particular Theorem 9.16) Consider the problem

$$(D) \quad \begin{cases} \Delta u + \lambda u(1-u) = 0 & \text{in } \Omega; \\ u = 0 & \text{on } \quad \partial\Omega. \end{cases}$$

(i) Show that (D) has no positive solution for $0 \le \lambda \le \lambda_1$. Hint: multiply the equation by u and use the fact that for functions which are zero on the boundary, $\int_\Omega |\nabla u|^2 \ge \lambda_1^2 \int_\Omega u^2$ (Theorem 9.16).

(ii) Show that (D) has a unique positive solution u_λ for every $\lambda \in (\lambda_1, \infty)$.

(ii) Show that u_λ is strictly increasing in λ and determine its limit as $\lambda \to \infty$.

7. The weak solution approach in one space dimension

Instead of the Dirichlet problem for Poisson's equation,

$$\begin{cases} -\Delta u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we first consider the one-dimensional version of

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is, for given f in say $C([a, b])$, we look for a function u satisfying

$$(P) \quad \begin{cases} -u'' + u = f & \text{in } (a, b); \\ u(a) = u(b) = 0 \end{cases}$$

Of course we can treat (P) as a linear second order inhomogeneous equation, and construct a solution by means of ordinary differential equation techniques, but that is not the point here. We use (P) to introduce a method that also works in more space dimensions. Let $\psi \in C^1([a,b])$ with $\psi(a) = \psi(b) = 0$ and suppose that $u \in C^2([a,b])$ is a classical solution of (P). Then

$$\int_a^b (-u'' + u)\psi = \int_a^b f\psi,$$

so that integrating by parts, and because $\psi(a) = \psi(b) = 0$,

$$\int_a^b (u'\psi' + u\psi) = \int_a^b f\psi.$$

For the moment we say that $u \in C^1([a,b])$ is a *weak solution* of (P) if

$$\forall \psi \in C^1([a,b]) \text{ with } \psi(a) = \psi(b) = 0 : \quad \int_a^b (u'\psi' + u\psi) = \int_a^b f\psi. \tag{7.1}$$

A *classical solution* is a function $u \in C^2([a,b])$ which satisfies (P).

The program to solve (P) is as follows.

A. Adjust the definition of a weak solution so that we can work with functions on a suitable Hilbert space.

B. Obtain the unique existence of a weak solution u by means of Riesz' Theorem or the Lax-Milgram Theorem.

C. Show that $u \in C^2([a,b])$ and $u(a) = u(b) = 0$, under appropriate conditions on f.

D. Show that a weak solution which is in $C^2([a,b])$ is also a classical solution.

Step D is easy, for if $u \in C^2([a,b])$ with $u(a) = u(b) = 0$ is a weak solution, then

$$\int_a^b (-u'' + u - f)\psi = \int_a^b (u'\psi' + u\psi - f\psi) = 0$$

for all $\psi \in C^1([a,b])$ with $\psi(a) = \psi(b) = 0$, and this implies $-u'' + u = f$ on $[a,b]$, so u is a classical solution of (P).

For step A we introduce the *Sobolev spaces* $W^{1,p}$.

7.1 Definition Let $\emptyset \neq I = (a,b) \subset \mathbb{R}, 1 \leq p \leq \infty$. Recall that $D(I)$ is the set of all smooth functions with compact support in I. Then $W^{1,p}(I)$ consists of all $u \in L^p(I)$ such that the distributional derivative of u can be represented by a function in $v \in L^p(I)$, i.e.

$$\int_I v\psi = -\int_I u\psi' \quad \forall \psi \in D(I).$$

We write $u' = v$.

7.2 Exercise Show that u' is unique.

7.3 Remark For I bounded it is immediate that

$$C^1(\overline{I}) \subset W^{1,p}(I) \;\; \forall p \in [1, \infty].$$

7.4 Exercise Show that $W^{1,p}(I) \not\subset C^1(I)$.

7.5 Definition $H^1(I) = W^{1,2}(I)$.

7.6 Theorem $W^{1,p}(I)$ is a Banach space with respect to the norm

$$\|u\|_{1,p} = |u|_p + |u'|_p,$$

where $|\cdot|_p$ denotes the L^p-norm.

7.7 Theorem $H^1(I)$ is a Hilbert space with respect to the inner product

$$(u, v)_1 = (u, v) + (u', v') = \int_I (uv + u'v').$$

The inner product norm is equivalent to the $W^{1,2}$-norm.

7.8 Theorem $W^{1,p}(I)$ is reflexive for $1 < p < \infty$.

7.9 Theorem $W^{1,p}(I)$ is separable for $1 \leq p < \infty$. In particular $H^1(I)$ is separable.

7.10 Theorem For $1 \leq p \leq \infty$ and $x, y \in I$ we have

$$u(x) - u(y) = \int_y^x u'(s)ds,$$

for every $u \in W^{1,p}(I)$, possibly after redefining u on a set of Lebesque measure zero.

We recall the concept of Hölder continuity.

7.11 Remark In particular we have by Hölders inequality ($1/p + 1/q = 1$),

$$|u(x) - u(y)| = \Big| \int_I \chi_{[x,y]}(s) u'(s) ds \Big| \le |\chi_{[x,y]}|_q |u'|_p \le$$

$$|x - y|^{1/q} \|u\|_{1,p} = \|u\|_{1,p} |x - y|^{\frac{p-1}{p}} \quad \text{if} \quad p > 1.$$

7.12 Definition For $0 < \alpha \le 1$, I bounded, and $f \in C(\overline{I})$, let the Hölder seminorm be defined by

$$[u]_\alpha = \sup_{\substack{x,y \in I \\ x \ne y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Then

$$C^\alpha(\overline{I}) = \{u \in C(\overline{I}) : [u]_\alpha < \infty\}$$

is called the class of *uniformly Hölder continuous* functions with *exponent* α.

7.13 Theorem $C^\alpha(\overline{I})$ is a Banach space with respect to the norm

$$\|u\| = |u|_\infty + [u]_\alpha,$$

and for $1 < p \le \infty$, and I bounded, $W^{1,p}(I) \subset C^{1-1/p}(\overline{I})$.

7.14 Corollary For $1 < p \le \infty$, and I bounded, the injection $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is compact.

7.15 Theorem Let $u \in W^{1,p}(\mathbb{R})$, $1 \le p < \infty$. Then there exists a sequence $(u_n)_{n=1}^\infty \subset D(\mathbb{R})$ with $\|u_n - u\|_{1,p} \to 0$. In other words, $D(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$.

7.16 Corollary Let $u, v \in W^{1,p}(I)$, $1 \le p \le \infty$. Then $uv \in W^{1,p}(I)$ and $(uv)' = uv' + u'v$. Moreover, for all $x, y \in I$

$$\int_x^y u'v = [uv]_x^y - \int_x^y uv'.$$

7.17 Definition Let $1 \le p < \infty$. Then the space $W_0^{1,p}(I)$ is defined as the closure of $D(I)$ in $W^{1,p}(I)$.

7.18 Theorem Let $1 \leq p < \infty$ and I bounded. Then

$$W_0^{1,p}(I) = \{u \in W^{1,p}(I) : u = 0 \text{ on } \partial I\}$$

7.19 Remark For $1 \leq p < \infty$, $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$.

7.20 Proposition (Poincaré) Let $1 \leq p < \infty$, and I bounded. Then there exists a $C > 0$, depending on I, such that for all $u \in W_0^{1,p}(I)$:

$$\|u\|_{1,p} \leq C|u'|_p.$$

Proof We have

$$|u(x)| = |u(x) - u(a)| = |\int_a^x u'(s)ds| \leq \int_a^b |u'(s)|ds \leq |u'|_p|1|_q,$$

implying

$$|u|_p \leq (b-a)^{\frac{1}{p}}|1|_q|u'|_p. \blacksquare$$

7.21 Corollary Let $1 \leq p < \infty$ and I bounded. Then $\|u\|_{1,p} = \|u'\|_p$ defines an equivalent norm on $W_0^{1,p}(I)$. Also

$$((u,v)) = \int_I u'v'$$

defines an equivalent inner product on $H_0^1(I) = W_0^{1,2}(I)$. (Two inner products are called equivalent if their inner product norms are equivalent.)

We now focus on the spaces $H_0^1(I)$ and $L^2(I)$ with I bounded. We have established the (compact) embedding $H_0^1(I) \hookrightarrow L^2(I)$. Thus every bounded linear functional on $L^2(I)$ is automatically also a bounded linear functional on $H_0^1(I)$, if we consider $H_0^1(I)$ as being contained in $L^2(I)$, but having a stronger topology. On the other hand, not every bounded functional on $H_0^1(I)$ can be extended to L^2, e.g. if $\psi \in L^2 \backslash H^1$, then $\varphi(f) = \int_I \psi f'$ defines a bounded functional on $H_0^1(I)$ which cannot be extended. This implies that if we want to consider $H_0^1(I)$ as being contained in $L^2(I)$, we *cannot simultaneously apply* Riesz' Theorem to both spaces and identify them with their dual spaces. If we identify $L^2(I)$ and $L^2(I)^*$, we obtain the triplet

$$H_0^1(I) \overset{i}{\hookrightarrow} L^2(I) = L^2(I)^* \overset{i^*}{\hookrightarrow} H_0^1(I)^*.$$

Here i is the natural embedding, and i^* its adjoint. One usually writes

$$H_0^1(I)^* = H^{-1}(I).$$

Then

$$H_0^1(I) \hookrightarrow L^2(I) \hookrightarrow H^{-1}(I). \tag{7.2}$$

The action of H^{-1} on H_0^1 is made precise by

7.22 Theorem Suppose $F \in H^{-1}(I)$. Then there exist $f_0, f_1 \in L^2(I)$ such that

$$F(v) = \int_I f_0 v - \int_I f_1 v' \qquad \forall v \in H_0^1(I).$$

Thus $H^{-1}(I)$ consists of L^2 functions and their first order distributional derivatives. Note however that this characterization depends on the (standard) identification of L^2 and its Hilbert space dual. Also F does not determine f_0 and f_1 uniquely (e.g. $f_0 \equiv 0, f_1 \equiv 1$ gives $F(v) = 0 \quad \forall v \in H_0^1(I)$).

7.23 Remark Still identifying L^2 and L^{2^*} we have for $1 < p < \infty$, writing $W_0^{1,p}(I)^* = W^{-1,p}(I)$,

$$W_0^{1,p}(I) \hookrightarrow L^2(I) \hookrightarrow W^{-1,p}(I), \tag{7.3}$$

and Theorem 7.22 remains true but now with f_0 and f_1 in $L^q(I)$, where $1/p+1/q = 1$.

We return to

$$(P) \quad \begin{cases} -u'' + u = f & \text{in } I = (a,b); \\ u = 0 \ \text{ on } \delta I = \{a,b\}. \end{cases}$$

7.24 Definition A *weak solution* of (P) is a function $u \in H_0^1(I)$ such that

$$\int_I (u'v' + uv) = \int_I fv \qquad \forall v \in H_0^1(I). \tag{7.4}$$

Since $D(I)$ is dense in $H_0^1(I)$ it suffices to check this integral identity for all $\psi \in D(I)$. Thus a weak solution is in fact a function $u \in H_0^1(I)$ which satisfies $-u'' + u = f$ *in the sense of distributions*. Note that the boundary condition $u = 0$ on ∂I follows from the fact that $u \in H_0^1(I)$.

7.25 Theorem Let $f \in L^2(I)$. Then (P) has a unique weak solution $u \in H_0^1(I)$, and

$$\frac{1}{2}\int_I (u'^2 + u^2) - \int_I fu = \min_{v \in H_0^1(I)} \Big\{ \frac{1}{2}\int_I (v'^2 + v^2) - \int_I fv \Big\}.$$

Proof The left hand side of (7.4) is the inner product on $H_0^1(I)$. The right hand side defines a bounded linear functional

$$\varphi(v) = \int_I fv$$

on $L^2(I)$, and since $H_0^1(I) \hookrightarrow L^2(I)$, φ is also a bounded linear functional on $H_0^1(I)$. Thus the unique existence of u follows immediately from Riesz' Theorem. It also follows from Lax-Milgram applied with

$$A(u,v) = \int_I (u'v' + uv),$$

and then A being symmetric, the minimum formula is also immediate. ∎

How regular is this solution? We have $u \in H_0^1(I)$, so that $u' \in L^2(I)$ and also $u'' = u - f \in L^2(I)$. Thus

$$u \in H^2(I) = \{u \in H^1(I) : \quad u' \in H^1(I)\}. \tag{7.5}$$

Clearly if $f \in C(\overline{I})$, then $u'' \in C(\overline{I})$.

