Functional Analysis

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October 16, 2007

1 Basic concepts

Definition 1.1 A point set X is called a metric space if there exists a map

$$d : X \times X \to \overline{\mathbb{R}^+},$$

such that, for all $x, y, z \in X$, (i) $d(x, y) = 0 \Leftrightarrow x = y$; (ii) $d(x, y) = d(y, x)$; (iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality). The map d is called the metric.

Definition 1.2 A real vector space X is called a real normed space if there exists a map

$$|| \cdot || : X \to \overline{\mathbb{R}^+},$$

such that, for all $\lambda \in \mathbb{R}$ and $x, y \in X$, (i) $||x|| = 0 \Leftrightarrow x = 0$; (ii) $||\lambda x|| = |\lambda| \, ||x||$; (iii) $||x + y|| \leq ||x|| + ||y||$ (triangle inequality). The map $|| \cdot ||$ is called the norm. If $|| \cdot ||$ only satisfies (ii) and (iii) then it is called a seminorm. Note that X is automatically a metric space with respect to the metric defined by $d(x, y) = ||x - y||$.

Exercise 1 In $\mathbb{R}^2$ we consider for $x = (x_1, x_2)$ the norms

$$\begin{aligned} ||x||_1 &= |x_1| + |x_2|, \\ ||x||_2 &= \sqrt{x_1^2 + x_2^2}, \\ ||x||_\infty &= \max\{|x_1|, |x_2|\}. \end{aligned}$$

Prove that $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_\infty$ are norms in $\mathbb{R}^2$ and draw the corresponding unit balls.

Exercise 2 Prove that in $\mathbb{R}^k$ the norms $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_\infty$ satisfy

$$||x||_\infty \leq ||x||_2 \leq ||x||_1 \leq \sqrt{k}||x||_2 \leq k||x||_\infty \text{ for all } x \in \mathbb{R}^k.$$

Definition 1.3 Two normed spaces X and Y are called isometric if there exists an isometry $T : X \to Y$, i.e. T is linear, bijective and norm preserving ($\forall x \in X : ||T(x)|| = ||x||$).

Definition 1.4 The set $B(y, R) = \{x \in X : d(x, y) < R\}$ is called an open ball with center y and radius R.

Definition 1.5 Let S be a subset of a metric or normed space X. S is called open ($\Leftrightarrow X \backslash S$ is closed) if for every $y \in S$ there exists $R > 0$ such that $B(y, R) \subset S$. The open sets form a topology on X, i.e. (i) $\emptyset$ and X are open; (ii) unions of open sets are open; (iii) finite intersections of open sets are open. This topology is denoted by τ.

Proposition 1.6 *Equivalent norms on a vector space X define the same topology. Two norms $||\cdot||$ and $|||\cdot|||$ are called equivalent if there exist $A, B > 0$ such that for all $x \in X$*

$$A||x|| \leq |||x||| \leq B||x||.$$

Exercise 3 If X and Y are normed spaces then $X \times Y$ is also a normed space. The norms $||(x, y)|| = ||x|| + ||y||$, $||(x, y)|| = \sqrt{||x||^2 + ||y||^2}$ and $||(x, y)|| = \max(||x||, ||y||)$ are all equivalent.

Definition 1.7 Let X be a metric space, and $(x_n)_{n=1}^{\infty} \subset X$ a sequence. Then $(x_n)_{n=1}^{\infty}$ is called convergent with limit $\overline{x} \in X$ (notation $x_n \to \overline{x}$) if $d(x_n, \overline{x}) \to 0$ as $n \to \infty$. If $d(x_n, x_m) \to 0$ as $m, n \to \infty$, then $(x_n)_{n=1}^{\infty}$ is called a Cauchy sequence.

Theorem 1.8 *(Riesz' Lemma) Let X be a normed space and $L \subset X$, $L \neq X$, a closed suspspace. For every $\theta \in (0, 1)$ there exists $x_\theta \in X$ with $||x_\theta|| = 1$ such that $d(x_\theta, L) = \inf\{||x - x_\theta|| : x \in L\} > \theta$.*

Definition 1.9 Let X be a normed space. Then X has the Heine-Borel property if every bounded sequence in X has a convergent subsequence (with limit in X).

Theorem 1.10 *Let X be a normed space. Then X has the Heine-Borel property if and only if X is finite dimensional.*

Theorem 1.11 *Let X be a finite dimensional normed space. Then all norms on X are equivalent and there is norm on X such that with this norm X is isometric to $\mathbb{R}^N$ with the standard Euclidean norm where N is the dimension of X.*

Definition 1.12 A metric space X is called complete if every Cauchy sequence in X is convergent. A complete normed space is called a Banach space.

Theorem 1.13 *(Cantor) Let X be a metric space. Then X is complete if and only if $\cap_{n=1}^{\infty} F_n$ is a singleton for every sequence of nonempty closed sets $F_1 \supset F_2 \supset \cdots$ with* $\text{diam}(F_n) = \sup\{d(x, y) : x, y \in F_n\} \to 0$.

Exercise 4 Let X be a finite dimensional normed space. Then X is complete.

Exercise 5 (convergence of abolutely convergent series) Let X be a Banach space and $x_n \in X$ for $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} ||x_n|| < \infty \ \Rightarrow \ \sum_{n=1}^{\infty} x_n \ \text{ is convergent in } X.$$

Proposition 1.14 *A closed subset of a complete metric space is also a complete metric space (with the same metric). In particular is every closed subset of a Banach space a complete metric space and every closed subspace of a Banach space a Banach space.*

Theorem 1.15 *(Banach contraction theorem) Let X be a complete metric space and $T : X \to X$ a contraction, i.e. a map satisfying*

$$d(Tx, Ty) \leq \theta d(x, y) \quad \forall x, y \in X,$$

for some fixed $\theta \in [0, 1)$. Then T has a unique fixed point $\overline{x} \in X$. Moreover, if $x_0 \in X$ is arbitrary, and $(x_n)_{n=1}^{\infty}$ is defined by

$$x_n = Tx_{n-1} \qquad \forall n \in \mathbb{N},$$

then $x_n \to \overline{x}$ as $n \to \infty$.

Theorem 1.16 *(Baire) Let X be a complete metric space and suppose that $X_n \subset X$ with $n \in \mathbb{N}$ are closed sets with empty interior. Then $\cup_{n\in\mathbb{N}} X_n$ also has empty interior.*

Exercise 6 Let X be an infinite dimensional Banach space. Prove that X is not the union of a countable family of finite dimensional subspaces.

Definition 1.17 Let X be a vector space. A set $S \subset X$ is called linearly independent if $\lambda_1 x_1 + \cdots \lambda_N x_N = 0$ implies $\lambda_1 = \cdots = \lambda_N = 0$ for any choice of $N \in \mathbb{N}$, $x_1, \ldots, x_N \in S$ and $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$. A linearly independent set $S \subset X$ is called a Hamel basis of X if every $x \in X$ can be written als $x = \lambda_1 x_1 + \cdots \lambda_N x_N$ with $N \in \mathbb{N}$, $x_1, \ldots, x_N \in S$ and $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$.

Exercise 7 Let X be an infinite dimensional Banach space. Prove that X does not have a countable Hamel basis.

2 Examples of normed spaces

2.1 $\mathbb{R}^N$, sequence spaces

On $\mathbb{R}^N$ we have the norms

$$||x||_p = \Big(\sum_{n=1}^{N} |x_n|^p\Big)^{\frac{1}{p}} \ \ (1 \leq p < \infty) \ \text{ and } \ ||x||_\infty = \max_{n=1,\ldots,N} |x_n|.$$

Definition 2.1

$$1 \le p < \infty : \quad l^p = \{x = (x_1, x_2, x_3, \ldots) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

$$l^\infty = \{x = (x_1, x_2, x_3, \ldots) : \sup_{n \in \mathbb{N}} |x_n| < \infty\}.$$

$$(c) = \{x = (x_1, x_2, x_3, \ldots) : \lim_{n \to \infty} x_n \text{ exists}\}.$$

$$(c_0) = \{x = (x_1, x_2, x_3, \ldots) : \lim_{n \to \infty} x_n = 0\}.$$

Theorem 2.2 *l^p is a Banach space for $1 \le p < \infty$ w.r.t. the norm $||x||_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$. l^∞ is a Banach space w.r.t. the norm $||x||_\infty = \sup_{n \in \mathbb{N}} |x_n|$. (c) and (c_0) are closed subspaces of l^∞.*

Exercise 8

$$\text{If } 1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad x, y \ge 0, \quad \text{then } xy \le \frac{x^p}{p} + \frac{y^q}{q},$$

with equality if and only if $y = x^{p-1}$. (Note: $(p-1)(q-1) = 1$.)

Exercise 9 (Hölder's inequality)

$$\text{If } 1 \le p, q \le \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad x \in l^p, y \in l^q, \quad \text{then } \sum_{n=1}^{\infty} |x_n y_n| \le ||x||_p ||y||_q.$$

Exercise 10 ($1 \le p \le \infty$) Prove the triangle inequality for $||\cdot||_p$. Hint: $\sum |x_n + y_n|^p = \sum |x_n + y_n||x_n + y_n|^{p-1}$, apply Hölder's inequality.

Exercise 11 ($1 \le p \le \infty$) Prove that l^p is complete w.r.t. $|| \cdot ||_p$.

Exercise 12 Prove that (c) and (c_0) are closed in l^∞.

Exercise 13 Prove that $1 < p < r < \infty$ implies $l^1 \subset l^p \subset l^r \subset l^\infty$ (strict inclusions), that $||x||_1 \ge ||x||_p \ge ||x||_r \ge ||x||_\infty$ and that $||x||_p \to ||x||_\infty$ as $p \to \infty$. Here we use the convention that $||x||_p = \infty$ if $x \notin l^p$.

2.2 Spaces of measurable functions

Throughout this subsection (Ω, Λ, μ) is a measure space, i.e. Ω is a set, Λ is a σ-algebra of measurable subsets of Ω and $\mu : \Lambda \to [0, \infty]$ a measure. We assume that subsets of measurable sets with measure zero are measurable. (The Borel σ-algebra's used in probability in general do not have this property). We also assume that (Ω, Λ, μ) is σ-finite, i.e. $\Omega = \cup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \in \Lambda$ and $\mu(\Omega_n) < \infty$. We denote by $\mathcal{M} = \mathcal{M}(\Omega, \Lambda, \mu) = \mathcal{M}(\Omega)$ the vector space of all real valued measurable functions on Ω. On this set we define an equivalence relation by

$$f \sim g \iff \mu(\{x \in \Omega : f(x) \ne g(x)\}) = 0.$$

The set $\mathcal{M}/_\sim$ of all equivalence classes $[f]$ with $f \in \mathcal{M}$ is denoted by $M = M(\Omega, \Lambda, \mu) = M(\Omega)$. It is customary to write f instead of $[f]$. The set M is again a vector space, with the obvious definitions of addition and scalar multliplication in terms of representatives of equivalence classes.