7.26 Corollary Let $f \in C(\overline{I})$. Then (P) has a unique classical solution $u \in C^2(\overline{I})$.

7.27 Exercise Let $\alpha, \beta \in \mathbb{R}$. Use Stampachia's Theorem applied to $K = \{u \in H^1(I) : u(0) = \alpha,\ u(1) = \beta\}$ with $A(u,v) = ((u,v))$ and $\varphi(v) = \int fv$ to generalize the method above to

$$(P') \quad \begin{cases} -u'' + u = f & \text{in } (0,1); \\ u(0) = \alpha; \quad u(1) = \beta. \end{cases}$$

Next we consider the Sturm-Liouville problem

$$(SL) \quad \begin{cases} -(pu')' + qu = f & \text{in } (0,1); \\ u(0) = u(1) = 0, \end{cases}$$

where $p, q \in C([0,1])$, $p, q > 0$, and $f \in L^2(0,1)$.

7.28 Definition $u \in H_0^1(0,1)$ is a weak solution of (SL) if

$$A(u,v) = \int_0^1 (pu'v' + quv) = \int_0^1 fv \qquad \forall v \in H_0^1(0,1). \tag{7.6}$$

7.29 Exercise Prove that (SL) has a unique weak solution.

Finally we consider the Neumann problem

$$(N) \qquad \begin{cases} -u'' + u = f & \text{in } (0,1); \\ u'(0) = u'(1) = 0. \end{cases}$$

7.30 Definition $u \in H^1(0,1)$ is a weak solution of (N) if

$$\int_0^1 (u'v' + uv) = \int_0^1 fv \qquad \forall v \in H^1(0,1).$$

7.31 Exercise Explain the difference between Definitions 7.24 and 7.30, and prove that for $f \in L^2(0,1)$, (N) has a unique weak solution. Show that if $f \in C([0,1])$ there exists a unique classical solution $u \in C^2([0,1])$. (Don't forget the boundary conditions.)

8. Eigenfunctions for the Sturm-Liouville problem

Recall that (SL) was formulated weakly as

$$A(u,v) = \varphi(v) \qquad \forall v \in H_0^1(0,1),$$

where

$$A(u,v) = \int_0^1 pu'v' + quv \quad \text{and} \quad \varphi(v) = \int_0^1 fv.$$

For $p, q \in C([0,1])$, $p, q > 0$, $A(\cdot,\cdot)$ defines an equivalent inner product on $H_0^1(0,1)$, and for $f \in L^2(0,1)$ (in fact $f \in H^{-1}(0,1)$ is sufficient), φ belongs to the dual of $H_0^1(0,1)$.

8.1 Exercise Define $T : L^2(0,1) \to L^2(0,1)$ by $Tf = u$, where u is the (weak) solution of (SL) corresponding to f. Show that T is linear, compact and symmetric.

8.2 Theorem Let $p, q \in C([0,1])$, $p, q > 0$. Then there exists a Hilbert basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2(I)$, such that φ_n is a weak solution of

$$\begin{cases} -(pu')' + qu = \lambda_n u & \text{in } (0,1); \\ u(0) = u(1) = 0, \end{cases}$$

where $(\lambda_n)_{n=1}^\infty$ is a nondecreasing unbounded sequence of positive numbers.

8.3 Exercise Prove this theorem (to show that $\lambda_n > 0$ use φ_n as testfunction).

8.4 Remark By (the Krein-Rutman) Theorem A.23, $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, and φ_1 can be chosen positive. In fact all λ_i are simple. This follows from the theory of ordinary differential equations, see e.g. [CL].

8.5 Exercise Show that $T : H_0^1(0,1) \to H_0^1(0,1)$ is also symmetric with respect to the inner product $A(\cdot,\cdot)$. Derive, using the eigenvalue formulas in Theorem B.14 for compact symmetric operators, that

$$\lambda_1 = \min_{0 \neq u \in H_0^1(0,1)} \frac{\int_0^1 pu'^2 + qu^2}{\int_0^1 u^2}, \quad \lambda_2 = \min_{(u,\varphi_1)=0} \frac{\int_0^1 pu'^2 + qu^2}{\int u^2},$$

etcetera.

9. Generalization to more dimensions

Throughout this section $\Omega \subset \mathbb{R}^n$ is an open connected set.

9.1 Definition Let $1 \leq p \leq \infty$. Then the *Sobolev space* $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \in L^p(\Omega)\}.$$

9.2 Theorem With respect to the norm

$$\|u\|_{1,p} = |u|_p + \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|_p$$

$W^{1,p}(\Omega)$ is a Banach space, which is reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$.

9.3 Proposition Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \qquad (i = 1, \ldots, n).$$

9.4 Theorem (Sobolev embedding) Let $1 \leq p < n$. Define p^* by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Then $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, and the embedding is continuous, i.e.

$$|u|_{p^*} \leq C\ \|u\|_{1,p} \quad \forall u \in W^{1,p}(\Omega).$$

To get some feeling for the relation between p and p^*, we consider the scaling $u_\lambda(x) = u(\lambda x)$. This scaling implies that on $\mathbb{R}^n$ an estimate of the form

$$|u|_q \leq C(n,p)|\nabla u|_p$$

for all sufficiently smooth functions $u : \mathbb{R}^n \to \mathbb{R}$, implies necessarily that $q = p^*$, and indeed, for this value of q, this (Sobolev) inequality can be proved.

9.5 Theorem Let Ω be bounded with $\partial\Omega \in C^1$. Then

$$p < n \quad \Rightarrow \quad W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in (1, p^*);$$

$$p = n \quad \Rightarrow \quad W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in (1, \infty);$$

$$p > n \quad \Rightarrow \quad W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}),$$

and all three injections are compact.

9.6 Definition Let $1 \leq p < \infty$. Then $W_0^{1,p}(\Omega)$ is the closure of $D(\Omega)$ in the $\|\cdot\|_{1,p}$-norm. Here $D(\Omega)$ is the space of all smooth functions with compact support in Ω.

9.7 Theorem Suppose $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $\partial\Omega \in C^1$, where $1 \leq p < \infty$. Then

$$u = 0 \text{ on } \partial\Omega \ \text{ if and only if } \ u \in W_0^{1,p}(\Omega).$$

9.8 Theorem (Poincaré inequality) For all bounded $\Omega \subset \mathbb{R}^n$ there exists $C = C(p,\Omega)$, such that for all $1 \leq p < \infty$,

$$|u|_p \leq C|\nabla u|_p \qquad \forall u \in W_0^{1,p}(\Omega).$$

Thus $|||u|||_p = |\nabla u|_p$ is an equivalent norm on $W_0^{1,p}(\Omega)$.

Next we turn our attention to the Hilbert space case $p = 2$. We write $H^1(\Omega) = W^{1,2}(\Omega)$ and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

9.9 Proposition $H^1(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,v) = \int_\Omega uv + \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i},$$

and $H_0^1(\Omega) \subset H^1(\Omega)$ is a closed subspace, and has

$$((u,v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}$$

as an equivalent inner product.

9.10 Corollary For any bounded domain $\Omega \subset \mathbb{R}^n$, the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

From now on we assume that Ω is bounded. Consider for $f \in L^2(\Omega)$ the problem

$$(D) \begin{cases} -\Delta u = f & \text{in } \Omega; \\ u = 0 \quad \text{in} & \partial\Omega. \end{cases}$$

9.11 Definition $u \in H_0^1(\Omega)$ is called a weak solution of (D) if

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v \qquad \forall v \in H_0^1(\Omega).$$

Note that, defining $\varphi \in (H_0^1(\Omega))^*$ by

$$\varphi(v) = \int_\Omega f v,$$

this inequality is equivalent to

$$((u,v)) = \varphi(v) \qquad \forall v \in H_0^1(\Omega).$$

As in the one-dimensional case we have from Riesz' Theorem (or Lax-Milgram):

9.12 Theorem Let Ω be a bounded domain, $f \in L^2(\Omega)$. The (D) has a unique weak solution $u \in H_0^1(\Omega)$, and the function $E : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$E(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega f v,$$

attains its minimum in u.

9.13 Theorem (regularity) Suppose $\partial\Omega \in C^\infty$ and $f \in C^\infty(\overline{\Omega})$. Then $u \in C^\infty(\overline{\Omega})$.

Next we consider the operator $T : L^2(\Omega) \to H_0^1(\Omega)$ defined by $Tf = u$. Clearly T is a bounded linear operator. Since we may also consider T as $T : L^2(\Omega) \to L^2(\Omega)$ or $T : H_0^1(\Omega) \to H_0^1(\Omega)$, and since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we have

9.14 Proposition $T : H_0^1(\Omega) \to H_0^1(\Omega)$ is compact, and also $T : L^2(\Omega) \to L^2(\Omega)$ is compact.

9.15 Theorem (i) $T : L^2(\Omega) \to L^2(\Omega)$ is symmetric with respect to the standard inner product in $L^2(\Omega)$; (ii) $T : H_0^1(\Omega) \to H_0^1(\Omega)$ is symmetric with respect to the inner product $((\cdot,\cdot))$.

Proof (i) $(Tf, g) = (g, Tf) = ((Tg, Tf)) = ((Tf, Tg)) = (f, Tg)$. (ii) $((Tf, g)) = (f, g) = (g, f) = ((Tg, f)) = ((f, Tg))$. ∎

9.16 Theorem Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exists a sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \ldots \uparrow \infty,$$

and a Hilbert basis of eigenfunctions

$$\varphi_1, \varphi_2, \varphi_3, \ldots$$

of $L^2(\Omega)$, such that

$$(E_\lambda) \begin{cases} -\Delta u = \lambda u & \text{in } \Omega; \\ u = 0 \quad \text{on} & \partial\Omega, \end{cases}$$

has nontrivial weak solutions if and only if $\lambda = \lambda_i$ for some $i \in \mathbf{N}$. Moreover φ_i is a weak solution of (E_{λ_i}), and

$$\lambda_1 = \min_{0 \neq \varphi \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} \qquad (\text{attained in } \ \varphi = \varphi_1);$$

$$\lambda_{m+1} = \min_{\substack{0 \neq \varphi \in H_0^1(\Omega) \\ (\varphi,\varphi_1) = \ldots = (\varphi,\varphi_m) = 0}} \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} \qquad (\text{attained in } \ \varphi = \varphi_{m+1}).$$

The function $\varphi_i \in C^\infty(\Omega)$ satisfies the partial differential equation in a classical way, and if $\partial\Omega \in C^\infty$, then also $\varphi_i \in C^\infty(\overline{\Omega})$, for all $i \in \mathbf{N}$. Finally $\varphi_1 > 0$ in Ω.

Proof Applying the spectral decomposition theorem to $T : L^2(\Omega) \to L^2(\Omega)$ we obtain a Hilbert basis $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \ldots\}$ corresponding to eigenvalues (counted with multiplicity) $\mu_1, \mu_2, \mu_3, \mu_4, \ldots$ of $T : L^2(\Omega) \to L^2(\Omega)$, with $|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq \ldots \downarrow 0$. Since φ_i satisfies

$$\int_\Omega \nabla\mu_i\varphi_i\nabla v = \int_\Omega \varphi_i v \qquad \forall v \in H_0^1(\Omega),$$

we have, putting $v = \varphi_i$,

$$\mu_i \int_\Omega |\nabla \varphi_i|^2 = \int_\Omega {\varphi_i}^2,$$

so that clearly $\mu_i > 0$. Thus setting $\lambda_i = 1/\mu_i$, we obtain $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \uparrow \infty$ as desired. The eigenvalue formulas for λ_i follow from the eigenvalue formulas for μ_i applied to $T : H_0^1(\Omega) \to H_0^1(\Omega)$, since

$$\frac{((T\varphi, \varphi))}{((\varphi, \varphi))} = \frac{(\varphi, \varphi)}{((\varphi, \varphi))} = \frac{\int \varphi^2}{\int |\nabla \varphi|^2}.$$

We have completed the proof except for regularity, and the fact that $\lambda_1 < \lambda_2$, and $\varphi_1 > 0$ in Ω. Both these properties follow from the Krein-Rutman theorem applied to the cone C of nonnegative functions. Unfortunately $H_0^1(\Omega)$ is not the space for which T satisfies the conditions of this theorem with C as above, so one has to choose another space, with 'smoother' functions. Here we do not go into the details of this argument, and restrict ourselves to the following observation.

9.17 Proposition Let u be the weak solution of (D). Then

$$f \ge 0 \;\; a.e. \text{ in } \Omega \;\Rightarrow\; u \ge 0 \;\; a.e. \text{ in } \Omega.$$

Proof We use the following fact. Let $u \in H^1(\Omega)$. Define

$$u^+(x) = \begin{cases} u(x) & \text{if } u(x) > 0; \\ 0 & \text{if } u(x) \le 0, \end{cases}$$

and

$$u^-(x) = \begin{cases} -u(x) & \text{if } u(x) < 0; \\ 0 & \text{if } u(x) \ge 0. \end{cases}$$

Then $u^+, u^- \in H^1(\Omega)$, and

$$\nabla u^+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0; \\ 0 & \text{if } u(x) \le 0, \end{cases}$$

and

$$\nabla u^-(x) = \begin{cases} -\nabla u(x) & \text{if } u(x) < 0 \\ 0 & \text{if } u(x) \ge 0. \end{cases}$$

Now taking u^- as test function we obtain

$$0 \le \int_\Omega |\nabla u^-|^2 = \int_\Omega \nabla u^- \nabla u^- = -\int_\Omega \nabla u \nabla u^- = -\int f u^- \le 0,$$

implying $u^- \equiv 0$. ∎

We conclude this section with some remarks.

Consider the problem

$$(D_1) \quad \begin{cases} -\Delta u + u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem can be dealt with in the same manner as (D) with $((u, v)) = \int \nabla u \nabla v$ replaced by

$$(u, v) = \int_\Omega (uv + \nabla u \nabla v).$$

Consider the Neumann problem

$$(N_1) \quad \begin{cases} -\Delta u + u = f & \text{in } \Omega; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where ν is the outward normal on $\partial\Omega$. Here the approach is as in $N = 1$.

9.18 Definition $u \in H^1(\Omega)$ is a weak solution of (N_1) if

$$\int_\Omega \nabla u \nabla v + uv = \int fv \quad \forall u \in H^1(\Omega).$$

Unique existence of a weak solution follows as before.

Consider the problem

$$(N) \quad \begin{cases} -\Delta u = f & \text{in } \Omega; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here we have a complication. By Gauss' theorem, a solution should satisfy

$$0 = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_\Omega \Delta u = -\int_\Omega f,$$

so we must restrict to f with $\int f = 0$. Also, if u is a solution, then so is $u + C$ for any constant C. Therefore we introduce the spaces

$$\tilde{L}^2(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f = 0\} \quad \text{and} \quad \tilde{H}^1(\Omega) = \{u \in H^1(\Omega) : \int_\Omega u = 0\}.$$

9.19 Definition Let $f \in \tilde{L}^2(\Omega)$. Then $u \in \tilde{H}^1(\Omega)$ is called a weak solution if

$$\int_\Omega \nabla u \nabla v = \int fv \qquad \forall v \in \tilde{H}^1(\Omega).$$

Observe that if this relation holds for all $v \in \tilde{H}^1(\Omega)$, it also holds for all $v \in H^1(\Omega)$.

9.20 Proposition The brackets $((\cdot,\cdot))$ also define an inner product on $\tilde{H}^1(\Omega)$, and $((\cdot,\cdot))$ is equivalent (on $\tilde{H}^1(\Omega)$) to the standard inner product.

9.21 Corollary For all $f \in \tilde{L}^2(\Omega)$, there exists a unique weak solution in $\tilde{H}^1(\Omega)$ of (N).

In the problems above, as well as in the methods, we can replace $-\Delta$ by any linear second order operator in divergence form

$$-\mathrm{div}(A\,\nabla) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}),$$

where

$$A(x) = (a_{ij}(x))_{i,j=1,\ldots,n}$$

is a symmetric x-dependent matrix with eigenvalues

$$0 < \delta < \lambda_1(x) \leq \lambda_2(x) \leq \ldots \leq \lambda_n(x) < M < \infty,$$

for all $x \in \Omega$, and $a_{ij} \in C(\overline{\Omega})$, $i.j = 1,\ldots,n$. In all the statements and proofs $\int \nabla u \nabla v$ then has to be replaced by

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}.$$

10. Maximum principles for parabolic equations

We consider solutions $u(x,t)$ of the equation

$$u_t = Lu, \tag{10.1}$$

where

$$Lu = \sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} + cu. \tag{10.2}$$

Throughout this section we assume that L has bounded continuous coefficients $a_{ij}(x,t) = a_{ji}(x,t)$, $b_i(x,t)$, $c(x,t)$ defined for (x,t) in a set of the form $Q_T = \Omega \times (0,T]$, with $T > 0$, and Ω a domain in $\mathbb{R}^N$. The set $\Gamma_T = \overline{Q_T}\backslash Q_T$ is called the parabolic boundary of Q_T. If the elliptic part Lu is uniformly elliptic in Q_T, that is if there exist numbers $0 < \lambda \leq \Lambda < \infty$, independent of $(x,t) \in Q_T$, such that

$$\lambda|\xi|^2 \leq (A(x,t)\xi,\xi) \leq \Lambda|\xi|^2 \quad \forall (x,t) \in Q_T \quad \forall \xi \in \mathbb{R}^N, \tag{10.3}$$

where

$$A(x,t) = \big(a_{ij}(x,t)\big)_{i,j=1,\dots,N} = \begin{pmatrix} a_{11}(x,t) & \cdots & a_{1N}(x,t) \\ \vdots & & \vdots \\ a_{N1}(x,t) & \cdots & a_{NN}(x,t) \end{pmatrix},$$

then equation (10.1) is called uniformly parabolic in Q_T.

10.2 Notation

$$C^{2,1}(Q_T) = \{u : Q_T \to \mathbb{R};\ u, u_t, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(Q_T)\}.$$

Our goal is again to exclude the existence of maxima and minima of (sub- and super-) solutions u in Q_T.

10.3 Theorem Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and $c \equiv 0$, and suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies the inequality $u_t \le Lu$ in Q_T. Then

$$\sup_{Q_T} u = \max_{\overline{Q_T}} u = \max_{\Gamma_T} u.$$

Proof First we assume that $u_t < Lu$ in Q_T and that u achieves a maximum in $P = (x_0, t_0) \in Q_T$. Then the first order x-derivatives of u vanish in P, and

$$(Hu)(P) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(P)\right)_{i,j=1,\dots,N}$$

is negative semi-definite, i.e.

$$\sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j}(P)\xi_i\xi_j \le 0 \quad \forall \xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N.$$

Consequently

$$(Lu)(P) = \sum_{i,j=1}^{N} a_{ij}(P)\frac{\partial^2 u}{\partial x_i \partial x_j}(P) \le 0,$$

implying $u_t(P) < 0$, so u cannot have a maximum in P.

Now suppose that we only know that $u_t \le Lu$ in Q_T. Let

$$v(x) = e^{\gamma x_1}, \qquad \gamma > 0.$$

Then

$$(Lv)(x) = (a_{11}\gamma^2 + \gamma b_1)e^{\gamma x_1} \geq \gamma(\lambda\gamma - b_0)e^{\gamma x_1} > 0,$$

if $\gamma > b_0/\lambda$, where $b_0 = \sup_{Q_T} |b|$. Hence, by the first part of the proof, we have for all $\varepsilon > 0$ that

$$\sup_{Q_T}(u + \varepsilon v) = \max_{\Gamma_T}(u + \varepsilon v).$$

Letting $\varepsilon \downarrow 0$ completes the proof. ▪

10.4 Theorem Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and let $c \leq 0$, and suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies the inequality $u_t \leq Lu$ in Q_T. Then

$$\sup_{Q_T} u = \max_{\overline{Q_T}} u \leq \max_{\Gamma_T} u^+.$$

Proof Exercise. ▪

10.5 Corollary Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies the inequality $u_t \leq Lu$ in Q_T. If $u \leq 0$ on Γ_T. Then also $u \leq 0$ on Q_T.

Proof Exercise. Hint: consider the function

$$v(x,t) = e^{-kt}u(x,t),$$

with $k > 0$ sufficiently large. ▪

10.6 Remark Everything we have done so far remains valid if Q_T is replaced by $D_T = D \cap \{t \leq T\}$, where D is a bounded domain in $\mathbb{R}^{N+1}$.

As in the elliptic case we also have a strong maximum principle, but this requires a little more work.

10.7 Theorem (*Boundary Point Lemma*) Let L be a uniformly elliptic operator with bounded coefficients in a domain $D \subset \mathbb{R}^{N+1}$ and $c \leq 0$ in D. Suppose the interior ball condition in a point $P = (x_0, t_0) \in \partial D$ is satisfied by means of a ball $B = B_R((x_1, t_1))$ with $x_1 \neq x_0$. Let $u \in C^{2,1}(D) \cap C(D \cup \{P\})$ satisfy

$$Lu - u_t \geq 0 \quad \text{in} \quad D \;\; \text{and} \;\; u(x,t) < u(P) \quad \forall (x,t) \in D.$$

Then, if $u(P) \geq 0$, we have

$$\lim \inf_{\substack{(x,t)\to P \\ (x,t)\in S_\theta}} \frac{u(P) - u(x,t)}{|(x,t) - P|} > 0,$$

where S_θ is a cone with top P and opening angle $\theta < \pi$, intersecting D and radially symmetric around the line through P and (x_1, t_1). For $c \equiv 0$ in D the same conclusion holds if $u(P) < 0$, and if $u(P) = 0$ the sign condition on c may be omitted. N.B. If the outward normal ν on ∂D and the normal derivative $\frac{\partial u}{\partial \nu}$ exist in P, then $\frac{\partial u}{\partial \nu}(P) > 0$.

Proof Without loss of generality we assume that the center of the disk B is in the origin. Let B_0 be a small disk centered at P such that $\overline{B_0}$ does not have any point in common with the t-axis. This is possible because P is not at the top or the bottom of B. Consequently there exists $\rho > 0$ such that $|x| \geq \rho$ for all $(x, t) \in K = B \cap B_0$. Consider

$$v(x,t) = e^{-\alpha(x^2+t^2)} - e^{-\alpha R^2}, \tag{10.4}$$

which is zero on ∂B. Then

$$Lv - v_t = e^{-\alpha(x^2+t^2)}\Big\{\sum_{i,j=1}^{N} 4\alpha^2 x_i x_j a_{ij} - \sum_{i=1}^{N} 2\alpha(a_{ii} + b_i x_i) + c + 2\alpha t\Big\} - ce^{-\alpha R^2}.$$

Hence, if c is nonpositive, we have on K that

$$Lv - v_t \geq e^{-\alpha R^2}\Big\{4\alpha^2\lambda\rho^2 - 2\alpha(N\Lambda + Nb_0 R) + c - 2\alpha R\Big\} > 0, \tag{10.5}$$

choosing $\alpha > 0$ sufficiently large, because c is bounded. For the function

$$w_\varepsilon(x,t) = u(x,t) + \varepsilon v(x,t),$$

it then also follows that $Lw_\varepsilon - w_t > 0$ in K. Since $v = 0$ on ∂B, and $u < u(P)$ on B, we can choose $\varepsilon > 0$ so small that $w_\varepsilon \leq u(P)$ on ∂K. Applying Theorem 10.3 (10.4) and keeping in mind Remark 10.6, it follows that $w_\varepsilon \leq u(P)$ on K, so that

$$u(x,t) \leq u(P) - \varepsilon v(x,t) \quad \forall (x,t) \in K. \tag{10.6}$$

This completes the proof. ∎

10.8 Theorem Let L be uniformly elliptic with bounded coefficients in a domain $D \subset \mathbb{R}^{N+1}$ and $c \leq 0$ in D. Suppose $u \in C^{2,1}(D)$ satisfies

$$Lu - u_t \geq 0 \text{ and } u(x,t) \leq M \quad \forall (x,t) \in D.$$

If $u(P) = M \geq 0$ for some $P = (x_0, t_0) \in D$, then $u \equiv M$ on the component of $D \cap \{t = t_0\}$ containing P. For $c \equiv 0$ in D the condition $M \geq 0$ can be omitted.

Proof Suppose the result is false. Then there exist two points $P = (x_0, t_0)$ and $P_1 = (x_1, t_0)$ in D such that $u(P) = M$, $u(P_1) < M$, and $u < M$ on the line segment l joining P and P_1 We can choose P and P_1 in such a way that the distance of l to the boundary of D is larger then the length of l. For notational convenience we argue in the remainder of the proof as if $N = 1$ and x_1 is to the right of x_0. Then for every $x \in (x_0, x_1)$ let $\rho = \rho(x)$ be the largest radius such that $u < M$ on the ball $B_\rho(x)$. Clearly $\rho(x)$ is well defined. By definition, $u = M$ in some point P_x on the boundary of $B_{\rho(x)}(x)$, and, applying the boundary point lemma, it follows that P_x is either the top or the bottom of $B_\rho(x)$, so $P_x = (x, t_0 \pm \rho(x))$. Now let $\delta > 0$ be small, and consider $x + \delta$. Then, again by the boundary point lemma, P_x cannot be in the closure of $B_{\rho(x+\delta)}(x + \delta)$. Hence

$$\rho(x + \delta)^2 < \rho(x)^2 + \delta^2. \tag{10.7}$$

Substituting δ/m for δ, and $x + j\delta/m$ for $j = 0, \dots, m - 1$ in (10.7), we obtain, summing over j,

$$\rho(x + \delta)^2 < \rho(x)^2 + \frac{\delta^2}{m}, \tag{10.8}$$

for all m so that $\rho(x)$ is nonincreasing in x. Letting $x \downarrow x_0$ it follows that $u(x_0, t_0) < M$, a contradiction. ■

10.9 Theorem Let L be uniformly elliptic with bounded coefficients in D_T where D is a domain in $\mathbb{R}^{N+1}$ and $c \le 0$ in D_T. Suppose $u \in C^{2,1}(D_T)$ satisfies $u_t \le Lu$ in D_T and, for $M \ge 0$ and $t_0 < t_1 \le T$, that $u < M$ in $D \cap \{t_0 < t < t_1\}$. Then also $u < M$ on $D \cap \{t = t_1\}$. For $c \equiv 0$ in D the condition $M \ge 0$ can be omitted.