Definition 2.3

$$1 \leq p < \infty : \quad L^p(\Omega) = \{f = [f] \in M(\Omega) : \int_\Omega |f|^p < \infty\}, \quad ||f||_p = \big(\int_\Omega |f|^p\big)^{\frac{1}{p}}.$$

$$L^\infty(\Omega) = \{f = [f] \in M(\Omega) : \exists g \in [f] \text{ with } \sup_\Omega |g| < \infty\}, \quad ||f||_\infty = \sup_\Omega |g|.$$

Theorem 2.4 *$L^p = L^p(\Omega)$ is a Banach space for $1 \leq p \leq \infty$ w.r.t. the norm $||f||_p$.*

Exercise 14 (Hölder's inequality)

$$\text{If } 1 \leq p, q \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad f \in L^p, g \in L^q, \quad \text{then } ||fg||_1 \leq ||f||_p ||g||_q.$$

Exercise 15 ($1 \leq p \leq \infty$) Prove the triangle inequality for $|| \cdot ||_p$.

Exercise 16 ($1 \leq p \leq \infty$) Prove that L^p is complete w.r.t. $|| \cdot ||_p$. Hint: for $p < \infty$ take a Cauchy sequence (f_n) and assume first, taking a subsequence if necessary, that $||f_{n+1} - f_n|| < 2^{-n}$. Consider $g_n = |f_1| + |f_2 - f_1| + \cdots + |f_n - f_{n-1}|$ and show, using the monotonic convergence theorem, that $0 \leq g_n \uparrow g \in L^p$ outside a set of measure zero. Hence also f_n converges outside this set to a limit function f. Use the dominated convergence theorem to conclude that $f \in L^p$ and $||f_n - f||_p \to 0$. For $p = \infty$: take a Cauchy sequence (f_n) and first cut out a set of measure zero such that on the complement $|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty$ for all $m, n \in \mathbb{N}$.

Exercise 17 If $\Omega = \mathbb{N}$, $\Lambda = \mathcal{P}(\mathbb{N})$ is the collection of all subsets of $\mathbb{N}$ and $\mu(A)$ is the number of elements in A for every $A \in \Lambda$, then L^p is isometric to l^p.

Exercise 18 Assume that $\mu(\Omega) = 1$. Prove that $1 < p < r < \infty$ implies $L^1 \supset L^p \supset L^r \supset L^\infty$ (strict inclusions) and that $||f||_1 \leq ||f||_p \leq ||f||_r \leq ||f||_\infty$.

2.3 Spaces of continuous functions

Theorem 2.5 *Let $T > 0$. Then $C([0,T])$, the linear space of real valued continuous functions on $[0,T]$ endowed with the norm $||f||_\infty = \max_{0 \leq t \leq T} |f(t)|$, is a Banach space. More generally, if X is a compact topological space, then $C(X)$, the linear space of real valued continuous functions on X, is a Banach space with $||f||_\infty = \max_{x \in X} |f(x)|$.*

Theorem 2.6 *If X is a topological space, then $C_b(X)$, the linear space of bounded real valued continuous functions on X, is a Banach space with $||f||_\infty = \sup_{x \in X} |f(x)|$. If X is compact, then $C_b(X) = C(X)$.*

In fact $C_b(X)$ is a Banach algebra.

Exercise 19 Let $V = C([a,b])$. Prove that $||f||_1 = \int_a^b |f(x)|$ is a norm on V, but that V is not complete. Show that the completion $\hat{V}$ of V is $L_1([a,b])$.

Exercise 20 Show that the set $\mathcal{P}([a,b])$ of polynomials on $[a,b]$ is a normed linear space with norm $||\cdot||_\infty$ which is not Banach. However, $\mathcal{P}_n([a,b])$, the set of all polynomials of order $\leq n$, is Banach.

Exercise 21 Let X be a non-empty set and let $B(X)$ be the set of all bounded functions on X. Prove that $B(X)$ is Banach w.r.t. the supremum norm.

Exercise 22 Let $C^1([a,b])$ be the vector space of all functions that are continuously differentiable on $[a,b]$. For $f \in C^1([a,b])$ let

$$\rho_1(f) = \max_{a \leq x \leq b} |f'(x)| = ||f'||_\infty.$$

Show that ρ_1 is a semi-norm but not a norm. Show that

$$||f|| = ||f||_\infty + \rho_1(f) = ||f||_\infty + ||f'||_\infty$$

is a norm on $C^1([a,b])$ and that $C^1([a,b])$ is a Banach space w.r.t. this norm. Show that $C^1([a,b])$ is not a Banach space w.r.t. $||\cdot||_\infty$.

Exercise 23 Give an example of a sequence $(f_n)_{n=1}^\infty$ in $C([0,1])$ and $f \in C([0,1])$ such that $f_n(x) \to f(x)$ point-wise on $[0,1]$, but $||f_n - f||_\infty \not\to 0$.
Prove that if $\{f_n\}_{n=1}^\infty$ is a sequence in $C([0,1])$ and $f \in C([0,1])$ is such that $f_n(x) \uparrow f(x)$ for all $x \in [0,1]$, then $||f_n - f||_\infty \to 0$ (*Dini's theorem*).

Exercise 24 For $0 < \alpha \leq 1$ let $C^{0,\alpha}([a,b])$ be the vector space of all functions that are uniformly Hölder continuous on $[a,b]$, i.e. there exists $C > 0$ such that $|f(x) - f(y)| \leq C|x-y|^\alpha$ for all $x, y \in [a,b]$. For $f \in C^{0,\alpha}([a,b])$ let

$$[f]_\alpha = \sup_{a \leq x < y \leq b} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Show that $[\cdot]_\alpha$ is a semi-norm but not a norm. Show that

$$||f||_\alpha = ||f||_\infty + [f]_\alpha$$

is a norm on $C^{0,\alpha}([a,b])$ and that $C^{0,\alpha}([a,b])$ is a Banach space w.r.t. this norm. Show that $C^{0,\alpha}([a,b])$ is not a Banach space w.r.t. $||\cdot||_\infty$.

3 Axiom of choice and Zorn's lemma

Axiom. (Axiom of choice) If $(X_\alpha)_\alpha \in A$ is a family of nonempty sets indexed by $\alpha \in A$, then there exists a (choice) function $f : A \to \cup_{\alpha \in A} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for every $\alpha \in A$.

Definition 3.1 Let X be a point set. A relation $<$ on X is called a partial order if, for all $x, y, z \in X$, (i) $x < x$; (ii) $x < y$ and $y < x$ implies $x = y$; (iii) $x < y$ and $y < z$ implies $x < z$.

If $A \subset X$ has the property that for every $x, y \in A$ with $x \neq y$ either $x < y$ or $y < x$ holds, then A is called a chain in X.

If $A \subset X$ and $m \in X$ are such that $a < m$ for all $a \in A$, then m is called an upper bound for A.

If $a \in X$ has the property that for every $x \in X$ the implication $a < x \Rightarrow a = x$ holds, then a is called a maximal element in X.

Axiom. (Zorn's Lemma) If X is a partially ordered set such that every chain in X has an upper bound, then X has a maximal element.

Theorem 3.2 *The axiom of choice and Zorn's lemma are equivalent.*

Exercise 25 Prove that every vector space X has a Hamel basis. Hint: consider the set of all linearly independent sets $S \subset X$, partially ordered by inclusion, i.e. $S_1 < S_2 \iff S_1 \subset S_2$, and apply Zorn's lemma.

Theorem 3.3 *(Hahn-Banach) Let X be a vector space and suppose that $p : X \to \mathbb{R}$ satisfies (i) $p(\lambda x) = \lambda p(x)$; (ii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ and for all $\lambda \geq 0$. If $L \subset X$ is a linear subspace of X and $f : L \to \mathbb{R}$ is a linear map with $f(x) \leq p(x)$ for all $x \in L$, then there exists a linear map $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in L$ and $F(x) \leq p(x)$ for all $x \in X$.*

4 Separable spaces

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Definition 4.1 A normed space is called separable if there exists a map $f : \mathbb{N} \to X$ such that X is the closure of $f(\mathbb{N})$.

Usually theorems for separable spaces can be proved without using the (uncountable) axiom of choice (e.g. Hahn-Banach theorems).

Theorem 4.2 *Let (Ω, Λ, μ) be a σ-finite measure space. Then L^p is separable for $1 \leq p < \infty$.*

In general L^∞ is not separable.

Exercise 26 Prove that l^p is separable for $1 \leq p < \infty$ and that l^∞ is not separable.

Exercise 27 Prove that (c) and (c_0) are separable.

Exercise 28 Prove that $C([0, 1])$ is separable.

Exercise 29 Prove that $L^\infty([0, 1])$ is not separable.

Exercise 30 Prove that $C^\alpha([0, 1])$ is not separable.

5 Bounded linear maps, dual spaces

Definition 5.1 Let X and Y be normed spaces, and $T : X \to Y$ a linear map, i.e.

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y) \quad \text{for all} \quad \lambda, \mu \in \mathbb{R} \;\; \text{and} \;\; x, y \in X.$$

Then T is called bounded if

$$||T|| = \sup_{0 \neq x \in X} \frac{||T(x)||_Y}{||x||_X} < \infty.$$

The map $T \to ||T||$ defines a norm on the vector space $B(X,Y)$ of bounded linear maps $T : X \to Y$, so that $B(X,Y)$ is a normed space. In the case that $Y = \mathbb{R}$, the space $X^* = B(X, \mathbb{R})$ is called the dual space of X. Alternative notation: $< f, x >= f(x)$ for $f \in X^*$ and $x \in X$. The space $X^{**} = (X^*)^*$ is called the second dual space of X.

Exercise 31 Determine $B(\mathbb{R}^m, \mathbb{R}^n)$. For the calculation of the norm (for different choices of the norms on $\mathbb{R}^m$ and $\mathbb{R}^n$) see the courses by Spijker et al.

Theorem 5.2 *Let X and Y be normed spaces, and $T : X \to Y$ a linear map. Then $T \in B(X,Y) \iff T$ is continuous in $0 \iff T : X \to Y$ is uniformly continuous.*

Theorem 5.3 *Let X be a normed space and Y a Banach space. Then $B(X,Y)$ is also a Banach space. In particular every dual space is a Banach space.*

Theorem 5.4 *Let X be a normed space. Then X is separable if X^* is separable.*

Theorem 5.5 *(Hahn-Banach extension theorem for bounded linear functionals) Let X be a normed space, $L \subset X$ a subspace and $\phi \in L^*$. Then there exists $\Phi \in X^*$, such that*

$$\Phi|_L = \phi \;\; \text{and} \;\; ||\Phi||_{X^*} = ||\phi||_{L^*}.$$

Exercise 32 Let X be a normed space, $L \subset X$ a subspace and $x_0 \in X$ with $d = d(x_0, L) = \inf\{||x - x_0|| : x \in X\} > 0$. Then there exists $F \in X^*$ with $||F|| = 1$, $F|_L = 0$ and $F(x_0) = d$.

Exercise 33 Let X be a normed space and $x \in X$. Then there exists $F \in X^*$ with $||F|| = 1$ and $F(x) = ||x||$.

Exercise 34 Let X be a normed space. Denote the set of linear maps from X to $\mathbb{R}$ by $X^\#$. Then $X^* = X^\# \iff X$ is finite dimensional.

Exercise 35 Define the map $J : X \to X^{**}$ by

$$J(x) : f \to < J(x), f >=< f, x >$$

for every $x \in X$. Then $J : X \to J(X)$ is an isometry.

Definition 5.6 X is called reflexive if $J(X) = X^{**}$.