Proof Suppose there exists a point $P = (x_1, t_1)$ in $D \cap \{t = t_1\}$ with $u(P) = M$. For notational convenience we assume that P is the origin, so $x_1 = 0$ and $t_1 = 0$. Consider the function

$$v(x, t) = e^{-(|x|^2 + \alpha t)} - 1, \tag{10.9}$$

which is zero on, and positive below the parabola $\alpha t = -|x|^2$. Then

$$Lv - v_t = e^{-(|x|^2+\alpha t)} \Big\{ \sum_{i,j=1}^{N} 4x_i x_j a_{ij} - \sum_{i=1}^{N} 2(a_{ii} + b_i x_i) + c + \alpha \Big\} - c.$$

Hence, if c is nonpositive,

$$Lv - v_t \ge e^{-(|x|^2+\alpha t)} \Big\{ 4\lambda |x|^2 - 2(N\Lambda + N b_0)|x| + c + \alpha \Big\}. \tag{10.10}$$

Now let B be a small ball with center in the origin. Choosing $\alpha > 0$ sufficiently large, we can make the right hand side of (10.10) positive in B. Consider on $K = B \cap \{\alpha t < -|x|^2\}$ the function

$$w_\varepsilon(x, t) = u(x, t) + \varepsilon v(x, t),$$

then by similar reasoning as before we can choose $\varepsilon > 0$ so small that $w_\varepsilon \le M$ on ∂K. By Theorem 10.3/10.4 it follows again that $w_\varepsilon \le M$ thoughout K, so

$$u(x,t) \le M - \varepsilon v(x,t) \quad \forall (x,t) \in K, \tag{10.11}$$

implying $u_t > 0$ in the origin. Hence also $Lu > 0$ so that with respect to the x-variables u cannot have a local maximum in the origin, a contradiction. This completes the proof. Note that we did not need u to be defined for $t > 0$ in the proof, which corresponds to $t > t_1$ in the Theorem. Thus the proof applies equally well to Q_T with $t_1 = T$. ∎

10.10 Theorem (*Strong Maximum Principle*) Let L be uniformly elliptic with bounded coefficients in D_T and $c \le 0$ in D_T. Suppose $u \in C^{2,1}(D_T)$ satisfies $u_t \le Lu$ in D_T and, for some $M \ge 0$ that $u \le M$ in D_T. If $P \in D_T$ and $u(P) = M$, then $u = M$ in every point P' in D_T which can be joined to P by a continuous curve γ in D_T along which, running from P' to P, t is nondecreasing. For $c \equiv 0$ in D_T the condition $M \ge 0$ can be omitted.

Proof Suppose that $u(P') < M$. Then there must exist a point P_0 on γ between P' and P such that $u(P_0) = M$ and $u < M$ on the component of $\gamma - \{P_0\}$ containing P'. In view of the previous two lemmas this is impossible. ∎

10.11 Remark For Q_T the statement in Theorem 10.10 is simply that either $u < M$ in Q_T, or $u \equiv M$ in Q_T.

Next we give some applications of the results above to semilinear parabolic equations. Instead of (10.1) we now consider

$$u_t = Lu + f(x,t,u), \tag{10.12}$$

where f is a given continous function of the variables x and t, as well as the unknown u.

10.12 Proposition (*Comparison Principle*) Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and suppose that $f(x,t,u)$ is nonincreasing in u. If $u, v \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfy $u_t \le Lu + f(x,t,u)$, $v_t \ge Lv + f(x,t,v)$ in Q_T, and $u \le v$ in Γ_T, then $u \le v$ throughout Q_T.

Proof Suppose there is a point $P \in Q_T$ with $u(P) > v(P)$. We may assume that P has t-coordinate T. Let D_T be the connected component of $Q_T \cap \{u > v\}$ containing P. Writing $z = u - v$ we have $z_t \le Lz$ in D_T and $z = 0$ on the parabolic boundary of D_T. By Corollary 10.5 and Remark 10.6 it follows that $z \le 0$ in D_T, contradicting $z(P) > 0$. ∎

The condition that f is nonincreasing in u is rather restrictive. However, if f satisfies a one-sided uniform Lipschitz condition,

$$f(x,t,v) - f(x,t,u) \leq K(v-u), \quad \forall x,t,u,v, \quad v > u, \tag{10.13}$$

we can rewrite the equation as

$$u_t = Lu + Ku + f(x,t,u) - Ku = Lu - Ku + g(x,t,u). \tag{10.14}$$

Clearly $g(x,t,u) = f(x,t,u) - Ku$ is nonincreasing in u. Since in Proposition 10.12 there is no restriction on c, it applies equally well to (10.14).

To turn this weak comparison principle into a *strong* comparison principle we need a two-sided Lipschitz condition,

$$|f(x,t,v) - f(x,t,u)| \leq K|v-u|. \tag{10.15}$$

10.13 Theorem (*Strong Comparison Principle*) Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and suppose that f satisfies (10.15). If $u, v \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfy $u_t \leq Lu + f(x,t,u)$, $v_t \geq Lv + f(x,t,v)$ in Q_T, and $u \leq v$ in Γ_T, then $u < v$ throughout Q_T, unless $u \equiv v$ in Q_T.

Proof From the previous theorem we have $u \leq v$. Hence the statement follows from the following variant of Theorem 10.10. ∎

10.14 Theorem Let L be uniformly elliptic with bounded coefficients in D_T where D is a domain in $\mathbb{R}^{N+1}$, and let f satisfy (10.15). Suppose $u, v \in C^{2,1}(D_T)$ satisfy $u_t \leq Lu + f(x,t,u)$, $v_t \geq Lv + f(x,t,v)$, and $u \leq v$ in D_T. If $P \in D_T$ and $u(P) = v(P)$, then $u = v$ in every point P' in D_T which can be joined to P by a continuous curve in D_T along which, running from P' to P, t is nondecreasing.

Proof Writing again $z = u - v$ we have, in view of (10.15)

$$z_t \leq Lz - (f(x,t,v) - f(x,t,u)) \leq Lz + Kv.$$

Using the same trick as in the proof of Corollary 10.5 the theorem follows from Theorem 10.14. ∎

10.15 Remark The previous theorems generalize the weak and the strong comparison principle to the case of semilinear equations. In fact they can also be extended to the class of quasilinear equations. We may allow for instance coefficients a_{ij}, b_i and c which are smooth functions of x, t and u, satisfying (10.3) uniformly in all three variables. The proofs are always based on writing an equation for the difference $w = v - u$ of two solutions u and v, using also the integral form of the mean value theorem, i.e.

$$F(v) - F(u) = [F(u + t(v-u)]_{t=0}^{t=1} = (\int_0^1 F'(u+tw)dt)\, w.$$

Next we shall give a monotonicity property for sub- and supersolutions of semilinear *autonomous* equations, that is, the coefficients a, b, c, and nonlinearity f are independent of t.

10.16 Theorem (*Monotonicity*) Let L be uniformly elliptic in Q_T with bounded coefficients independent of t, Ω be bounded, and suppose that $f = f(x,u)$ satisfies (10.13). Suppose $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $0 \le L\underline{u} + f(x,\underline{u})$ in Ω, and $\underline{u} = 0$ on $\partial\Omega$, and suppose $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies $u_t = Lu + f(x,u)$ in Q_T, $u = 0$ on $\partial\Omega \times (0,T]$, and $u(x,0) = \underline{u}(x)$ for all $x \in \Omega$, then $u_t \ge 0$ in Q_T.

Proof By Theorem 10.13 we have $\underline{u} \le u$ in Q_T. Define $v(x,t) = u(x,t+h)$, where $0 < h < T$. Again applying Theorem 10.13 we have $u \le v$ in Q_{T-h}, i.e. $u(x,t+h) \ge u(x,t)$. ∎

10.17 Remark The assumption that $\underline{u} = 0$ on $\partial\Omega$ can be relaxed to $\underline{u} \le 0$, but then we can no longer talk about a solution $u \in C(\overline{Q_T})$, because obviously u will be discontinuous in the set of cornerpoints $\partial\Omega \times \{0\}$. The result however remains true for solutions $u \in C^{2,1}(Q_T)$ which are continuous up to both the lateral boundary $\partial\Omega \times (0,T]$, and $\Omega \times \{0\}$, and in addition have the property that for every cornerpoint $y_0 \in \partial\Omega \times \{0\}$,

$$0 \ge \limsup_{y \in Q_T, y \to y_0} u(y) \ge \liminf_{y \in Q_T, y \to y_0} u(y) \ge \underline{u}(y_0).$$

11. Potential theory and existence results

Most of the results in this section are taken from [LSU,Chapter IV]. The role of the Newton potential in the elliptic theory is taken over by the *volume potential*

$$w(x,t) = \int_0^t \int_{\mathbb{R}^n} E(x-y, t-\tau) f(y,\tau)\, dy\, d\tau. \tag{11.1}$$

Here

$$E(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-|x|^2/4t}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{11.2}$$

is the fundamental solution of the heat equation, and $f(x,t)$ is a bounded measurable function. More precisely, if we extend the definition of $E(x,t)$ to all values of x and t by setting $E(x,t)=0$ for $t<0$, we have that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} E(x,t)(\psi_t(x,t)+\Delta\psi(x,t))dxdt = -\psi(0,0) \tag{11.3}$$

for all compactly supported smooth functions $\psi(x,t)$. Also

$$\int_{\mathbb{R}^n} E(x,t)dx = 0 \quad \forall t>0, \tag{11.4}$$

and

$$\lim_{t\downarrow 0}\int_{\mathbb{R}} E(x,t)\psi(x,t)dx \to \psi(0,0), \tag{11.5}$$

for all bounded continuous functions $\psi(x,t)$. In particular the Cauchy problem

$$(CP)\begin{cases} u_t = \Delta u \quad & x\in\mathbb{R}^n\ ,\ t>0; \\ u(x,0)=u_0(x) \quad & x\in\mathbb{R}^n \end{cases}$$

is treated just as in the one-dimensional case, and has a unique bounded classical solution if u_0 is continuous and bounded, and this solution is given by the convolution

$$u(x,t) = \int_{\mathbb{R}^n} E(x-y,t)u_0(y)dy. \tag{11.6}$$

The volume potential (11.1) is a continuous weak solution of

$$(I)\begin{cases} u_t = \Delta u + f \quad & x\in\mathbb{R}^n\ ,\ t>0; \\ u(x,0)=0 \quad & x\in\mathbb{R}^n. \end{cases}$$

The volume potential plays the same role in the theory of parabolic equations as the Newton potential in the theory of elliptic equations. Recall that the Newton potential allows only one differentiation with respect to x under the integral and therefore its Laplacian cannot be computed by bringing Δ under the integral. For the volume potential we encounter the same difficulty if we want to compute

$$(\frac{\partial}{\partial t}-\Delta)w.$$

because E_t and $E_{x_ix_j}$ are not locally integrable in space and time. Indeed, if we compute integrals of E and its derivatives using polar coordinates $r,\theta_1,\dots,\theta_{n-1}$ and set $r=2s\sqrt{t}$, we find

$$\int_0^T\int_{\mathbb{R}^n} E(x,t)dxdt = \int_0^T\int_{\mathbb{R}^n}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-|x|^2/4t}dxdt =$$

$$\int_0^T \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_0^\infty e^{-r^2/4t} n\omega_n r^{n-1} dr dt = \int_0^T \frac{n\omega_n}{\pi^{\frac{n}{2}}} \int_0^\infty e^{-s^2} s^{n-1} ds dt =$$

$$\int_0^T \frac{2}{\Gamma(\frac{n}{2})} \int_0^\infty e^{-s^2} s^{n-1} ds dt = \int_0^T dt = T, \tag{11.7}$$

and similarly

$$|\int_0^T \int_{\mathbb{R}^n} E_{x_i}(x,t) dx dt| = |\int_0^T \int_{\mathbb{R}^n} \frac{x_i}{2t} E(x,t) dx dt| \le \int_0^T \int_{\mathbb{R}^n} \frac{|x|}{2t} E(x,t) dx dt =$$

$$\int_0^T \frac{n\omega_n}{\pi^{\frac{n}{2}}} \int_0^\infty \frac{y}{\sqrt{t}} e^{-s^2} s^{n-1} ds dt = \int_0^T \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \frac{dt}{\sqrt{t}} < \infty. \tag{11.8}$$

However for the time derivative we get

$$\int_0^T \int_{\mathbb{R}^n} E_t(x,t) dx dt = \int_0^T \int_{\mathbb{R}^n} (\frac{|x|^2}{4t^2} - \frac{n}{2t}) E(x,t) dx dt =$$

$$\int_0^T \frac{n\omega_n}{\pi^{\frac{n}{2}}} \int_0^\infty (\frac{s^2}{4} - \frac{n}{2}) e^{-s^2} s^{n-1} ds \frac{dt}{t}, \tag{11.9}$$

and this integral does not exist. The same difficulty occurs for

$$\int_0^T \int_{\mathbb{R}^n} E_{x_i x_j}(x,t) dx dt.$$

Thus only first order space derivatives of the volume potential may be computed by differentiating under the integral directly. However, if we consider instead of w the function

$$w_h(x,t) = \int_0^{t-h} \int_{\mathbb{R}^n} E(x-y, t-\tau) f(y,\tau) dy d\tau, \tag{11.10}$$

the singularity is gone and differentiation under the integral permitted. Hence

$$\frac{\partial w_h}{\partial t}(x,t) = \int_0^{t-h} \int_{\mathbb{R}^n} \frac{\partial E}{\partial t}(x-y, t-\tau) f(y,\tau) dy d\tau + \int_{\mathbb{R}^n} E(x-y,h) f(y,t-h) dy =$$

$$\int_0^{t-h} \int_{\mathbb{R}^n} \frac{\partial E}{\partial t}(x-y, t-\tau)(f(y,\tau) - f(x,\tau)) dy d\tau + \int_{\mathbb{R}^n} E(x-y,h) f(y,t-h) dy$$

$$\to \int_0^{t-h} \int_{\mathbb{R}^n} \frac{\partial E}{\partial t}(x-y, t-\tau)(f(y,\tau) - f(x,\tau)) dy d\tau + f(x,t),$$

provided the integral exists, in which case we conclude that

$$\frac{\partial w}{\partial t}(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{\partial E}{\partial t}(x-y, t-\tau)(f(y,\tau) - f(x,\tau)) dy d\tau + f(x,t), \tag{11.11}$$

and similarly

$$\frac{\partial^2 w}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{\partial^2 E}{\partial x_i \partial x_j}(x-y,t-\tau)(f(y,\tau)-f(x,\tau))dyd\tau. \qquad (11.12)$$

It is straight forward to check that the derivation above is valid for functions which are Hölder continuous in x, uniformly in x and t, i.e.

$$|f(x,t)-f(y,t)| \leq C|x-y|^\alpha. \qquad (11.13)$$

Here as always the Hölder exponent α is assumed to be strictly beteen 0 and 1. Using polarcoordinates centered at $y=x$, with $r=|x-y|=2s\sqrt{t-\tau}$, we see that (11.13) implies that

$$|f(x,t)-f(y,t)| \leq C|x-y|^\alpha = C(2s\sqrt{t-\tau})^\alpha, \qquad (11.14)$$

which makes the integral finite, whence (11.11) and like wise (11.12) are is valid. Consequently the volume potential w of f is then a classical solution of Problem (I), so that combined with the solution of the homogeneous Cauchy problem, we can now formulate

11.1 Theorem Let $u_0 : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}\times[0,T] \to \mathbb{R}$ be bounded and continuous, and let f be uniformly Hölder continuous in x (11.13). Then

$$\begin{cases} u_t = \Delta u + f & x \in \mathbb{R}^n,\ 0<t\leq T; \\ u(x,0)=u_0 & x\in\mathbb{R}^n \end{cases}$$

has a bounded classical solution given by

$$u(x,t) = \int_{\mathbb{R}^n} E(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} E(x-y,t-\tau)f(y,\tau)dyd\tau.$$

We note that if f is Hölder continuous in t, we may also write

$$\frac{\partial w_h}{\partial t}(x,t) = \int_0^{t-h} \int_{\mathbb{R}^n} \frac{\partial E}{\partial t}(x-y,t-\tau)f(y,\tau)dyd\tau + \int_{\mathbb{R}^n} E(x-y,h)f(y,t-h)dy =$$

$$\int_0^{t-h} \int \frac{\partial E}{\partial t}(x-y,t-\tau)(f(y,\tau)-f(y,t))dyd\tau + \int E(x-y,h)(f(y,t-h)-f(y,t))dy$$

$$\to \int_0^t \int_{\mathbb{R}^n} \frac{\partial E}{\partial t}(x-y,t-\tau)(f(y,\tau)-f(y,t))dyd\tau = \frac{\partial w}{\partial t}(x,t). \qquad (11.15)$$

Just as for the Newton potential, we have used Hölder continuity of f with respect to x, to obtain that the volume potential of f has the desired continuous derivatives, second order in x, and first order in t. Again this is not the optimal result, because a Hölder modulus of continuity is still lacking. To formulate the optimal result we introduce some notation. For an arbitrary partial derivative

$$D = (\frac{\partial}{\partial t})^r (\frac{\partial}{\partial x_1})^{s_1} (\frac{\partial}{\partial x_2})^{s_2} \cdots (\frac{\partial}{\partial x_n})^{s_n}, \tag{11.16}$$

we shall say that the order of D is equal to

$$|D| = 2r + s_1 + s_2 + \cdots + s_n. \tag{11.17}$$

In other words, time-derivatives are counted with a double weight. Let $Q_T = \Omega \times (0,T]$ and let k be an integer. The space of all functions with bounded continuous derivatives upto order k on Q_T is denoted by $C^k(\overline{Q_T})$ (this notation differs from Notation 10.2). The norm on this space is

$$||u||_{k,Q_T} = \sum_{|D|\leq k} \sup_{(x,t)\in Q_T} |(Du)(x,t)|. \tag{11.18}$$

For $0 < \alpha < 1$ we shall consider Hölder regularity of the highest order derivatives with exponents α with respect to x, and $\alpha/2$ with respect to t. This is done using the seminorms

$$[u]_{\alpha,Q_T} = \sup_{x,y\in\Omega, 0<s,t\leq T} \frac{|u(x,t)-u(y,s)|}{|x-y|^\alpha + |s-t|^{\alpha/2}}; \quad [u]_{k+\alpha,Q_T} = \sum_{|D|=k} [Du]_{\alpha,Q_T}, \tag{11.19}$$

and the norms

$$||u||_{k+\alpha,Q_T} = ||u||_{k,Q_T} + [u]_{k+\alpha,Q_T}, \quad k = 0,1,2,\ldots \tag{11.20}$$

The corresponding spaces Banach spaces are denoted by $C^{k+\alpha}(\overline{Q_T})$.

11.2 Theorem Let $f \in C^\alpha(\mathbb{R}^n \times [0,T])$. Then the heat potential w defined by (11.1) belongs to $C^{2+\alpha}(\mathbb{R}^n \times [0,T])$. Moreover, the following estimate holds

$$||w||_{2+\alpha,\mathbb{R}^n\times[0,T]} \leq C(n,\alpha)||f||_{\alpha,\mathbb{R}^n\times[0,T]}. \tag{11.21}$$

Proof see [LSU, section IV.2]. ∎

We also mention the so-called *double-layer potential*

$$v(x,t) = -2\int_0^t \int_{\mathbb{R}^{n-1}} \frac{\partial E(x'-y', x_n, t-\tau)}{\partial x_n} g(y',\tau)dy'd\tau, \tag{11.21}$$

which solves the following problem for the heat equation on a half space.

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n_+, \ 0 < t \leq T; \\ u(x,0) = 0 & x \in \mathbb{R}^n, \ x_n \geq 0; \\ u(x,t) = g(x',t) & x' \in \mathbb{R}^{n-1}, x_n = 0, \ 0 < t \leq T. \end{cases}$$

Here

$$\mathbb{R}^n_+ = \{x = (x', x_n) = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \ x' \in \mathbb{R}^{n-1}, \ x_n > 0\},$$

and g is assumed to be bounded and integrable. Since for $x_n > 0$ the singularity of E falls outside the domain of integration it is clear from differentiating under the integral sign, that v solves the heat equation. Evidently also $v(x,0) = 0$.

11.3 Exercise Show that $v(x,t) \to g(x'_0, t_0)$ as $(x,t) \to (x'_0, t_0)$ in every point $(x'_0, t_0) \in \mathbb{R}^{n-1} \times [0, \infty)$ where g is continuous.

For v there is also a general estimate in terms of the Hölder norms.

11.4 Theorem Let $g \in C^\alpha(\mathbb{R}^{n-1} \times [0,T])$. Then the double-layer potential v defined by (11.21) belongs to $C^\alpha(\overline{\mathbb{R}^n_+} \times [0,T])$. Moreover, the following estimate holds

$$||w||_{\alpha, \overline{\mathbb{R}^n_+} \times [0,T]} \leq C(n, \alpha) ||g||_{\alpha, \overline{\mathbb{R}^{n-1}_+} \times [0,T]}. \tag{11.22}$$

Proof see [LSU, section IV.2]. ■

Finally we mention a similar estimate for the solution of the homogeneous Cauchy problem.

11.5 Theorem Let $u_0 \in C^\alpha(\mathbb{R}^n)$. Then the solution u of (CP) defined by (11.5) belongs to $C^\alpha(\mathbb{R}^n \times [0,T])$. Moreover

$$||u||_{\alpha, \mathbb{R}^n \times [0,T]} \leq C(n, \alpha) ||u_0||_{\alpha, \mathbb{R}^n}. \tag{11.23}$$

Proof see [LSU, section IV.2]. ■

Just as in the elliptic case these results are the starting point for a general existence theory of classical solutions to (boundary) initial value problems for L defined by (10.2). Sometime we shall write $L(x,t)$ to emphasize the dependence of the coefficients on x and t. We assume that L is a uniformly elliptic operator with

Hölder continuous coefficients. We begin with the formulation of these results for the Cauchy-Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^n$.

$$(CD) \begin{cases} u_t = Lu \ + f(x,t) & x \in \Omega,\ 0 < t \le T; \\ u(x,0) = u_0(x) & x \in \overline{\Omega}; \\ u(x,t) = \Phi(x,t) & x \in \partial\Omega,\ 0 < t \le T. \end{cases}$$

Observe that in the corner points of $Q_T = \Omega \times (0,T]$, which are the points in $\partial\Omega\times\{0\}$, we can, assuming the the solution is smooth upto the parabolic boundary $\Gamma_T = \overline{Q_T}\backslash Q_T$, compute values of u and its derivatives by taking limits along $\Omega\times\{0\}$ or along the lateral boundary $S_T = \partial\Omega\times(0,T]$. This leads to compatibility conditions whch have to be satisfied by u_0 and Φ.

The zero order compatibility condition comes from the continuity of u. It reads

$$\lim_{t\downarrow 0} \Phi(x_0,t) = \lim_{x\to x_0} u_0(x,0) \quad \forall x_0 \in \partial\Omega. \tag{11.24}$$

The first order compatiblity condition follows from the partial differential equation, which implies

$$\lim_{t\downarrow 0} \frac{\partial\Phi}{\partial t}(x_0,t) = \lim_{x\to x_0} (L(x,0)u_0(x,0) + f(x,0)) \quad \forall x_0 \in \partial\Omega. \tag{11.25}$$

The higher order compatibility conditions are obtained from differentiating the equation.

11.6 Theorem Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{2+\alpha}$ and let L be a uniform elliptic operator with coefficients $a_{ij}, b_i, c \in C^\alpha(\overline{Q_T})$ $(0 < \alpha < 1)$, let $f \in C^\alpha(\overline{Q_T})$, let $u_0 \in C^\alpha(\overline{\Omega})$, and let $\Phi \in C^{2+\alpha}(\overline{Q_T})$. If the compatibility conditions (11.24) and (11.25) are satisfied, then problem (CD) has a unique solution $u \in C^{2+\alpha}(\overline{Q_T})$. Moreover

$$||u||_{2+\alpha,Q_T} \le C(||u_0||_{2+\alpha,\Omega} + ||\Phi||_{2+\alpha,Q_T} + ||f||_{\alpha,Q_T}). \tag{11.26}$$

The constant depends on Q_T, n, α and L.

11.7 Remark If one only takes u_0 and Φ continuous and satisfying the zero order compatibility condition (11.24), one still has a unique solution $u \in C^2(Q_T) \cap C(\overline{Q_T})$. Continuity of f however is not sufficient, as is already indicated by our discussion of the volume potential.

11.8 Remark Similar results hold for bounded solutions of the Cauchy problem, where $\Omega = \mathbb{R}^n$ and no lateral boundary conditions are imposed. In Theorem 11.6 simply erase all terms involving Φ to obtain the correct result for $\Omega = \mathbb{R}^n$. Note that also the compatibility conditions disappear.

Next we give existence results for semilinear problems, where the function $f(x,t)$ is replaced by a function $f(x,t,u)$. As in the semilinear elliptic case we shall obtain solutions from an application of Schauder's fixed point theorem (Theorem A.36). The idea is again to substitute a function $v(x,t)$ for u in $f(x,t,u)$ and solve (CD) with $f(x,t)$ replaced by $f(x,t,v(x,t))$. This will define a mapping $T: v \to u$ whose fixed points are solutions to the semilinear problem. Let us make this precise.

11.9 Theorem In the context of Theorem 11.6 assume that on bounded subsets of $Q_T \times \mathbb{R}$ the functions f and $\frac{\partial f}{\partial u}$ are uniformly Hölder continuous with respect to x, t and u, respectively with exponents α, $\alpha/2$ and 1 (i.e. Lipschitz continuous in u). Also assume that the zero and first order compatibility conditions hold. The latter now reads

$$\lim_{t \downarrow 0} \frac{\partial \Phi}{\partial t}(x_0,t) = \lim_{x \to x_0} \left(L(x,0)u_0(x,0) + f(x,0,u_0(x))\right) \quad \forall x_0 \in \partial\Omega. \tag{11.27}$$

Finally assume that

$$\frac{\partial f}{\partial u} \text{ is bounded.} \tag{11.28}$$

Then problem (CD) with f replaced by $f(x,t,u)$ has a unique solution $u \in C^{2+\alpha}(\overline{Q_T})$.