Theorem 5.7 *Let X be a Banach space. Then X is reflexive if and only if X^* is reflexive. Also X is reflexive and separable if and only if X^* is reflexive and separable. If X is reflexive and $L \subset X$ is a closed subspace then L is reflexive.*

Theorem 5.8 *(Banach-Steinhaus) Let X and Y be Banach spaces and $T_n : X \to Y$ with $n \in \mathbb{N}$ a sequence of bounded linear maps. Then*

$$\sup_{n\in\mathbb{N}} ||T_n(x)||_Y < \infty \text{ for every fixed x} \in X \iff \sup_{n\in\mathbb{N}} ||T_n||_{B(X,Y)} < \infty.$$

Theorem 5.9 *(Open Mapping Theorem) Let X and Y be Banach spaces and let $T : X \to Y$ be bounded linear and surjective. Then $T(B_X(0,1))$ contains $B_Y(0,r)$ for some $r > 0$.*

Theorem 5.10 *(Closed Graph Theorem) Let X and Y be Banach spaces and let $T : X \to Y$ be linear. Denote the graph of T in $X \times Y$ by $G(T) = \{(x, T(x)) : x \in X\}$. Then*

$$G(T) \text{ is closed in } X \times Y \iff T \in B(X,Y).$$

Exercise 36 Let X and Y be Banach spaces and let $T : X \to Y$ be bounded linear and surjective. Then there exists $c > 0$ such that for all $y \in Y$ there exists $x \in X$ such that $T(x) = y$ and $||x||_X \leq c||y||_Y$.

Theorem 5.11 *Let X and Y be Banach spaces and $T \in B(X,Y)$. If T is bijective then $T^{-1} \in B(Y,X)$.*

Definition 5.12 Two normed spaces X and Y are called isomorphic if there exists a bijective $T \in B(X,Y)$ with $T^{-1} \in B(Y,X)$.

Exercise 37 Let X and Y be Banach spaces, $T \in B(X,Y)$ and $T_n \in B(X,Y)$ for $n \in \mathbb{N}$. If T is bijective and $T_n \to T$ in $B(X,Y)$, then T_n is bijective for n sufficiently large whence $T_n^{-1} \in B(Y,X)$. Moreover, $T_n^{-1} \to T^{-1}$ in $B(Y,X)$. Hint: first use Theorem 1.15 to solve $T_n x = y$.

Exercise 38 Let X be a Banach space, $I : X \to X$ the identity map and $T \in B(X) = B(X,X)$. If $||T|| < 1$ then $I - T$ is bijective and

$$(I-T)^{-1} = I + T + T \circ T + T \circ T \circ T + \cdots = \sum_{n=0}^{\infty} T^n \in B(X,X).$$

Exercise 39 Let X and Y be Banach spaces and $T_n \in B(X,Y)$ for $n \in \mathbb{N}$. If $T(x) = \lim_{n\to\infty} T_n(x)$ exists in Y for every $x \in X$, then $T \in B(X,Y)$ and $||T|| \leq \liminf ||T_n||$.

6 Differentiation and integration

Definition 6.1 Let X and Y be normed spaces, and $f : X \to Y$ a map. Then f is called differentiable in $x_0 \in X$ if there exists a $T \in B(X, Y)$ such that $R(x)$ defined by

$$f(x) = f(x_0) + T(x - x_0) + R(x),$$

satisfies $||R(x)|| = o(||x - x_0||)$ as $x \to x_0$. Notation: $T = f'(x_0)$. In the special case that $X = \mathbb{R}$ we identify the map $f'(x_0) : \mathbb{R} \to Y$ with the image of 1, i.e. the usual derivative defined as the limit of the differential quotient. If $A \subset X$ then f is called continuously differentiable on A if f is differentiable in every x in A and the map $f' : A \to B(X, Y)$ is continuous.

Theorem 6.2 *Let $[a, b]$ be a closed bounded interval, Y a Banach space, and $f : [a, b] \to Y$ a continuous map. Then f is Riemann integrable on $[a, b]$, i.e.*

$$\int_a^b f(t)dt = \lim_{\max\{t_i - t_{i-1}, i=1\ldots,N\} \to 0} \sum_{i=1}^{N} f(\theta_i)(t_i - t_{i-1}),$$

with $a = t_0 \le \theta_1 \le t_1 \le \theta_2 \le \cdots \le t_N = b$, exists and the limit is uniform in the choice of the partition and the intermediate points. Moreover,

$$|| \int_a^b f(t)dt|| \le \int_a^b ||f(t)||dt.$$

Theorem 6.3 *Let $[a, b]$ be a closed bounded interval, Y a Banach space, and $f : [a, b] \to Y$ differentiable with $f' : [a, b] \to Y$ continuous. Then*

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Theorem 6.4 *(Mean Value Theorem) Let X be a normed space, Y a Banach space, and $f : X \to Y$ continuously differentiable. Then*

$$f(x) - f(y) = \int_0^1 f'(tx + (1 - t)y)(x - y)dt = \int_0^1 f'(tx + (1 - t)y)dt(x - y),$$

for every $x, y \in X$.

Theorem 6.5 *(Implicit Function Theorem) Let X, Y and Z be Banach spaces, and $f : X \times Y \to Z$ continuously differentiable. Define $i_X : X \to X \times Y$ and $i_Y : Y \to X \times Y$ by $i_X(x) = (x, 0)$ and $i_Y(y) = (0, y)$. Suppose that $f(\tilde{x}, \tilde{y}) = 0$ and that the partial derivative*

$$D_y f(\tilde{x}, \tilde{y}) = f'(\tilde{x}, \tilde{y}) \circ i_Y \ : Y \ \to \ Z,$$

is a bijection. Then there exist $\delta_1, \delta_2 > 0$ and a continuously differentiable function

$$\phi : B_X(\tilde{x}, \delta_1) = \{x \in X : ||x - \tilde{x}|| < \delta_1\} \to B_Y(\tilde{y}, \delta_2) = \{x \in Y : ||y - \tilde{y}|| < \delta_2\},$$

such that $f(x, \phi(x)) = 0$ *for every* $x \in B_X(\tilde{x}, \delta_1)$ *and such that there are no other solutions of* $f(x, y) = 0$ *in* $B_X(\tilde{x}, \delta_1) \times B_Y(\tilde{y}, \delta_2)$. *Finally,*

$$\phi'(x) = -(D_y f(x, \phi(x)))^{-1}(D_x f(x, \phi(x))),$$

where $D_x f(x, y) = f'(x, y) \circ i_X$.

7 Applications

Exercise 40 Let $T > 0$ and $X = C([0, T])$. Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, i.e. $\exists L > 0\, \forall x, y \in \mathbb{R} : |f(x) - f(y)| \leq L|x - y|$ and let $\xi \in \mathbb{R}$. Then $\Phi : X \to X$ defined by

$$\Phi(x)(t) = \xi + \int_0^t f(x(s))ds,$$

is a contraction if $T < \frac{1}{L}$. What is the initial value problem satisfied by the unique fix point of Φ?

Exercise 41 Let $T > 0$ and $X = C([0, T])$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Then $\Phi : X \to X$ defined by

$$\Phi(x)(t) = \xi + \int_0^t f(x(s))ds,$$

is continuously differentiable. Determine $\Phi'(x)$. Do the same for Φ considered as function of $\xi \in \mathbb{R}$ and $x \in C([0, T])$ and determine also the partial derivatives $D_\xi \Phi(\xi, x)$ and $D_x \Phi(\xi, x)$.

Exercise 42 Let $T > 0$ and $X = C([0, T])$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Let $\Phi : \mathbb{R} \times X \to X$ be defined by

$$\Psi(\xi, x)(t) = x(t) - \xi - \int_0^t f(x(s))ds,$$

and let $\Psi(\tilde{\xi}, \tilde{x}) = 0$. Apply the implicit function theorem and conclude that there exists a continuously differentiable function $\xi \to x(\xi) \in C([0, T])$, defined in a neighbourhood of $\tilde{\xi}$, such that $\Psi(\xi, x(\xi)) = 0$. For the invertibility condition you may assume that T is small. Determine an integral equation that defines $y = x'(\tilde{\xi})$. What is the initial value problem solved by y?

Exercise 43 Let $T > 0$ and $X = C([0, T])$. Let $f : (x, \lambda) \in \mathbb{R} \times \mathbb{R} \to f(x, \lambda) \in \mathbb{R}$ be continuously differentiable. Let $\Phi : \mathbb{R} \times X \to X$ be defined by

$$\Psi(\lambda, x)(t) = x(t) - \int_0^t f(x(s), \lambda)ds,$$

and let $\Psi(\tilde{\lambda}, \tilde{x}) = 0$. Apply the implicit function theorem and conclude that there exists a continuously differentiable function $\lambda \to x(\lambda) \in C([0, T])$, defined in a neighbourhood of $\tilde{\lambda}$, such that $\Psi(\lambda, x(\lambda)) = 0$. For the invertibility condition you may assume that T is small. Determine an integral equation that defines $y = x'(\tilde{\lambda})$. What is the initial value problem solved by y?

8 Examples of dual spaces

8.1 L^p

Theorem 8.1 *Let $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and (Ω, Λ, μ) a σ-finite measure space. Define $\Phi : L^q \to (L^p)^*$ by*

$$\Phi(g) : f \in L^p \to \int_\Omega fg d\mu \in \mathbb{R},$$

for every $g \in L^q$. Then Φ is linear and norm preserving, i.e. $||\Phi(g)||_{(L^p)^*} = ||g||_q$. Moreover, if $1 \leq p < \infty$, then Φ is surjective.

In short, we say that $(L^p)^* = L^q$, if $1 \leq p, q \leq \infty$ and $p \neq \infty$, and $(L^\infty)^* \supset L^1$. As a special case we have of course

Theorem 8.2 *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ Define $\Phi : l^q \to (l^p)^*$ by*

$$\Phi(y) : x = (x_1, x_2, \ldots) \in l^p \to \sum_{n=1}^{\infty} x_n y_n \in \mathbb{R},$$

for every $y = (y_1, y_2, \ldots) \in l^q$. Then Φ is linear and norm preserving. If $1 \leq p < \infty$, then Φ is surjective.

Note that $L^1 \subset (L^\infty)^*$ (strict inclusion) because otherwise L^∞ would be separable.

Exercise 44 Prove Theorem 8.2.

Exercise 45 Prove that l^1 is isometric to the duals of (c) and (c_0).

9 Weak topologies

Definition 9.1 Let X be a normed space. The weak topology $\sigma(X, X^*)$ on X is the smallest topology on X for which every $f \in X^*$ is a continuous function from X to $\mathbb{R}$ (with respect to this topology).

Theorem 9.2 *Let X be a normed space. The weak topology coincides with the norm topology if and only if X is finite dimensional.*

Exercise 46 Let X be an infinite dimensional normed space. Show that for any $A \subset X$ weakly open and $a \in A$, there must be a line through a completely contained in A. In particular A is unbounded if $A \neq \emptyset$.

Exercise 47 Let X be an infinite dimensional normed space. Show that the weak closure of the sphere $\{x \in X : ||x|| = 1\}$ is $\{x \in X : ||x|| \leq 1\}$.

In every topology one can define the concept of convergence. For x_n converging to x in the weak topology we use the notation $x_n \rightharpoonup x$.