Proof The uniqueness follows from the results in the previous section. For existence we first assume that f is nonincreasing in u and that the coefficient c of the zero order term in L is identically equal to zero. Writing

$$f(x,t,u) = f(x,t,0) + g(x,t,u)u, \tag{11.29}$$

where

$$g(x,t,u) = \begin{cases} \frac{f(x,t,u)-f(x,t,0)}{u} & u \neq 0 \\ \frac{\partial f}{\partial u}(x,t,0) & u = 0, \end{cases} \tag{11.30}$$

we have $g \leq 0$. We will apply Schauder's fixed point theorem to a sufficiently large ball in the space $X = C^{\alpha}(\overline{Q_T})$. For $v \in X$ consider the problem

$$(I_v) \begin{cases} u_t = Lu \; + g(x,t,v)u + f(x,t,0) & x \in \Omega, \; 0 < t \leq T; \\ u(x,0) = u_0(x) & x \in \overline{\Omega}; \\ u(x,t) = \Phi(x,t) & x \in \partial\Omega, \; 0 < t \leq T. \end{cases}$$

By Theorem 11.6 this defines a map

$$T : X \to X \; ; \qquad u = Tv. \tag{11.31}$$

If A is a bound on the initial data $|u_0|$, and B on the source term $|f(x,t,0)|$, then the functions $\overline{u}(x,t) = A + Bt$ and $\underline{u}(x,t) = -A - Bt$ are clearly super- and subsolutions for (I_v). Hence

$$|u(x,t)| \leq A + Bt. \tag{11.32}$$

Next we use the estimate (see [Friedman])

$$||u||_{\alpha,Q_T} \le C\{\sup_{Q_T}|u| + \sup_{Q_T}|f|\}. \tag{11.33}$$

Here the constant C depends on the boundary conditions.

As a consequence of (11.33) and (11.32) we have that $T(X)$ is bounded. But then, in view of (11.26), also $||u||_{2+\alpha,Q_T}$ is bounded independent of $v \in X$. It follows that, taking for A a large ball in X, the assumptions of Theorem A.36 are satisfied. Hence there exists a fixed point of T, which belongs to $C^{2+\alpha}(\overline{Q_T})$ and is a solution of our semilinear problem. This completes the proof for the particular case that $c \equiv 0$ and f is nonincreasing.

The general case is first reduced to zero parabolic boundary data, and then, by means of the transformation

$$z(x,t) = e^{-At}u(x,t),$$

reduced to the previous. ∎

We conclude this session with a remark on the general existence/uniqueness result for quasilinear equations on a bounded domain with Dirichlet boundary data. In fact the coefficients a_{ij} may be allowed to depend on x, t, u and u_x. Here $u_x = (u_{x_1}, \ldots, u_{x_n})$. The remaining terms of L are collected in one single term $b(x,t,u,u_x)$. Provided the uniform ellipticity conditions and some additional growth conditions are satisfied, one can obtain again the unique existence of a classical solution. This can be found in Chapters 5 and 6 of [LSU]. The proofs are basically continuation arguments to the linear theory and rely heavily on the appropiate apriori (Schauder type) estimates.

12. Asymptotic behaviour of solutions to the semilinear heat equation

For a bounded domain with boundary $\partial\Omega \in C^{2,\alpha}$ we consider the problem

$$(P)\begin{cases} u_t = \Delta u + f(u) & \text{in } Q = \Omega \times (0,\infty); \\ u = 0 & \text{on } \Gamma = \partial\Omega \times (0,\infty); \\ u = \phi & \text{on } \overline{\Omega}. \end{cases}$$

We shall be dealing with classical solutions, that is, solutions which satisfy at least $u \in C^{2,1}(Q) \cap C(\overline{Q})$. For sufficiently smooth f with $f'(u)$ bounded we have seen (Remark 10.7) that (P) has a unique classical solution, provided the initial profile ϕ is uniformly continuous on $\overline{\Omega}$, with $\phi = 0$ on $\partial\Omega$. To control the global behaviour of solution orbits we make the assumption

$$\exists M > 0 : \qquad |u| > M \Rightarrow uf(u) < 0.$$

Thus large positive constants are supersolutions and large negative constants are subsolutions. This implies that the solution is uniformly bounded.

We can view the map which assigns the solution $u(\cdot,t)$ at time $t \geq 0$ to the function ϕ as a continuous semigroup $\{T(t),\ t \geq 0\}$, provided we choose an appropiate Banach space to work in. This choice also depends on the Liapounov functional for this problem, which is obtained from formally multiplying the equation by $-u_t$ and integrating by parts over Ω. Writing $F(u) = \int_0^u f(s)ds$, this yields,

$$-\int_\Omega u_t^2 = -\int_\Omega u_t \Delta u - \int_\Omega f(u)u_t = \int_\Omega \frac{\partial}{\partial t}(\frac{1}{2}|\nabla u|^2 - F(u))$$

$$= \frac{d}{dt}\int_\Omega (\frac{1}{2}|\nabla u|^2 - F(u)).$$

Hence the Liapounov functional should be

$$V(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 - F(u)).$$

Thus we need a little more regularity for u then just $u \in C^{2,1}(Q) \cap C(\overline{Q})$.

We choose to work in the space

$$X = \{\phi \in C^1(\overline{\Omega}),\ \phi = 0 \text{ on } \partial\Omega\}.$$

The following theorem then follows from the basic results on existence and uniqueness of classical solutions.

12.1 Theorem Let $\{T(t) : X \to X,\quad t \geq 0\}$ be defined by $\phi \to u(t)$, where $u(t)$ is the solution of Problem (P) with initial value ϕ. Then T is a continuous semigroup on X, and every orbit $\omega(\phi)$ is precompact. Moreover, the function V defined above is a strict Liapounov functional for T. Thus the ω-limit set $\omega(\phi)$ of $\gamma(\phi)$ is nonempty, compact, connected, and consists of equilibria, on which the function V takes the same value.

As a consequence of this theorem we have that, if the equilibria of (P) are isolated, every solution stabilizes to an equilibrium.

Another question is: how fast does a solution stabilize to an equilibrium? This is related to the *principle of linearized stability.* We consider an equilibrium v of (P), and a solution $u(x,t)$ of (P) with initial value ϕ "close" to v. Writing $w(x,t) = u(x,t) - v(x)$, i.e. $u = v + w$, we obtain the equation

$$w_t = \Delta w + f(v(x) + w) - f(v(x))$$

for w. Replacing the nonlinear part of this equation by the linear approximation $f'v(x))w$, we obtain the linear parabolic equation

$$w_t = \Delta w + qw, \quad q(x) = f'(v(x)),$$

for which we can try solutions of the form

$$w(x,t) = e^{-\lambda t}\psi(x).$$

This leads us to the eigenvalue problem

$$(E_\lambda) \begin{cases} \Delta\psi + q\psi = -\lambda\psi & \text{in } \Omega; \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

This eigenvalue problem allows a unique sequence of eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \uparrow \infty,$$

and corresponding eigenfunctions

$$\psi_1, \psi_2, \psi_3, \psi_4 \ldots,$$

such that, upto scalar multiples of ψ_i, the only solutions of (E_λ) are the pairs

$$(\lambda_1, \psi_1), (\lambda_2, \psi_2), (\lambda_3\psi_3), (\lambda_4, \psi_4) \ldots$$

The function ψ_1 is strictly positive in Ω, and is the only eigenfunction which will appear in the proofs below. Clearly the sign of λ_1 will be important for the stability of v. We shall prove that it actually determines the stability of v. First however we introduce a new Banach space to work in.

Let $e \in C^2(\overline{\Omega})$ be the solution of

$$\begin{cases} \Delta e = -1 & \text{in } \Omega; \\ e = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $e > 0$ in Ω, and on the boundary $\partial\Omega$ we have for the normal derivative, because of the boundary point lemma, that $\partial e/\partial\nu < 0$. We introduce the space

$$Y = C_e(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : \exists k > 0 : |u(x)| \leq ke(x) \ \forall x \in \Omega\},$$

with corresponding norm

$$||u||_e = \inf\{k > 0 : |u(x)| \leq ke(x) \forall \in \Omega\}.$$

With this norm Y is a Banach space.

12.2 Theorem Let v be an equilibrium for (P), and suppose that λ_1 is the first eigenvalue of the corresponding problem (E_λ). Then, if $\lambda_1 > 0$, the equilibrium v is asymptotically stable in the space Y, and moreover,

$$\exists C > 0\ \forall t \geq 0 :\ ||\phi - v||_e < \delta \Rightarrow ||T(t)\phi - v||_e \leq Ce^{-\lambda_1 t}.$$

If $\lambda_1 < 0$, the equilibrium v is unstable.

Proof Suppose that $\lambda_1 > 0$. Writing (λ, ψ) for (λ_1, ψ_1), we look for a supersolution of the form

$$\overline{u}(x,t) = v(x) + g(t)\psi(x),$$

with $g(t)$ positive and decreasing. We compute

$$-\overline{u}_t + \Delta\overline{u} + f(\overline{u}) = -g'\psi + \Delta v + g\Delta\psi + f(v + g\psi) =$$

$$-g'\psi - f(v) - g(f'(v) + \lambda)\psi + f(v + g\psi) =$$

$$-g'\psi + g\psi\big(\frac{f(v + g\psi) - f(v)}{g\psi} - f'(v) - \lambda\big) =$$

(for some $\theta = \theta(x) \in (0,1)$)

$$-g'\psi + g\psi\big(f'(v + \theta g\psi) - f'(v) - \lambda\big) \leq$$

$$\phi(-g' - \lambda g + Cg^2).$$

Thus, if $g(t)$ is a solution of

$$\begin{cases} g' = -\lambda g + Cg^2 & \text{for } t > 0; \\ g(0) = \mu > 0, \end{cases}$$

it follows that $\overline{u}$ is a supersolution. Now observe that for every $0 < \mu < \lambda/C$ the unique solution $g(t)$ satisfies an estimate of the form

$$g(t) \leq Ce^{-\lambda_1 t}.$$

Thus the statement of the theorem now follows from the maximum principle, provided we take

$$\phi(x) \leq v(x) + \mu\psi(x).$$

This proves the asymptotic and exponential stablility from above.

The remainder of the proof is left as an exercise. ∎

Appendix: Functional Analysis

A. Banach spaces

A.1 Definition A real vector space X is called a *real normed space* if there exists a map

$$\|\cdot\| : X \to \overline{\mathbb{R}}^+,$$

such that, for all $\lambda \in \mathbb{R}$ and $x, y \in X$, (i) $\|x\| = 0 \Leftrightarrow x = 0$; (ii) $\|\lambda x\| = |\lambda|\,\|x\|$; (iii) $\|x + y\| \leq \|x\| + \|y\|$. The map $\|\cdot\|$ is called the *norm*.

A.2 Definition Suppose X is a real normed space with norm $\|\cdot\|$, and that $|||\cdot|||$ is also a norm on X. Then $\|\cdot\|$ and $|||\cdot|||$ are called equivalent if there exist $A, B > 0$ such that for all $x \in X$

$$A\|x\| \leq |||x||| \leq B\|x\|.$$

A.3 Notation $B_R(y) = \{x \in X : \|x - y\| < R\}$.

A.4 Definition Let S be a subset of a normed space X. S is called *open* ($\Leftrightarrow X\backslash S$ is closed) if for every $y \in S$ there exists $R > 0$ such that $B_R(y) \subset S$. The open sets form a *topology* on X, i.e. (i) $\emptyset$ and X are open; (ii) unions of open sets are open; (iii) finite intersections of open sets are open.

A.5 Remark Equivalent norms define the same topology.

A.6 Definition Let X be a normed space, and $(x_n)_{n=1}^\infty \subset X$ a sequence. Then $(x_n)_{n=1}^\infty$ is called *convergent* with limit $\overline{x} \in X$ (notation $x_n \to \overline{x}$) if $\|x_n - \overline{x}\| \to 0$ as $n \to \infty$. If $\|x_n - x_m\| \to 0$ as $m, n \to \infty$, then $(x_n)_{n=1}^\infty$ is called a *Cauchy sequence*.

A.7 Definition A normed space X is called a *Banach space* if every Cauchy sequence in X is convergent.

A.8 Theorem (Banach contraction theorem) Let X be a Banach space and $T : X \to X$ a contraction, i.e. a map satisfying

$$\|Tx - Ty\| \leq \theta\|x - y\| \quad \forall x, y \in X,$$

for some fixed $\theta \in [0, 1)$. Then T has a unique fixed point $\overline{x} \in X$. Moreover, if $x_0 \in X$ is arbitrary, and $(x_n)_{n=1}^\infty$ is defined by

$$x_n = Tx_{n-1} \qquad \forall n \in \mathbf{N},$$

then $x_n \to \overline{x}$ as $n \to \infty$.

A.9 Definition Let X and Y be normed spaces, and $T : X \to Y$ a linear map, i.e.

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty \quad \text{for all} \quad \lambda, \mu \in \mathbb{R} \;\text{ and }\; x, y \in X.$$

Then T is called *bounded* if

$$\|T\| = \sup_{0 \neq x \in X} \frac{\|Tx\|_Y}{\|x\|_X} < \infty.$$

The map $T \to \|T\|$ defines a norm on the vector space $\mathcal{B}(X, Y)$ of bounded linear maps $T : X \to Y$, so that $\mathcal{B}(X, Y)$ is a normed space. In the case that $Y = \mathbb{R}$, the space $X^* = \mathcal{B}(X, \mathbb{R})$ is called the *dual space* of X.

A.10 Theorem Let X be a normed space and Y a Banach space. Then $\mathcal{B}(X, Y)$ is also a Banach space. In particular every dual space is a Banach space.

Many problems in linear partial differential equations boil down to the question as to whether a given linear map $T : X \to Y$ is invertible.

A.11 Theorem Let X and Y be Banach spaces, and let $T \in \mathcal{B}(X, Y)$ a bijection. Then $T^{-1} \in \mathcal{B}(Y, X)$.

A.12 Theorem (method of continuity) Let X be a Banach space and Y a normed space, and T_0 and $T_1 \in B(X, Y)$. For $t \in [0, 1]$ let $T_t \in \mathcal{B}(X, Y)$ be defined by

$$T_t x = (1 - t) T_0 x + t T_1 x.$$

Suppose there exists $C > 0$ such that

$$\|x\|_X \leq C \|T_t x\|_Y \qquad \forall x \in X.$$

Then T_0 is surjective if and only if T_1 is surjective, in which case all T_t are invertible with

$$T_t^{-1} \in \mathcal{B}(Y, X) \quad \text{and} \quad \|T_t^{-1}\| \leq C.$$

A.13 Definition Let X, Y be normed spaces, and $T \in \mathcal{B}(X, Y)$. The *adjoint* T^* of T is defined by

$$T^* f = f \circ T \qquad \forall f \in Y^*,$$

i.e.

$$(T^* f)(x) = f(Tx) \quad \forall x \in X.$$

A.14 Remark Observe that if $X = Y$ and I is the identity on X, then I^* is the identity on X^*.

A.15 Theorem Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Then $T^* \in \mathcal{B}(Y^*, X^*)$ and

$$\|T\|_{B(X,Y)} = \|T^*\|_{\mathcal{B}(Y^*,X^*)}.$$

A.15 Definition Let X, Y be normed spaces and $T \in \mathcal{B}(X, Y)$. Then T is called *compact* if, for every bounded sequence $(x_n)_{n=1}^\infty \subset X$, the sequence $(Tx_n)_{n=1}^\infty$ contains a convergent (in Y) subsequence. The linear subspace of compact bounded linear maps is denoted by $\mathcal{K}(X, Y)$.

A.16 Theorem Let X be a normed space and Y a Banach space. Then $\mathcal{K}(X, Y)$ is a closed linear subspace of $\mathcal{B}(X, Y)$. Furthermore: $T \in \mathcal{K}(X, Y) \Leftrightarrow T^* \in \mathcal{K}(Y^*, X^*)$.

In all practical cases one can only verify that $T \in \mathcal{K}(X, Y)$ if Y is Banach, because then it suffices to extract a Cauchy sequence from $(Tx_n)_{n=1}^\infty$.

A.17 Definition Let X be a normed space and $M \subset X$. Then

$$M^\perp = \{f \in X^* : \quad f(x) = 0 \quad \forall x \in M\}.$$

A.18 Definition Let X, Y be vector spaces, and $T : X \to Y$ linear. Then

$$N(T) = \{x \in X : Tx = 0\} \quad \text{(kernel of } T\text{)},$$

$$R(T) = \{y \in Y : \exists x \in X \ \text{ with } \ Y = Tx\} \quad \text{(range of } T\text{)}.$$

Clearly these are linear subspaces of X and Y respectively.

A.18 Definition Let X be a normed space and $M \subset X^*$. Then

$$^\perp M = \{x \in X : f(x) = 0 \qquad \forall f \in M\}.$$

A.19 Theorem (Fredholm alternative) Let X be a Banach space and $T \in \mathcal{K}(X) = \mathcal{K}(X, X)$. Let $I \in \mathcal{B}(X) = \mathcal{B}(X, X)$ denote the identity. Then

(i) $\dim N(I - T) = \dim N(I^* - T^*) < \infty$;

(ii) $R(I - T) =^\perp N(I^* - T^*)$ is closed;

(iii) $N(I-T)=\{0\} \;\Leftrightarrow\; R(I-T)=X$.

Thus $I-T$ has properties resembling those of *matrices*.

A.20 Definition Let X be a Banach space and $T \in \mathcal{B}(X)$. Then

$$\rho(t)=\{\lambda\in\mathbb{R} : T-\lambda I \text{ is a bijection}\}$$

is called the *resolvent set* of T, and $\sigma(T)=\mathbb{R}\backslash\rho(T)$ the *spectrum* of T. A subset of the spectrum is

$$\sigma_E(T)=\{\lambda\in\sigma(T) : \lambda \text{ is an eigenvalue of } T\}=\{\lambda\in\mathbb{R} : N(T-\lambda I)\neq\{0\}\}.$$

A.21 Theorem Let X be a Banach space and $T\in\mathcal{K}(X)$. Then

(i) $\sigma(T)\subset[-\|T\|,\|T\|]$ is compact;

(ii) $\dim X=\infty \;\Rightarrow\; 0\in\sigma(T)$;

(iii) $\sigma(T)\backslash\{0\}\subset\sigma_E(T)$;

(iv) either $\sigma(T)\backslash\{0\}$ is finite or $\sigma(T)\backslash\{0\}$ consists of a sequence converging to zero.

A.22 Definition Let X be a vector space. A *convex cone* in X is a set $C\subset X$ with

$$\lambda x+\mu y\in C \quad \forall\lambda,\mu\in\overline{\mathbb{R}}^{+}\ \forall x,y\in C.$$

A.23 Theorem (Krein-Rutman) Let X be a Banach space and $C\subset X$ a closed convex cone with

$$\mathrm{int}C\neq\emptyset \quad\text{and}\quad C\cap(-C)=\{0\}.$$

Suppose $T\in\mathcal{K}(X)$ satisfies

$$T(C\backslash\{0\})\subset\mathrm{int}C.$$

Then $\overline{\lambda}=\sup\sigma(T)$ is the only eigenvalue with an eigenvector in C, and its multiplicity is one.

A.24 Theorem Let X,Y,X be Banach spaces, and $T\in\mathcal{B}(X,Y)$, $S\in\mathcal{B}(Y,Z)$. If $T\in\mathcal{K}(X,Y)$ or $S\in\mathcal{K}(Y,Z)$, then $S\circ T\in\mathcal{K}(X,Z)$.

A.25 Definition Let X be a normed space. The *weak topology* on X is the smallest topology on X for which every $f\in X^*$ is a continuous function from X to $\mathbb{R}$ (with respect to this topology).

The weak topology is weaker then the norm topology, i.e. every norm open set is also weakly open. If X is finite dimensional, the converse also holds, but never if $\dim X = \infty$.

A.26 Notation In every topology one can define the concept of convergence. For x_n converging to x in the weak topology we use the notation $x_n \rightharpoonup x$.

A.27 Proposition Let X be a Banach space, and $(x_n)_{n=1}^{\infty}$ a sequence in X. Then

(i) $x_n \rightharpoonup x \;\Leftrightarrow\; f(x_n) \to f(x) \;\; \forall f \in X^*$;

(ii) $x_n \rightharpoonup x \Rightarrow \|x\|$ is bounded and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

A.28 Theorem Let X be a Banach space, and $K \subset X$ a convex set, i.e. $\lambda x + (1-\lambda)y \in K \; \forall x, y \in K \; \forall \lambda \in [0,1]$. Then K is weakly closed if and only if K is norm closed.

A.29 Notation Let X be a normed space, $x \in X$ and $f \in X^*$. Then we shall frequently write $< f, x >= f(x)$. Thus $< \cdot, \cdot >: X^* \times X \to \mathbb{R}$. Note that for every fixed $x \in X$ this expression defines a function from X^* to $\mathbb{R}$.

A.30 Definition The *weak** topology on X^* is the smallest topology for which all $x \in X$ considered as functions from X^* to $\mathbb{R}$ are continuous.

A.31 Notation For convergence in the weak* topology we write $f_n \overset{*}{\rightharpoonup} f$, and again this is equivalent to $< f_n, x > \to < f, x >$ for all $x \in X$. The importance of the weak* topology lies in

A.32 Theorem (Alaoglu) Let X be a Banach space. Then the closed unit ball in X^* is compact in the *weak** topology.

A.33 Definition A Banach space X is called *reflexive* if every $\varphi \in (X^*)^*$ is of the form

$$\varphi(f) = f(x) =< f, x > \qquad \forall f \in X^*$$

for some $x \in X$.

A.34 Corollary Let X be a separable reflexive space. Then every bounded sequence in X has a weakly convergent subsequence.

A.35 Theorem (Schauder's fixed point theorem, first version) Let X be a Banach space, let $A \subset X$ be convex and compact, and let $T : A \to A$ be a continuous mapping. Then T has at least one fixed point.

Proof Since A is compact, we can, given any integer k, make a covering of A with finitely many balls $B_1, B_2, \dots, B_N$ with radius $\frac{1}{k}$. The centers of these balls are denoted by $x_1, x_2, \dots, x_N$. Let A_k be the convex hull of these points, i.e. the set of all convex combinations. This is a subset of A because A is convex.

For any $x \in A$ let $d_i = d(x, A - B_i)$ (here d stands for distance). Then $d_i = 0$, unless $x \in B_i$. With this observation in mind, we define a continuous map $J_k : A \to A_k$ by the convex combination

$$J_k(x) = \frac{\sum_{i=1}^n d_i x_i}{\sum_{i=1}^n d_i}.$$

Next we observe that Brouwer's fixed point theorem applies to A_k, because A_k is the convex hull of a finite number of points. Thus the composition $J_k \cdot T$, which takes A_k to itself, must have a fixed point $\overline{x}_k$.

Since the construction of J_k implies that $||J_k(x) - x|| < \frac{1}{k}$, it follows that

$$||T(\overline{x}_k) - \overline{x}_k|| = ||T(\overline{x}_k) - J_k(T(\overline{x}_k))|| < \frac{1}{k}.$$

By compactness we can then conclude that a subsequence of $(\overline{x}_k)$ converges to a fixed point of T.

A.36 Theorem (Schauder's fixed point theorem, second version) Let X be a Banach space, and let $A \subset X$ be convex and closed. Moreover, let $T : A \to A$ be a continuous mapping with the property that $T(A)$ is precompact in X. Then T has at least one fixed point in A.

Proof Exercise.

B. Hilbert spaces

B.1 Definition Let H be a (real) vector space. A function

$$(\cdot,\cdot) : H \times H \to \mathbb{R}$$

is called an *inner product* if, for all $u, v, w \in H$ and for all $\lambda, \mu \in \mathbb{R}$, (i) $(u, u) \geq 0$, and $(u, u) = 0 \Leftrightarrow u = 0$; (ii) $(u, v) = (v, u)$; (iii) $(\lambda u + \mu v, w) = \lambda(u, w) + \mu(v, w)$.

B.2 Remark Any inner product satisfies

$$|(u, v)| \leq \sqrt{(u, u)(v, v)} \quad \forall u, v \in H, \quad \text{(Schwartz)}$$

and also
$$\sqrt{(u+v,u+v)} \le \sqrt{(u,u)} + \sqrt{(v,v)} \quad \forall u,v \in H.$$
Consequently, $\|u\| = \sqrt{(u,u)}$ defines a norm on H, called the *inner product norm.*

B.3 Definition If H is a Banach space with respect to this inner product norm, then H is called a *Hilbert space.*

B.4 Theorem For every closed convex subset K of a Hilbert space H, and for every $f \in H$, there exists a unique $u \in K$ such that

$$\|f-u\| = \min_{v\in K} \|f-v\|,$$

or, equivalently,
$$(f-u, v-u) \le 0 \qquad \forall v \in K.$$
Moreover the map $P_K : f \in H \to u \in K$ is contractive in the sense that

$$\|P_K f_1 - P_K f_2\| \le \|f_1 - f_2\|.$$

B.5 Theorem (Riesz) For fixed $f \in H$ define $\varphi \in H^*$ by

$$\varphi(v) = (f,v) \quad \forall v \in H.$$

Then the map $f \to \varphi$ defines an *isometry* between H and H^*, which allows one to identify H and H^*.

B.6 Corollary Every Hilbert space is reflexive. In particular bounded sequences of Hilbert spaces have weakly convergent subsequences.