Proposition 9.3 *Let X be a Banach space, and $(x_n)_{n=1}^{\infty}$ a sequence in X. Then (i) $x_n \rightharpoonup x \;\Leftrightarrow\; f(x_n) \to f(x) \;\; \forall f \in X^*$; (ii) $x_n \rightharpoonup x \Rightarrow ||x||$ is bounded and $||x|| \leq \liminf_{n\to\infty} ||x_n||$.*

Exercise 48 Let $1 < p < \infty$ and $e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, \ldots), \ldots$. Then $e_n \rightharpoonup 0$ as $n \to \infty$.

Exercise 49 (difficult) In l^1 weakly convergent sequences are also convergent in the norm topology. Why does this not contradict Theorem 9.2?

Definition 9.4 The weak* topology $\sigma(X^*, X)$ on X^* is the smallest topology for which all $x \in X$ considered as functions from X^* to $\mathbb{R}$ are continuous.

Theorem 9.5 *Let X be a separable Banach space and let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of $B = \{x \in X : ||x|| \leq 1\}$. Define for $f, g \in B^* = \{f \in X^* : ||f|| \leq 1\}$*

$$d(f,g) = \sum_{n=1}^{\infty} \frac{f(x_n) - g(x_n)}{2^n}.$$

Then d is a metric on B^* and d induces a topology on B^* which coincides with the relative topology of the weak topology. In the same fashion: if X^* is separable, then there is a metric on B which induces on B the relative topology of the weak topology on X.*

For convergence in the weak* topology we write $f_n \overset{*}{\rightharpoonup} f$, and again this is equivalent to $< f_n, x > \to < f, x >$ for all $x \in X$. The importance of the weak* topology lies in

Theorem 9.6 *(Alaoglu) Let X be a Banach space and let $B^* = \{f \in X^* : ||f|| \leq 1\}$. Then B^* is compact in the* weak* *topology.*

The proof is based on the observation that X^* may be considered as a closed subset of the product $\Pi_{x\in X}\mathbb{R}$ equipped with the product topology and that B^* is a closed subset of $\Pi_{x\in X}[-||x||, ||x||]$ which is compact, being a product of compact sets.

Exercise 50 Let X be a separable reflexive space. Then every bounded sequence in X has a weakly convergent subsequence.

10 Locally convex topological vector spaces

If we consider a normed space X with its weak topology $\sigma(X, X^*)$ we have that the topology is completely defined by a set of neighbourhoods of the origin which we may take to be of the form $N = \{x \in X : |f_i(x)| < r_i, \, i = 1 \ldots n\}$ where $f_1, \ldots, f_n \in X^*$, $r_1, \ldots, r_n > 0$ and $n \in \mathbb{N}$. These neighbourhoods N have the following properties

(convexity) $x, y \in N, \, t \in [0, 1] \;\Rightarrow\; tx + (1-t)y \in N$

(symmetry) $x \in N$, $|t| = 1$, $\Rightarrow$ $tx \in N$

(absorbing) $\forall x \in X \, \exists \lambda \in [0, \infty) : x \in \lambda N$

(scaling) If N is a neighbourhood then so is λN for every $\lambda > 0$.

Neighbourhoods of points $x \neq 0$ are simply of the form $x + N$ where N is neighbourhood of 0. With these neighbourhoods X is a locally convex topological vector space. The algebraic vector space operations are continuous. The $\sigma(X, X^*)$-dual of X, i.e. the space of $\sigma(X, X^*)$-continuous linear functionals $f : X \to \mathbb{R}$, coincides with X^*.

Similar statements hold for X^* and the $\sigma(X^*, X)$-topology. Here neighbourhoods of the origin may be taken of the form $N = \{f \in X^* : |f(x_i)| < r_i, \, i = 1 \ldots n\}$ where $x_1, \ldots, x_n \in X$, $r_1, \ldots, r_n > 0$ and $n \in \mathbb{N}$. In particular X^* with the $\sigma(X^*, X)$-topology is a locally convex topological vector space and its $\sigma(X^*, X)$-dual is X (or better, $J(X)$, where J is the embedding of X in X^{**}).

We summarize the above observations as

Theorem 10.1 *Let X be a normed space. Then the weak dual of X is X^* and the weak* dual of X^* is X.*

To prove that the duals in Theorem 10.1 cannot be larger than the ones stated one uses a basic elementary lemma from linear algebra:

Lemma 10.2 *Let X be a vector space and let $f : X \to \mathbb{R}$ and $f_1, \ldots, f_n : X \to \mathbb{R}$ be linear functionals and let*

$$N = \{x \in X : \, f_1(x) = \cdots = f_n(x) = 0\}.$$

Then the following statements are equivalent.

(i) f is a linear combination of $f_1, \ldots, f_n$.

(ii) $\exists \gamma < \infty \, \forall x \in X \, |f(x)| \leq \gamma \max(|f_1(x)|, \ldots, |f_n(x)|)$.

(iii) $f(x) = 0$ for all $x \in N$.

11 Lower semi-continuity

Definition 11.1 Let X be a topological space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$. Then ϕ is lower semi-continuous if $\phi^{-1}((-\infty, y])$ is closed in X for every $y \in \mathbb{R}$. Equivalent statements are that $\phi^{-1}((y, +\infty])$ is open for every $y \in \mathbb{R}$ and that $\mathrm{epi}(\phi) = \{(x, y) : \, x \in X, y \geq \phi(x)\}$ is closed.

Exercise 51 Show that the sum of two lower semi-continuous functions is lower semi-continuous.

Exercise 52 Show that the pointwise supremum of any collection of lower semi-continuous functions is lower semi-continuous.

Exercise 53 Show that $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous implies that $\liminf_{n\to\infty} \phi(x_n) \geq \phi(x)$ if $x_n \to x$.

Theorem 11.2 *Let X be compact and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ lower semi-continuous. Then ϕ has a minimum on X.*

12 Convex sets and functions

Definition 12.1 Let X be a vector space, a subset $K \subset X$ is convex if $\lambda x + (1-\lambda)y \in K\ \forall x, y \in K\ \forall \lambda \in [0,1]$.

Theorem 12.2 *Let X be a normed space, $A, B \subset X$ two convex sets, $A \cap B = \emptyset$ and suppose that A is open. Then there exists $f \in X^*$ such that $\sup_A f \leq \inf_B f$.*

The proof is based on considering the convex open set $C = A - B - y$, where y is chosen such that $0 \in C$, and its Minkowski functional

$$\mu_C(x) = \inf\{t > 0 : \frac{x}{t} \in C\}.$$

Take $x \notin C$ and apply Theorem 3.3 to the functional $f : \lambda x \to \lambda \mu_C(x)$ with $p = \mu_C$. The argument is valid for any topological vector space.

Theorem 12.3 *Let X be a normed space, $A, B \subset X$ two convex sets, $A \cap B = \emptyset$ and suppose that A is closed and B is compact. Then there exists $f \in X^*$ such that $\sup_A f < \inf_B f$.*

The proof is based on applying the previous theorem to the sets $A + N$ and $B + N$ where N is a convex neighbourhood of 0 such that $A + N$ and $B + N$ have empty intersection. This choice of N is possible in locally convex spaces. If X is a normed space then N is a sufficiently small ball.

Theorem 12.4 *Let X be a normed space, and $A \subset X$ a convex set. Then A is weakly closed if and only if A is norm closed.*

Another important consequence is

Theorem 12.5 *(Goldstine) Let X be a normed space and let $B \subset X$ be the closed unit ball in X. Then the weak*-closure of $J(B)$ (where J is the embedding of X in X^{**}) is the closed unit ball in X^{**}.*

The proof derives a contradiction from applying Theorem 12.3 in X^{**} with the weak* topology $\sigma(X^{**}, X^*)$ to the weak*-closure of $J(B)$ and any singleton $\{y\} \subset \{z \in X^{**} : ||z|| \leq 1\}$ outside this closure.

Definition 12.6 Let X be a normed space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$. Then ϕ is called convex if $\phi(tx + (1-t)y) \geq t\phi(x) + (1-t)\phi(y)$ for all $x, y \in X$ and for all $0 \leq t \leq 1$.

Exercise 54 Show that ϕ is convex if and only if $\text{epi}(\phi) = \{(x,y) \in X \times \mathbb{R} : y \geq \phi(x)\}$ is convex.

Exercise 55 Show that the sum of two convex functions is convex.

Exercise 56 Show that the pointwise supremum of any collection of convex functions is convex.

Exercise 57 Show that weak lower semi-continuity and lower semi-continuity are equivalent properties for convex functions.

Theorem 12.7 *Let X be reflexive, $A \subset X$ convex, closed and bounded, $\phi : X \to \mathbb{R} \cup \{+\infty\}$ weakly lower semi-continuous (this includes the case that ϕ is convex and lower semi-continuous) and $\phi \not\equiv +\infty$ on A. Then ϕ has a global minimum on A.*

13 Uniformly convex spaces

Definition 13.1 Let X be a normed space. Then X is called uniformly convex if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B = \{x \in X : ||x|| \leq 1\}$:

$$||x - y|| > \epsilon \;\Rightarrow\; ||\frac{x+y}{2}|| < 1 - \delta.$$

Exercise 58 Which of the p-norms on $X = \mathbb{R}^N$ make X uniformly convex?

Theorem 13.2 *Every uniformly convex Banach space is reflexive.*

The proof uses Theorem 12.5.

14 Hilbert spaces

Definition 14.1 Let H be a (real) vector space. A function

$$(\cdot,\cdot) : H \times H \to \mathbb{R}$$

is called an inner product if, for all $u, v, w \in H$ and for all $\lambda, \mu \in \mathbb{R}$, (i) $(u,u) \geq 0$, and $(u,u) = 0 \Leftrightarrow u = 0$; (ii) $(u,v) = (v,u)$; (iii) $(\lambda u + \mu v, w) = \lambda(u,w) + \mu(v,w)$.

Any inner product satisfies

$$|(u,v)| \leq \sqrt{(u,u)(v,v)} \quad \forall u, v \in H, \quad \text{(Schwartz)}$$

and also

$$\sqrt{(u+v, u+v)} \leq \sqrt{(u,u)} + \sqrt{(v,v)} \quad \forall u, v \in H.$$

Consequently, $||u|| = \sqrt{(u,u)}$ defines a norm on H, called the inner product norm. This norm satisfies identity

$$||\frac{u+v}{2}||^2 + ||\frac{u-v}{2}||^2 = \frac{1}{2}(||u||^2 + ||v||^2) \quad \forall u, v \in H.$$

Definition 14.2 If H is a Banach space with respect to this inner product norm, then H is called a Hilbert space.

Theorem 14.3 *Every inner product space is uniformly convex. Hence every Hilbert space is reflexive and bounded sequences have weakly convergent subsequences.*

Theorem 14.4 *For every closed convex subset K of a Hilbert space H, and for every $f \in H$, there exists a unique $u \in K$ such that*

$$||f - u|| = \min_{v \in K} ||f - v||,$$

or, equivalently,

$$(f - u, v - u) \leq 0 \quad \forall v \in K.$$

Moreover the map $P_K : f \in H \to u \in K$ is contractive in the sense that

$$||P_K f_1 - P_K f_2|| \leq ||f_1 - f_2||.$$

Theorem 14.5 *(Riesz) For fixed $f \in H$ define $\varphi \in H^*$ by*

$$\varphi(v) = (f, v) \quad \forall v \in H.$$

Then the map $f \to \varphi$ defines an isometry between H and H^*, which allows one to identify H and H^*.

Theorem 14.6 *Let H be a Hilbert space, and $M \subset H$ a closed subspace. Let*

$$M^\perp = \{u \in H : (u, v) = 0 \, \forall v \in M\}.$$

Then $H = M \oplus M^\perp$, i.e. every $w \in H$ can be uniquely written as

$$w = u + v, \; u \in M, \, v \in M^\perp.$$

Definition 14.7 Let H be a Hilbert space. A bilinear form $a : H \times H \to \mathbb{R}$ is called bounded if, for some $C > 0$,

$$|a(u, v)| \leq C||u|| \, ||v|| \quad \forall u, v \in H,$$

coercive if, for some $\alpha > 0$,

$$a(u, u) \geq \alpha ||u||^2 \quad \forall u \in H,$$

and symmetric if

$$a(u, v) = a(v, u) \quad \forall u, v \in H.$$

A symmetric bounded coercive bilinear form on H defines an equivalent inner product on H.

Theorem 14.8 *(Stampacchia) Let K be a closed convex subset of a Hilbert space H, and $a : H \times H \to \mathbb{R}$ bounded coercive bilinear form. Let $\varphi \in H^*$. Then there exists a unique $u \in K$ such that*

$$a(u, v-u) \geq \varphi(v-u) \quad \forall v \in K.$$

Moreover, if a is also symmetric, then u is uniquely determined by

$$\frac{1}{2}a(u,u) - \varphi(u) = \min_{v \in K}\left\{\frac{1}{2}a(v,v) - \varphi(v)\right\}.$$

Taking $K = H$ there exists a unique $u \in H$ such that

$$a(u,v) = \varphi(v) \quad \forall v \in H.$$

Moreover, if a is symmetric, then u is uniquely determined by

$$\frac{1}{2}a(u,u) - \varphi(u) = \min_{v \in H}\left\{\frac{1}{2}a(v,v) - \varphi(v)\right\}.$$

Definition 14.9 Let H be a Hilbert space and $T : H \to H$ a bounded linear operator. Then T is called symmetric if

$$(Tx, y) = (x, Ty) \quad \forall x, y \in H.$$

T is called compact if the closure of the image of the unit ball is compact.

Exercise 59 Let H be a Hilbert space and $T : H \to H$ a compact bounded linear operator. Show that T is sequentially compact, i.e. for every bounded sequence x_n in H there exists a convergent subsequence of Tx_n.

Theorem 14.10 *Every separable Hilbert space has a Hilbert basis, i.e. a countable, possibly finite set $\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$ such that (i) $(\varphi_i, \varphi_j) = \delta_{ij}$; (ii) every $x \in H$ can be written uniquely as*

$$x = x_1\varphi_1 + x_2\varphi_2 + x_3\varphi_3 + \ldots,$$

where $x_1, x_2, x_3, \ldots \in \mathbb{R}$. Moreover,

$$||x||^2 = x_1^2 + x_2^2 + x_3^2 + \ldots,$$

and $x_i = (x, \varphi_i)$.

Theorem 14.11 *Let H be a Hilbert space, $T : H \to H$ a compact bounded symmetric linear operator and $N(T) = \{u \in H : Tu = 0\}$. Then $H = N(T) \oplus N(T)^\perp$ and $N(T)^\perp$ has a Hilbert basis $\{\varphi_1, \varphi_2, \ldots\}$ consisting of eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots \in \mathbb{R}$ with*

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots (\downarrow 0 \text{ if } \dim N(T)^\perp = \infty).$$

Moreover

$$|\lambda_1| = \sup_{0\neq x\in H} |\frac{(Tx,x)}{(x,x)}| = |(T\varphi_1,\varphi_1)|;$$

$$|\lambda_2| = \sup_{\substack{0\neq x\in H\\(x,\varphi_1)=0}} |\frac{(Tx,x)}{(x,x)}| = |(T\varphi_2,\varphi_2)|;$$

$$|\lambda_3| = \sup_{\substack{0\neq x\in H\\(x,\varphi)=(x,\varphi_2)=0}} |\frac{(Tx,x)}{(x,x)}| = |(T\varphi_3,\varphi_3)|,$$

etcetera. If $\psi \in H$ *satisfies* $(\psi,\varphi_1) = (\psi,\varphi_2) = \ldots = (\psi,\varphi_n) = 0$, *and* $(T\psi,\psi) = \lambda_{n+1}(\psi,\psi)$, *then* ψ *is an eigenvector for* λ_{n+1}.

15 Sobolev spaces and applications to BVP's

Consider the one-dimensional version of

$$-\Delta u + u = f \ \text{ in } \Omega; \quad u = 0 \ \text{ on } \partial\Omega,$$

that is, for given f in say $C([a,b])$, we look for a function u satisfying

$$(P) \ -u'' + u = f \ \text{ in } (a,b); \quad u(a) = u(b) = 0.$$

Of course we can treat (P) as a linear second order inhomogeneous equation, and construct a solution by means of ordinary differential equation techniques, but that is not the point here. We use (P) to introduce a method that also works in more space dimensions. Let $\psi \in C^1([a,b])$ with $\psi(a) = \psi(b) = 0$ and suppose that u is a classical solution of (P). i.e. $u \in C^2([a,b])$ and u satisfies the differential equation and the boundary conditions. Then

$$\int_a^b (-u'' + u)\psi = \int_a^b f\psi,$$

so that integrating by parts, and because $\psi(a) = \psi(b) = 0$,

$$\int_a^b (u'\psi' + u\psi) = \int_a^b f\psi.$$

For a suitable function space X we want to define a weak solution of (P) as a function $u \in X$ such that

$$\forall \psi \in X : \quad \int_a^b (u'\psi' + u\psi) = \int_a^b f\psi. \tag{14.1}$$

Definition 15.1 Let $I = (a,b) \subset \mathbb{R}, 1 \leq p \leq \infty$. Let $C_c^\infty(I)$ be the set of all smooth functions with compact support in I. Then $W^{1,p}(I)$ consists of all

$u \in L^p(I)$ such that the distributional derivative of u can be represented by a function in $v \in L^p(I)$, i.e.

$$\int_I v\psi = -\int_I u\psi' \quad \forall \psi \in C_c^\infty(I).$$

We write $u' = v$. For $p = 2$ we write $H^1(I) = W^{1,2}(I)$.

Exercise 60 Show that

$$v \in L^1_{loc}(I) = \{v : I \to \mathbb{R};\ v \in L^1(K) \text{ for every compact } K \subset I\}$$

and $\int_I v\psi = 0$ for every $\psi \in C_c^\infty(I)$ implies that $v = 0$ a.e. in I.

Exercise 61 Show that u' in Definition 15.1 is unique, i.e. there is no other function $v \in L^p(I)$ with the same properties.

Exercise 62 For I bounded show that $C^1(\overline{I})$ is contained in $W^{1,p}(I)$, but not the other way around.

Theorem 15.2 *$W^{1,p}(I)$ is a Banach space with respect to the norm*

$$||u||_{1,p} = ||u||_p + ||u'||_p,$$

where $||\cdot||_p$ denotes the L^p-norm. $H^1(I)$ is a Hilbert space with respect to the inner product

$$(u, v)_1 = (u, v) + (u', v') = \int_I (uv + u'v').$$

The inner product norm is equivalent to the $W^{1,2}$-norm. $W^{1,p}(I)$ is reflexive for $1 < p < \infty$. $W^{1,p}(I)$ is separable for $1 \le p < \infty$.

Exercise 63 Prove Theorem 15.2.

Exercise 64 Show that $v \in L^1_{loc}(I)$ and $\int_I v\psi' = 0$ for every $\psi \in C_c^\infty(I)$ implies that $v = C$ a.e. in I for some constant C.

Exercise 65 Let $g \in L^1_{loc}(I)$ and $y \in I$. Define the function v by $v(x) = \int_{[y,x]} g$. Use Fubini's theorem to show that $\int_I v\psi' = -\int_I g\psi$ for every $\psi \in C_c^\infty(I)$.

Theorem 15.3 *Let $u \in W^{1,p}(I)$ $(1 \le p \le \infty)$. Then, possibly after redefining u on a set of Lebesque measure zero,*

$$u(x) - u(y) = \int_y^x u'(s)ds,$$

for all $x, y \in I$.

Exercise 66 Prove Theorem 15.3 using exercises 64 and 65.

Exercise 67 Use Hölders inequality to show that

$$|u(x) - u(y)| \leq |x - y|^{\frac{p-1}{p}} ||u||_{1,p} \quad \text{if} \quad p > 1,$$

for $u \in W^{1,p}(I)$. ($W^{1,p}(I) \subset C^{0,1-1/p}(\overline{I})$ for I bounded).

Exercise 68 Show that the injection $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is compact if I is bounded and $1 < p \leq \infty$.

Exercise 69 For I bounded show that the injection $W^{1,1}(I) \hookrightarrow C(\overline{I})$ is bounded but not compact. Remark: the injection $W^{1,1}(I) \hookrightarrow L^q(I)$ is compact for every $1 \leq q < \infty$, see [1].

Theorem 15.4 *Let $u \in W^{1,p}(I)$, $1 \leq p < \infty$. Then there exists a sequence $(u_n)_{n=1}^{\infty} \subset C_c^{\infty}(\mathbb{R})$ with $||u_n - u||_{W^{1,p}(I)} \to 0$. In particular, when $I = \mathbb{R}$, $C_c^{\infty}(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$.*

Exercise 70 Let $u, v \in W^{1,p}(I)$, $1 \leq p \leq \infty$. Show that $uv \in W^{1,p}(I)$ and $(uv)' = uv' + u'v$. Moreover, for all $x, y \in I$

$$\int_x^y u'v = [uv]_x^y - \int_x^y uv'.$$

Definition 15.5 Let $1 \leq p < \infty$. The space $W_0^{1,p}(I)$ is defined as the closure of $C_c^{\infty}(I)$ in $W^{1,p}(I)$.

Exercise 71 Let $1 \leq p < \infty$ and I bounded. Show that

$$W_0^{1,p}(I) = \{u \in W^{1,p}(I) : u = 0 \ \text{ on } \ \partial I\}.$$

Theorem 15.6 *Let $1 \leq p < \infty$. Then $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$.*

Exercise 72 (Poincaré) Let $1 \leq p < \infty$, and I bounded. Show there exists a constant $C > 0$, depending on I, such that for all $u \in W_0^{1,p}(I)$:

$$||u||_{1,p} \leq C||u'||_p.$$

Proposition 15.7 *Let $1 \leq p < \infty$ and I bounded. Then $||u||_{1,p} = ||u'||_p$ defines an equivalent norm on $W_0^{1,p}(I)$. Also*

$$((u, v)) = \int_I u'v'$$

defines an equivalent inner product on $H_0^1(I) = W_0^{1,2}(I)$.