B.7 Theorem Let H be a Hilbert space, and $M \subset H$ a closed subspace. Let

$$M^\perp = \{u \in H : \quad (u,v) = 0 \quad \forall v \in M\}.$$

Then $H = M \oplus M^\perp$, i.e. every $w \in H$ can be uniquely written as

$$w = u + v, \quad u \in M, \quad v \in M^\perp.$$

B.8 Definition Let H be a Hilbert space. A *bilinear form* $A : H \times H \to \mathbb{R}$ is called *bounded* if, for some $C > 0$,

$$|A(u,v)| \le C\|u\|\,\|v\| \quad \forall u,v \in H,$$

coercive if, for some $\alpha > 0$,

$$A(u,u) \geq \alpha \|u\|^2 \qquad \forall u \in H,$$

and *symmetric* if

$$A(u,v) = A(v,u) \qquad \forall u, v \in H.$$

B.9 Remark A symmetric bounded coercive bilinear form on H defines an equivalent inner product on H.

B.10 Theorem (Stampacchia) Let K be a closed convex subset of a Hilbert space H, and $A : H \times H \to \mathbb{R}$ bounded coercive bilinear form. Let $\varphi \in H^*$. Then there exists a unique $u \in K$ such that

$$A(u, v-u) \geq \varphi(v-u) \qquad \forall v \in K.$$

Moreover, if A is also symmetric, then u is uniquely determined by

$$\frac{1}{2}A(u,u) - \varphi(u) = \min_{v \in K}\{\frac{1}{2}A(v,v) - \varphi(v)\}.$$

B.11 Corollary (Lax -Hilgram) Under the same conditions there exists a unique $u \in H$ such that

$$A(u,v) = \varphi(v) \qquad \forall v \in H.$$

Moreover, if A is symmetric, then u is uniquely determined by

$$\frac{1}{2}A(u,u) - \varphi(u) = \min_{v \in H}\{\frac{1}{2}A(v,v) - \varphi(v)\}.$$

Proof of Theorem B.10 Let (Riesz) φ correspond to $f \in H$. Fix $u \in H$. Then the map $v \to A(u,v)$ belongs to H^*. Thus, again by the Riesz Theorem, there exists a unique element in H, denoted by $\hat{A}u$, such that

$$A(u,v) = (\hat{A}u, v).$$

Clearly $\|\hat{A}u\| \leq C\|u\|$, and $(\hat{A}u, u) \geq \alpha\|u\|^2$ for all $u \in H$. We want to find $u \in K$ such that

$$A(u, v-u) = (\hat{A}u, v-u) \geq (f, v-u) \qquad \forall v \in K.$$

For $\rho > 0$ to be fixed later, this is equivalent to

$$(\rho f - \rho\hat{A}u + u - u,\ v - u) \leq 0 \qquad \forall v \in K,$$

i.e.

$$u = P_K(\rho f - \rho \hat{A} u + u).$$

Thus we have to find a fixed point of the map S defined by

$$S: \ u \to P_K(\rho f - \rho \hat{A} u + u),$$

so it suffices to show that S is a strict contraction. We have,

$$\begin{aligned} \|Su_1 - Su_2\| &= \|P_K(\rho f - \rho \hat{A} u_1 + u_1) - P_K(\rho f - \rho \hat{A} u_2 + u_2) \\ &\le \|(u_1 - u_2) - \rho(\hat{A} u_1 - \hat{A} u_2)\|, \end{aligned}$$

so that

$$\begin{aligned} \|Su_1 - Su_2\|^2 &\le \|u_1 - u_2\|^2 - 2\rho(\hat{A} u_1 - \hat{A} u_2, u_1 - u_2) + \rho^2 \|\hat{A} u_1 - \hat{A} u_2\|^2 \\ &\le \|u_1 - u_2\|^2 (1 - 2\rho\alpha + \rho^2 C^2). \end{aligned}$$

Thus for $\rho > 0$ sufficiently small, S is a strict contraction, and has a unique fixed point. This completes the first part of the theorem.

Next, if A is symmetric, then by Remark B.9 above and Riesz' theorem, there is a unique $g \in H$ such that

$$\varphi(v) = A(g, v) \qquad \forall v \in H.$$

So we must find u such that $A(g - u, v - u) \le 0 \quad \forall v \in K$, i.e. $u = P_K g$, if we replace the scalar product by $A(\cdot, \cdot)$, or, equivalently, $u \in K$ is the minimizer for

$$\min_{v \in K} A(g - v, g - v)^{\frac{1}{2}} = (\min_{v \in K} A(g, g) + A(v, v) - 2A(g, v))^{\frac{1}{2}}. \ \blacksquare$$

B.12 Definition Let H be a Hilbert space. Then $T \in \mathcal{B}(H)$ is called *symmetric* if

$$(Tx, y) = (x, Ty) \qquad \forall x, y \in H.$$

B.13 Definition A Hilbert space H is called *separable* if there exists a countable subset $S \subset H$ such that for every $x \in H$ there exists a sequence $(x_n)_{n=1}^{\infty} \subset S$ with $x_n \to x$.

B.14 Theorem Every separable Hilbert space has a *orthonormal Schauderbasis* or *Hilbert basis*, i.e. a countable set $\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$ such that

(i) $(\varphi_i, \varphi_j) = \delta_{ij}$;

(ii) every $x \in H$ can be written uniquely as

$$x = x_1\varphi_1 + x_2\varphi_2 + x_3\varphi_3 + \ldots,$$

where $x_1, x_2, x_3, \ldots \in \mathbb{R}$. Moreover,

$$\|x\|^2 = x_1^2 + x_2^2 + x_3^2 + \ldots \ ,$$

and $x_i = (x, \varphi_i)$.

B.14 Theorem Let H be a Hilbert space, and $T \in \mathcal{K}(H)$ symmetric. Then H has a Hilbert basis $\{\varphi_1, \varphi_2, ...\}$ consisting of eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots \in \mathbb{R}$ with

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \downarrow 0,$$

and such that

$$|\lambda_1| = \sup_{0 \neq x \in H} \left|\frac{(Tx,x)}{(x,x)}\right| = \left|\frac{(T\varphi_1,\varphi_1)}{(\varphi_1,\varphi_1)}\right|;$$

$$|\lambda_2| = \sup_{\substack{0 \neq x \in H \\ (x,\varphi_1)=0}} \left|\frac{(Tx,x)}{(x,x)}\right| = \left|\frac{(T\varphi_2,\varphi_2)}{(\varphi_2,\varphi_2)}\right|;$$

$$|\lambda_3| = \sup_{\substack{0 \neq x \in H \\ (x,\varphi)=(x,\varphi_2)=0}} \left|\frac{(Tx,x)}{(x,x)}\right| = \left|\frac{(T\varphi_3,\varphi_3)}{(\varphi_3,\varphi_3)}\right|,$$

etcetera. Moreover, if $\psi \in H$ satisfies $(\psi, \varphi_1) = (\psi, \varphi_2) = \ldots = (\psi, \varphi_n) = 0$, and $(T\psi, \psi) = \lambda_{n+1}(\psi, \psi)$, then ψ is an eigenvector for λ_{n+1}.

C. Continous semigroups and Liapounov functionals

In this section we discuss an abstract framework which serves as a tool to establish stabilization of solutions to evolution equations for large times.

C.1 Definition Let X be a normed space. Then a one-parameter family of mappings $\{T(t) : X \to X, \ \ t \geq 0\}$, T for short, is called a *continuous semigroup on* X if: (i) $T(0)$ is the identity; (ii) $T(t+s) = T(t)T(s) \ \ \forall t, s \geq 0$; (iii) for all fixed $\phi \in X$ the map $t \to T(t)\phi$ is continuous from $[0, \infty)$ to X; (iv) for all fixed $t \geq 0$ the map $\phi \to T(t)\phi$ is continuous from X into itself. The set $\gamma(\phi) = \{T(t)\phi, \ \ t \geq 0\}$ is called the *orbit* through ϕ. The *ω-limit set* of $\gamma(\phi)$, denoted by $\omega(\phi)$, is the set of all limit points of sequences $(T(t_n)\phi, \ n = 1, 2, \ldots)$ in X with $t_n \to \infty$.

We note that an orbit $\gamma(\phi)$ converges to an element χ in X if and only if the corresponding ω-limit set $\omega(\phi)$ is the singleton $\{\chi\}$. Such an element is easily seen to satisfy $T(t)\chi = \chi$ for all $t \geq 0$, i.e. χ is an equilibrium for T. Thus our ultimate goal is to find conditions guaranteeing that $\omega(\phi)$ consists of one single equilibrium.

C.2 Theorem Let T be a continuous semigroup on a normed space X. For every ϕ in X, the semigroup leaves $\omega(\phi)$ invariant. Moreover, if the orbit $\gamma(\phi)$ is precompact, then $\omega(\phi)$ is nonempty, compact, and connected.

Proof Clearly $\omega(\phi)$ is invariant, because for $\psi = \lim_{n\to\infty} T(t_n)\phi$ we have $T(t)\psi = T(t)\lim_{n\to\infty} T(t_n)\phi = \lim_{n\to\infty} T(t)T(t_n)\phi = \lim_{n\to\infty} T(t+t_n)\phi \in \omega(\phi)$.

By the compactness assumption $\omega(\phi)$ is nonempty. Moreover, $\omega(\phi)$ is contained in its own (compact) closure. Hence $\omega(\phi)$ will be compact if it can be shown to be closed. So suppose that $(\psi_n,\ n = 1, 2, \ldots)$ is a sequence in $\omega(\phi)$, which converges to some limit $\psi \in X$. By definition there exists for every n a $t_n > n$ such that $||T(t_n)\phi - \psi_n|| < 1/n$. Hence $||T(t_n)\phi - \psi|| < 1/n + ||\psi - \psi_n|| \to 0$. Since $t_n \to \infty$, it follows that ψ belongs to $\omega(\phi)$. Therefore $\omega(\phi)$ is closed.

Finally we show that $\omega(\phi)$ is connected. Arguing by contradiction we suppose that $\omega(\phi)$ is contained in the union of two disjoint open sets A and B, both having nonempty intersection with $\omega(\phi)$. Thus there exist two sequences $a_n \to \infty$ and $b_n \to \infty$ such that $T(a_n)\phi$ converges to a limit in A, and $T(b_n)\phi$ to a limit in B. Since the map $t \to T(t)\phi$ is continuous, it follows that there must be a sequence $t_n \to \infty$, with t_n between a_n and b_n, such that $T(t_n)\phi$, and hence also the limit points of this latter sequence, belong neither to A nor B. By assumption, there exists at least one such limit point. However, $A \cup B$ was supposed to contain all the limitpoints, a contradiction.

This completes the proof. ∎

C.3 Definition Let T be a continuous semigroup on a normed space X. A function $V : X \to \mathbb{R}$ is called a *Liapounov functional* for T, if for every ϕ in X the function $t \to V(T(t)\phi)$ is nonincreasing on $[0, \infty)$. If it is *strictly* decreasing in every point $t = t_0$ *unless* $T(t_0)\phi$ is an equilibrium, V is called a *strict* Liapounov functional.

Here a function $f : [0, \infty) \to \mathbb{R}$ is understood to be strictly decreasing in $t = t_0$ if there exists a $\delta > 0$ such that $t_0 < t < t_0 + \delta$ implies that $f(t) < f(t_0)$.

C.4 Theorem Let T be a continuous semigroup on a normed space X, and suppose that $V : X \to \mathbb{R}$ is a *continuous* Liapounov functional for T. Then V is constant on every (possibly empty) ω-limit set $\omega(\phi)$. If V is also a *strict* Liapounov functional, $\omega(\phi)$ consists only of equilibria.

Proof For some fixed $\phi \in X$ assume that $\omega(\phi)$ contains at least two elements

ψ and χ. Then there exist sequences $t_n \to \infty$ and $s_n \to \infty$, $s_n > t_n$ for all n, such that $T(t_n)\phi \to \psi$ and $T(s_n)\phi \to \chi$. But then $V(\chi) = V(\lim T(s_n)\phi) = \lim V(T(s_n)\phi) \le \lim V(T(t_n)\phi) = V(\lim T(t_n)\phi) = V(\psi)$. Since we can interchange the role of ψ and χ, it follows that V is constant on $\omega(\phi)$. The second part of the theorem is now immediate. ∎

C.5 Theorem Let T be a continuous semigroup on a normed space X, suppose that $V : X \to \mathbb{R}$ is a continuous strict Liapounov functional for T, and suppose that for some $\phi \in X$, the orbit $\gamma(\phi)$ is precompact. Then, if the equilibria of T are all isolated, $T(t)\phi$ converges to an equilibrium as $t \to \infty$.

Proof Exercise. Hint: assume that $\omega(\phi)$ contains more than one element and derive a contradiction. ∎

C.6 Definition Let T be a continuous semigroup on a normed space X. An equilibrium χ of T is called *stable* if

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall t \ge 0 : \ ||\phi - \chi|| < \delta \Rightarrow ||T(t)\phi - \chi|| < \epsilon.$$

If in addition

$$||\phi - \chi|| < \delta \Rightarrow ||T(t)\phi - \chi|| \to 0,$$

then χ is called *asymptotically stable.* If χ is not stable, then it is called *unstable.*

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