We now focus on the spaces $H_0^1(I)$ and $L^2(I)$ with I bounded. We have established the (compact) embedding $i : H_0^1(I) \hookrightarrow L^2(I)$. Thus every bounded linear functional on $L^2(I)$ is automatically also a bounded linear functional on $H_0^1(I)$, if we consider $H_0^1(I)$ as being contained in $L^2(I)$, but having a stronger topology. On the other hand, not every bounded functional on $H_0^1(I)$ can be extended to L^2, e.g. if $\psi \in L^2\backslash H^1$, then $\varphi(f) = \int_I \psi f'$ defines a bounded functional on $H_0^1(I)$ which cannot be extended. This implies that if we want to consider $H_0^1(I)$ as being contained in $L^2(I)$, we *cannot simultaneously apply* Riesz' Theorem to both spaces and identify them with their dual spaces. If we identify $L^2(I)$ and $L^2(I)^*$, we obtain the triplet

$$H_0^1(I) \overset{i}{\hookrightarrow} L^2(I) = L^2(I)^* \overset{i^*}{\hookrightarrow} H_0^1(I)^*.$$

Here i is the natural embedding, and i^* its adjoint ($i^* f = f \circ i$). One usually writes $H_0^1(I)^* = H^{-1}(I)$. The action of H^{-1} on H_0^1 is made precise by

Theorem 15.8 *Suppose $F \in H^{-1}(I)$. Then there exist $f_0, f_1 \in L^2(I)$ such that*

$$F(v) = \int_I f_0 v - \int_I f_1 v' \qquad \forall v \in H_0^1(I).$$

If I is bounded then we may take $f_0 = 0$.

Thus $H^{-1}(I)$ consists of L^2 functions and their first order distributional derivatives. Note however that this characterization depends on the (standard) identification of L^2 and its Hilbert space dual. Also F does not determine f_0 and f_1 uniquely (e.g. $f_0 \equiv 0, f_1 \equiv 1$ gives $F(v) = 0 \quad \forall v \in H_0^1(I)$).

Still identifying L^2 and L^{2^*} we have for $1 < p < \infty$, writing $W_0^{1,p}(I)^* = W^{-1,p}(I)$,

$$W_0^{1,p}(I) \hookrightarrow L^2(I) \hookrightarrow W^{-1,p}(I),$$

and Theorem 15.8 remains true but now with f_0 and f_1 in $L^q(I)$, where $1/p + 1/q = 1$.

Definition 15.9 A weak solution of (P) is a function $u \in H_0^1(I)$ such that

$$\int_I (u'v' + uv) = \int_I fv \qquad \forall v \in H_0^1(I).$$

Since $C_c^\infty(I)$ is dense in $H_0^1(I)$ it suffices to check this integral identity for all $\psi \in C_c^\infty(I)$. Thus a weak solution is in fact a function $u \in H_0^1(I)$ which satisfies $-u'' + u = f$ in the sense of distributions. Note that the boundary condition $u = 0$ on ∂I follows from the fact that $u \in H_0^1(I)$.

Theorem 15.10 *Let $f \in L^2(I)$. Then (P) has a unique weak solution and*

$$\frac{1}{2}\int_I (u'^2 + u^2) - \int_I fu = \min_{v \in H_0^1(I)} \left\{ \frac{1}{2}\int_I (v'^2 + v^2) - \int_I fv \right\}.$$

Exercise 73 Show that the solution u in Theorem 15.10 belongs to $H^2(I) = \{u \in H^1(I) : u' \in H^1(I)\}$. Show that u is a classical solution if $f \in C(\overline{I})$.

Exercise 74 Let $\alpha, \beta \in \mathbb{R}$. Use Stampachia's Theorem applied to $K = \{u \in H^1(I) : u(0) = \alpha, \ u(1) = \beta\}$ with $a(u,v) = ((u,v))$ and $\varphi(v) = \int fv$ to generalize the method above to

$$(P') \quad -u'' + u = f \ \text{ in } \ (0,1); \quad u(0) = \alpha; \quad u(1) = \beta.$$

Exercise 75 Generalize the methods above to the Sturm-Liouville problem

$$(SL) \quad -(pu')' + qu = f \ \text{ in } (0,1); \quad u(0) = u(1) = 0,$$

where $p, q \in C([0,1])$, $p, q > 0$, and $f \in L^2(0,1)$.

Exercise 76 Generalize the methods above to the Neumann problem

$$(N) \quad -u'' + u = f \ \text{ in } (0,1); \quad u'(0) = u'(1) = 0.$$

Recall that (SL) was formulated weakly as

$$a(u,v) = \varphi(v) \qquad \forall v \in H_0^1(0,1),$$

where

$$a(u,v) = \int_0^1 pu'v' + quv \quad \text{and} \quad \varphi(v) = \int_0^1 fv.$$

For $p, q \in C([0,1])$, $p, q > 0$, $a(\cdot,\cdot)$ defines an equivalent inner product on $H_0^1(0,1)$, and for $f \in L^2(0,1)$ (in fact $f \in H^{-1}(0,1)$ is sufficient), φ belongs to the dual of $H_0^1(0,1)$.

Exercise 77 Define $T : L^2(0,1) \to L^2(0,1)$ by $Tf = u$, where u is the (weak) solution of (SL) corresponding to f. Show that T is linear, compact and symmetric.

Theorem 15.11 *Let $p, q \in C([0,1])$, $p, q > 0$. Then there exists a Hilbert basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(I)$, such that φ_n is a weak solution of*

$$-(pu')' + qu = \lambda_n u \ \text{ in } (0,1); \qquad u(0) = u(1) = 0,$$

where $(\lambda_n)_{n=1}^\infty$ is a nondecreasing unbounded sequence of positive numbers.

Exercise 78 Show that $\lambda_n > 0$.

Exercise 79 Show that $T : H_0^1(0,1) \to H_0^1(0,1)$ is also symmetric with respect to the inner product $a(\cdot,\cdot)$. Derive that

$$\lambda_1 = \min_{0 \neq u \in H_0^1(0,1)} \frac{\int_0^1 pu'^2 + qu^2}{\int_0^1 u^2}, \quad \lambda_2 = \min_{(u,\varphi_1)=0} \frac{\int_0^1 pu'^2 + qu^2}{\int u^2},$$

etcetera.

Exercise 80 Let I be a bounded interval and let $F : L^2(I) \to \mathbb{R}$ be a continuous function. Suppose that u_n is a sequence in $H_0^1(I)$ which converges weakly (in $H_0^1(I)$) to a limit $u \in H_0^1(I)$. Show that $F(u_n) \to F(u)$. Conclude that F is weakly continuous on the closed balls in $H_0^1(I)$.

Exercise 81 Consider for $f \in L^2((0,1))$ the nonlinear problem

$$-u'' + u^3 = f \text{ in } (0,1); \quad u(0) = u(1) = 0.$$

Give a definition of a weak solution and relate the definition to the functional

$$F(u) = \int (\frac{1}{2}u'^2 + \frac{1}{4}u^4 - fu).$$

Show that F has a global minimum on $H_0^1(I)$ and that the minimizer is a weak solution.

16 Annihilators

Definition 16.1 Let X be a normed space and $L \subset X$ a subspace. Then

$$L^\perp = \{f \in X^* : f(x) = 0 \;\forall x \in L\}.$$

If $N \subset X^*$ is a subspace, then

$$^\perp N = \{x \in X : f(x) = 0 \;\forall f \in N\}.$$

Proposition 16.2 *Let X be Banach and $L \subset X$ a subspace. Then $^\perp(L^\perp) = \overline{L}$. For every subspace $N \subset X^*$ we have $(^\perp N)^\perp \supset \overline{N}$.*

Theorem 16.3 *Let X be Banach. Then X is reflexive if and only if $(^\perp N)^\perp = \overline{N}$ for every subspace $N \subset X^*$.*

Proposition 16.4 *Let X be Banach and $L, M \subset X$ two closed subspaces. Then*

$$L \cap M =^\perp (L^\perp + M^\perp) \text{ and } L^\perp \cap M^\perp = (L + M)^\perp.$$

Theorem 16.5 *Let X be Banach and $L, M \subset X$ two closed subspaces. Then*

$$L + M \text{ is closed} \Leftrightarrow L^\perp + M^\perp \text{ is closed} \Leftrightarrow$$

$$L + M =^\perp (L^\perp \cap M^\perp) \Leftrightarrow L^\perp + M^\perp = (L \cap M)^\perp$$

17 Unbounded operators

Definition 17.1 Let X and Y be Banach spaces. An unbounded linear operator is a pair $(A, D(A))$ such that $D(A)$ is a linear subspace of X and $A : D(A) \to Y$ a linear map. Notation: $A : D(A) \subset X \to Y$.
$N(A) = \{x \in D(A) : Ax = 0\}$ is the kernel of A, $R(A) = \{Ax : x \in D(A)\}$ is the range of A, $G(A) = \{(x, Ax) : x \in D(A)\}$ is the graph of A.
A is called closed if $G(A)$ is closed in $X \times Y$. A is called densely defined if the closure of $D(A)$ is X.

Unbounded operators may be bounded. It would have been more correct to speak of possibly unbounded operators. By the closed graph theorem a closed unbounded operator is bounded if $D(A) = X$.

Exercise 82 Let $X = Y = C([0,1])$, $D(A) = C^1([0,1])$ and $Au = u'$. Show that A is a densely defined closed unbounded linear operator.

Definition 17.2 Let X and Y be Banach spaces and $A : D(A) \subset X \to Y$ a densely defined unbounded linear operator. Let

$$D(A^*) = \{f \in Y^* : g : u \in D(A) \to f(Au) \in \mathbb{R} \text{ is bounded with respect to } ||\cdot||_X\},$$

and let A^*f be the unique extension of g to X. Then $A^* : D(A^*) \subset Y^* \to X^*$ is called the adjoint operator of A.

Exercise 83 Let $X = Y = L^2((0,1))$, $D(A) = H^1((0,1))$ and $Au = u'$. Show that A is a densely defined closed unbounded linear operator. Determine A^* if X and X^* are identified using Riesz' theorem. Do the same if $D(A) = H_0^1((0,1))$ and if $D(A) = \{u \in H^1((0,1)) : u(0) = 0\}$.

Proposition 17.3 *Let X and Y be Banach spaces and $A : D(A) \subset X \to Y$ a densely defined unbounded linear operator. Then $A^* : D(A^*) \subset Y^* \to X^*$ is closed.*

Theorem 17.4 *Let X and Y be Banach spaces and $A : D(A) \subset X \to Y$ a densely defined closed unbounded linear operator. Then the weak*-closure of $D(A^*)$ is Y. If Y is reflexive then the norm closure of $D(A^*)$ is Y^* and $A^* : D(A^*) \subset Y^* \to X^*$ is also a densely defined closed unbounded linear operator.*

Proposition 17.5 *Let X and Y be Banach spaces and $A : D(A) \subset X \to Y$ a densely defined unbounded linear operator. Define $J : Y^* \times X^* \to X^* \times Y^* = (X \times Y)^*$ by $J(f,g) = (-g, f)$. Then $J(G(A^*)) = G(A)^\perp$.*

Applying Proposition 16.4 to $L = X \times \{0\}$ and $M = G(A)$ we obtain

Proposition 17.6 *Let X and Y be Banach spaces and $A : D(A) \subset X \to Y$ a densely defined closed unbounded linear operator. Then*

$$N(A) = {}^\perp R(A^*) \quad \text{and} \quad N(A^*) = R(A)^\perp.$$

Corollary 17.7 *$N(A)^\perp \supset \overline{R(A^*)}$ (equality if X is reflexive) and ${}^\perp N(A^*) = \overline{R(A)}$.*

From the considerably deeper Theorem 16.5 we have

Theorem 17.8 *Let X and Y be Banach spaces and $A : D(A) \subset X \to Y$ a densely defined closed unbounded linear operator. Then*

$$R(A) \text{ is closed} \Leftrightarrow R(A^*) \text{ is closed} \Leftrightarrow$$

$$R(A) = {}^\perp N(A^*) \Leftrightarrow R(A^*) = N(A)^\perp$$

Theorem 17.9 *Let X and Y be Banach spaces and $A : D(A) \subset X \to Y$ a densely defined closed unbounded linear operator. Then*

$$R(A) = Y \Leftrightarrow \exists C > 0 : ||f|| \leq C||A^* f|| \; \forall f \in D(A^*)$$

$$\Leftrightarrow N(A^*) = \{0\} \text{ and } R(A^*) \text{ is closed}$$

A similar statement holds for A^*. Finally we note that

Theorem 17.10 *Let X and Y be Banach spaces and $A : D(A) \subset X \to Y$ a densely defined closed unbounded linear operator. Then*

$$D(A) = X \Leftrightarrow A \text{ is bounded} \Leftrightarrow$$

$$D(A^*) = Y^* \Leftrightarrow A^* \text{ is bounded}$$

18 Evolution problems in Banach spaces

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a (possibly) unbounded linear operator. For a given initial value $u_0 \in X$ we consider the initial value problem

$$\text{(IVP)} \quad \frac{du}{dt} + Au = 0 \quad (t > 0), \quad u(0) = u_0.$$

Theorem 18.1 *Suppose that $A : X \to X$ is a bounded linear operator. Then (IVP) has a unique classical solution defined by*

$$u(t) = \exp(-tA)u_0, \quad \exp(tA) = I - tA + \frac{1}{2}t^2A^2 - \frac{1}{3!}t^3A^3 + \cdots.$$

Here classical means that $u := [0, \infty) \to X$ is continuously differentiable.

For the case that A is unbounded we define

Definition 18.2 Let X be a Banach space and let $A : D(A) \subset X \to X$ be a (possibly) unbounded linear operator. Then A is called m-accretive if the closure of $D(A)$ is X and if, for every $\lambda > 0$, the linear map $I + \lambda A : D(A) \to X$ is bijective with $||(I + \lambda A)^{-1}|| \leq 1$.

Theorem 18.3 *Let X be a Banach space, let $A : D(A) \subset X \to X$ be m-accretive and let $u_0 \in D(A)$. Then (IVP) has a unique classical solution defined by*

$$u(t) = \lim_{n\to\infty} (I + \frac{t}{n}A)^{-n}u_0 = S(t)u_0.$$

The operators $S(t)$ are bounded with respect to the norm in X and $||S(t)|| = 1$. Moreover, $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \geq 0$. The family $\{S(t) : t \geq 0\}$ is called a contraction semi-group on X. For the inhomogeneous problem

$$\frac{du}{dt} + Au = f(t) \quad (0 < t \leq T), \quad u(0) = u_0.$$

with $f : [0,T] \to X$ continuously differentiable and $u_0 \in D(A)$ there is also a unique solution, given by the variation of constants formula

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds.$$

In the remainder of this section we consider the Hilbert space case.

Definition 18.4 Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be a (possibly) unbounded linear operator. Then A is called monotone if $(Au, u) \geq 0$ for all $u \in D(A)$ and A is called maximal monotone if also $R(I + A) = H$.

Proposition 18.5 *If $A : D(A) \subset H \to H$ is maximal monotone then A is closed, densely defined and M-accretive.*

Definition 18.6 If A is maximal monotone we define, for $\lambda > 0$,

$$J_\lambda = (I + \lambda A)^{-1} \quad \text{and} \quad A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$$

(the resolvent and the Yosida approximation of A).

Proposition 18.7 *Let $A : D(A) \subset H \to H$ be maximal monotone. Then*

$$A_\lambda = A \circ J_\lambda, \quad A_\lambda = J_\lambda \circ A \text{ on } D(A), \quad ||A_\lambda u|| \leq ||Au|| \text{ on } D(A), \quad J_\lambda \to I \text{ pointwise },$$

$$A_\lambda \to A \text{ pointwise on } D(A), \quad (A_\lambda u, u) \geq 0, \quad ||A_\lambda u|| \leq \frac{1}{\lambda}||u||$$

Existence of a solution to (IVP) in the Hilbert space case with A maximal monotone is based on first replacing A by A_λ.

Exercise 84 Proof uniqueness of the solution to (IVP) in the Hilbert space case when A maximal monotone and $u_0 \in D(A)$.

Exercise 85 Let $X = C([0,1])$, $D(A) = \{u \in C^2([0,1]) : u(0) = u(1) = 0\}$ and $Au = -u''$. Show that A generates a contraction semigroup on X. Change the spaces and show the existence of a similar semigroup in $L^2(0,1)$).

19 Spectral analysis and functional calculus

In this section X is always a complex Banach space, $X \neq \{0\}$, and $A : D(A) \subset X \to X$ a linear operator. Also Ω is always an open subset of $\mathbb{C}$.

Definition 19.1 The resolvent set $\rho(A)$ is the set of all complex λ such that the range $R(\lambda - A)$ of $\lambda - A = \lambda I - A$ is dense in X and $(\lambda - A)^{-1} : R(\lambda - A) \to D(A)$ exists and is bounded with respect to the norm in X. If A is closed $\lambda \in \rho(A)$ implies that $R(\lambda - A) = X$ and $(\lambda - A)^{-1} \in B(X)$. The operator $R_\lambda = (\lambda - A)^{-1}$ is called the resolvent of A. The complement of $\rho(A)$ in $\mathbb{C}$ is called the spectrum of A, notation $\sigma(A)$.

Definition 19.2 A function $F : \Omega \to X$ is called analytic if

$$F'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0}$$

exists for every $\lambda_0 \in \Omega$.

Theorem 19.3 *A function $F : \Omega \to X$ is analytic if and only if $\lambda \to f(F(\lambda))$ is analytic on Ω for every $f \in X^*$.*

A function $F : \Omega \to X$ which is analytic has the same nice properties as an ordinary $\mathbb{C}$-valued analytic function: Coursat's theorem, Cauchy formula's for F and its derivatives, local powerseries representation, maximum modulus theorem, Liouville's theorem, etc.

Theorem 19.4 *Let A be closed. Then $\rho(A)$ is open. Moreover*

$$\lambda, \mu \in \rho(A) \;\Rightarrow\; R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu,$$

whence R_λ and R_μ commute. Finally

$$|\lambda - \mu| < \frac{1}{||R_\mu||} \;\Rightarrow\; R_\lambda = \sum_{n=0}^{\infty} (\mu - \lambda)^n R_\mu^{n+1},$$

and $\lambda \to R_\lambda$ is analytic on $\rho(A)$.

Theorem 19.5 *Let A be bounded. Then*

$$|\lambda| > ||A|| \;\Rightarrow\; \lambda \in \rho(A), \quad R_\lambda = \sum_{n=1}^{\infty} \lambda^{-n} A^{n-1} \text{ and } ||R_\lambda|| \leq \frac{1}{|\lambda| - ||A||}.$$

The spectrum $\sigma(A)$ is nonempty (by Liouville's theorem) and compact. Moreover, for every polynomial p,

$$\sigma(p(A)) = p(\sigma(A)),$$

and the powerseries representation of R_λ is valid for all $|\lambda| > r(A)$, where

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} = \lim_{n \to \infty} ||T^n||^{\frac{1}{n}}$$

is the spectral radius of A.

Theorem 19.6 *If A is a compact bounded operator then $\sigma(T)$ is a countable set (including $\lambda = 0$) having no accumulation points except possibly $\lambda = 0$. Every nonzero $\lambda \in \sigma(A)$ is an eigenvalue with a finite dimensional space of generalized eigenvectors.*

Theorem 19.7 *Let A be bounded, $\sigma(A) \subset \Omega$, $F : \Omega \to \mathbb{C}$ analytic and C a contour in $\Omega \setminus \sigma(A)$ winding once (counterclockwise) around $\sigma(A)$ (and containing no holes of Ω). Define*

$$F(A) = \frac{1}{2\pi i} \int_C F(\lambda) R_\lambda d\lambda = \frac{1}{2\pi i} \int_C F(\lambda)(\lambda - A)^{-1} d\lambda.$$

Then $F(A)$ is a bounded linear operator on X and $\sigma(F(T)) = F(\sigma(T))$. The definition is independent of the particular choice of C (and thus only depends on the values of F in a neighbourhood of $\sigma(A)$). If $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ has radius of convergence larger than $r(A)$, then $f(A) = \sum_{n=0}^{\infty} a_n A^n$. Any bounded linear operator on X which commutes with A also commutes with $f(A)$. Moreover, if $G : \Omega \to \mathbb{C}$ is another analytic function, then

$$F(A) + G(A) = (F + G)(A) \quad \text{and} \quad F(A)G(A) = (FG)(A).$$

If $FG \equiv 1$ on Ω then $G(A) = F(A)^{-1}$. Finally, the choice $F(\lambda) = \lambda^n$ ($n = 0, 1, 2, \ldots$) gives

$$A^n = \frac{1}{2\pi i} \int_C \lambda^n R_\lambda d\lambda = \frac{1}{2\pi i} \int_C \lambda^n (\lambda - A)^{-1} d\lambda.$$

This theorem can be generalized to closed operators A with a nonempty resolvent set, Ω a neighbourhood of $\sigma(A) \cup \{\infty\}$ and $F : \Omega \to \mathbb{C}$ analytic with $F(\infty) = 0$. Of course $f(\lambda) = \lambda^n$ has then to be excluded. If A is unbounded we define the extended spectrum by $\sigma_e(A) = \sigma(A) \cup \{\infty\}$, otherwise $\sigma_e(A) = \sigma(A)$. With this convention one has again that $f(\sigma_e(A)) = \sigma(f(A))$.

Suppose A is bounded and $\sigma(A)$ is disconnected. Then there exist disjoint open sets Ω_1, Ω_2 such that $\sigma(A) \subset \Omega_1 \cup \Omega_2$ but not $\sigma(A) \subset \Omega_1$ or $\sigma(A) \subset \Omega_2$. Thus me may take contours C_1 in Ω_1 around $\sigma(A) \cap \Omega_1$ and C_2 in Ω_2 around $\sigma(A) \cap \Omega_2$ and write

$$F(A) = \frac{1}{2\pi i} \int_{C_1} f_1(\lambda) R_\lambda d\lambda + \frac{1}{2\pi i} \int_{C_2} f_2(\lambda) R_\lambda d\lambda,$$

for F defined by

$$F(\lambda) = F_i(\lambda) \ \text{ if } \ \lambda \in \Omega_i,$$

where $F_i : \Omega_1 \to \mathbb{C}$ is analytic ($i = 1, 2$). If we choose $F_i = 1$ we obtain a splitting $I = E_1 + E_2$ of the indentity with $E_1 E_2 = 0$ and $E_i^2 = E_i$ and $X = R(E_1) \oplus R(E_2)$. The projections E_1 and E_2 are called spectral projections. If we choose $F_i(\lambda) = \lambda$ we obtain $A = A_1 E_1 + A_2 E_2$ where $A_i : R(E_i) \to R(E_i)$ is the restriction of A to $R(E_i)$.

It may happen that $\sigma(A) \cap \Omega_1$ is a singleton μ. In that case μ is either a pole or an isolated essential singularity of $\lambda \to R_\lambda$. If it is a pole then μ is an eigenvalue of A.

20 Banach algebra's

Definition 20.1 A complex Banach space A is called a Banach algebra if there exists a multiplication $(x,y) \in A \times A \to xy \in A$ such that, for all $x, y, z \in A$, $\lambda \in \mathbb{C}$,

$$(xy)z = x(yz), \quad x(y+z) = xy + xz, \quad (y+z)x = yx + zx,$$

$$\lambda(xy) = (\lambda x)y = x(\lambda y), ||xy|| \leq ||x|| ||y||.$$

A is called commutative if also $xy = yx$ for all $x, y \in A$. If A contains an element e with $||e|| = 1$ and $ex = xe = x$ for all $x \in A$ then e is called a unit. If A has a unit and if there exists a map $x \in A \to x^* \in A$ such that, for all $x, y \in A$ and $\lambda \in \mathbb{C}$,

$$(x^*)^* = x, \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \bar{\lambda} x^*, \quad (xy)^* = y^* x^*$$

(such a map is called an involution) and such that $||x^* x|| = ||x||^2$ for all $x \in A$, then A is called a C^*-algebra.

The simplest example is of course the field of complex numers $\mathbb{C}$ itself.

Exercise 86 Let X be a complex Banach space. Show that $B(X)$ is a Banach algebra with unit.

Exercise 87 Let X be a compact Hausdorff space. Show that $C(X) = \{f : X \to \mathbb{C},\ f \text{ is continuous}\}$ is a commutative C^*-algebra with unit.

Spectral theory in Banach algebra's with unit is much the same as spectral theory in the algebra of bounded linear operators on a Banach space X. This is outlined below.

Definition 20.2 Let A be a complex Banach algebra with unit e. Then $a \in A$ is called invertible if there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = e$. For $x \in A$ the resolvent set $\rho(x)$ is the set of complex λ such that $\lambda e - x$ is invertible. The spectrum $\sigma(x)$ is the complement of $\rho(x)$ in $\mathbb{C}$.

Exercise 88 Let A be a complex Banach algebra. For each $a \in A$ define $L_a \in B(A)$ by $L_a(x) = ax$ for all $x \in A$. Show that $a \to L_a$ is an isometric algebra isomorphism of A onto a closed subalgebra of $B(A)$.

Exercise 89 Let A be a complex Banach algebra with unit e. Show that $\sigma(x) = \sigma(L_x)$ is nonempty and compact. Show that $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\} = \lim_{n\to\infty} ||x^n||^{\frac{1}{n}}$.

In the next section we look at closed subalgebra's with nicer properties then the algebra A itself. In doing this we don't want the spectrum of an element to depend to much on the choice of the subalgebra. For this we need:

Proposition 20.3 *Let A be a complex Banach algebra with unit e. Then the group G of invertible elements is open and every boundary point x of G is a topological zero-divisor, i.e. there exists a sequence z_n with $||z_n|| = 1$ and $z_n x \to 0$ and $xz_n \to 0$.*

Corollary 20.4 *Let A be a complex Banach algebra with unit e and let B be a closed subalgebra which also contains e. For $x \in B$ denote the spectrum of x in B by $\sigma_B(x)$ and the spectrum of x in A by $\sigma_A(x)$. Then*

$$\sigma_A(x) \subset \sigma_B(x) \quad \text{but} \quad \delta\sigma_B(x) \subset \delta\sigma_A(x).$$

In particular, when $\rho_A(x)$ is connected, then $\sigma_A(x) = \sigma_B(x)$.

Exercise 90 Let A be a complex Banach algebra with unit e. Define for $\lambda \in \rho(x)$ the resolvent $R_\lambda(x) = (\lambda e - x)^{-1}$. Show for $\lambda, \mu \in \rho(x)$ that $(\mu - \lambda)R_\lambda(x)R_\mu(x) = R_\lambda(x) - R_\mu(x)$. Show that $\lambda \to R_\lambda(x) \in A$ is analytic on $\rho(x)$ and that its derivative is $-R_\lambda(x)^2$.

At this point we are in the same situation as in Theorem 19.7, which may be formulated in the abstract context with the formula

$$F(x) = \frac{1}{2\pi i} \int_C F(\lambda)(\lambda e - x)^{-1} d\lambda,$$

where C is now a contour around $\sigma(x)$ in Ω.

In the remainder of this section we establish that every commutative C^*-algebra with unit is essentially a space of continuous functions on a compact Hausdorff space denoted by $\mathcal{M}$. The following exercise indicates a consequence of this result which will be relevant for the subalgebra's in the next section. It shows that we do not have to restrict ourselves to analytic functions F.

Exercise 91 Let X be a compact Hausdorff space and let $A = C(X)$. For $g \in C(X)$ characterize $\sigma(g)$. For which functions F can we define a function $F(g)$ in $C(X)$ such that the same algebraic properties hold for the map $F \to F(g)$ as in Theorem 19.7 for the map $F \to F(T)$?

The next theorem is essential in what follows. It is based on the obervation that the spectrum of an element in a complex Banach algebra with unit is never empty, which in turn was a consequence of Liouville's theorem for (A-valued) analytic functions.

Theorem 20.5 *(Gelfand-Mazur) Let A be a complex Banach algebra with unit such that every nonzero element is invertible. Then A is isometrically isomorphic to $\mathbb{C}$.*

The compact Hausdorff space $\mathcal{M}$ announced above will be a weak* closed subset of the unit sphere in the dual A^*. The elements of this subset are the nonzero functionals in the following definition.

Definition 20.6 Let A be a complex Banach algebra with unit e. A linear functional $f : A \to \mathbb{C}$ is called multiplicative if $f(x)f(y) = f(xy)$ for all $x, y \in A$.

Proposition 20.7 *Let A be a complex Banach algebra with unit e and $f : A \to \mathbb{C}$ a multiplicative linear functional which is not identical to* 0. *Then $f(e) = 1$, f is bounded and $||f|| = 1$. The kernel of f is an ideal in A, i.e. a linear subspace I with $xy, yx \in I$ whenever $x \in A$ and $y \in I$. Moreover, I is a maximal ideal meaning that I is not $\{0\}$ or A and that there is no larger ideal containing I except A itself.*

Exercise 92 Let A be a complex Banach algebra with unit e and $f : A \to \mathbb{C}$ a multiplicative linear functional which is not identical to 0. Show that $f(x) \in \sigma(x)$ for all $x \in A$. Hint: $f(x)e - x \in N(f)$ cannot be invertible.

Definition 20.8 Let A be a complex Banach algebra with unit e. Then $\mathcal{M}$, the set of all nonzero multiplicative linear functionals on A, is called the carrier space of A. We endow $\mathcal{M}$ with the relative topology of the weak* topology $\sigma(A^*, A)$ on A^*. If $J : A \to A^{**}$ is the usual embedding of A in its second dual, then $\hat{x}$ is defined as the restriction of $J(x)$ to $\mathcal{M}$. By definition $\hat{x}$ is continuous on $\mathcal{M}$. The mapping $x \to \hat{x}$ is called the Gelfand transform. It is defined if $\mathcal{M}$ is nonempty.

Thus the Gelfand transform takes $x \in A$ to $\hat{x} \in C(X)$. For $C(X)$ to be a Banach space (and in fact a C^*-algebra) we need:

Theorem 20.9 *Let A be a complex Banach algebra with unit e. Then $\mathcal{M}$ is weak*-compact in A^*. If $\mathcal{M}$ is nonempty then the Gelfand transform $x \to \hat{x}$ is an algebra homomorphism of A onto a subalgebra of $C(\mathcal{M})$. Moreover, $||\hat{x}||_\infty \leq r(x) \leq ||x||$ for all $x \in A$, so the Gelfand transform is also bounded.*

This theorem does half of the work but it is only of use if $\mathcal{M}$ is sufficiently large. This requires that A contains many ideals, which is not always true. For example: by a theorem of Naimark the only nontrivial closed ideal in $B(H)$, when H is a separable Hilbert space, is $K(H)$, the set of all compact operators, whence $\mathcal{M} = \emptyset$ if $A = B(H)$. For commutative Banach algebra's the situation is much better, the basic tool being:

Lemma 20.10 *Let A be a commutative complex Banach algebra with unit e. If $x \in A$ is not invertible, then the set $Ax = \{wx : w \in A\}$ is a proper ideal in A and $x \in Ax$.*

As a consequence of this lemma applied to the quotient algebra A with respect to any maximal ideal and Theorem 20.5 we have

Theorem 20.11 *Every maximal ideal in a commutative complex Banach algebra with unit is the kernel of a nonzero multiplicative linear functional on A.*

In view of the lemma and a typical Zorn argument there are many maximal ideals in the commutative case:

Theorem 20.12 *Let A be a commutative complex Banach algebra with unit e. Then $\sigma(x) = \{f(x) : f \in \mathcal{M}\}$ and $||\hat{x}||_\infty = r(x)$ for all $x \in A$.*

The following three theorems establish a criterion for the equivalence of $||x||$ and $||\hat{x}||$ and a criterion for commutativity.

Theorem 20.13 *Let A be a complex Banach algebra with unit. There exists a constant $c > 0$ such that $c||x||^2 \leq ||x^2||$ for all $x \in A$ if and only if there exists a constant $d > 0$ such that $d||x|| \leq r(x)$ for all $x \in A$. For $c = 1$ the statement can be reformulated as: $||x||^2 = ||x^2||$ for all $x \in A$ if and only if $r(x) = ||x||$ for all $x \in A$.*

Theorem 20.14 *Let A be a complex Banach algebra with unit. Define*

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Suppose that for some $x, y \in A$ there exists a constant $M > 0$ such that $||\exp(\lambda y)x\exp(-\lambda y)|| \leq M$ for all $\lambda \in \mathbb{C}$. Then $xy = yx$.

Theorem 20.15 *Let A be a complex Banach algebra with unit. Suppose there exists a constant $c > 0$ such that $c||x||^2 \leq ||x^2||$ for all $x \in A$. Then A is commutative.*

For commutative algebra's we may still have that information is lost in the Gelfand transform. Therefore we introduce:

Definition 20.16 Let A be a commutative complex Banach algebra with unit e. Then A is called semi-simple if $r(x) = 0$ implies that $x = 0$ for all $x \in A$. This is equivalent to saying that $\mathcal{M}$ separates the points of A.

Theorem 20.17 *If A is a semi-simple commutative complex Banach algebra with unit e, then the Gelfand transform is an algebra isomorphism of A onto a subalgebra $\hat{A}$ of $C(\mathcal{M})$ which contains the constant functions and separates the points of $\mathcal{M}$. Moreover, $\hat{x} = r(x)$ for all $x \in A$.*

The subalgebra $\hat{A}$ does not have to be closed in $C(\mathcal{M})$ but if one of the statements in Theorem 20.13 holds, the norms $||x||$ and $||\hat{x}||$ are equivalent whence $\hat{A}$ must be closed. For C^*-algebra's this is guaranteed by:

Proposition 20.18 *In a C^*-algebra the implication $xx^* = x^*x \Rightarrow r(x) = ||x||$ holds for all $x \in A$.*

Thus, if A is a commutative C^*-algebra with unit, it follows that A and $\hat{A}$ are isometrically isomorphic through the Gelfand transform, which also preserves the involution. Hence $\hat{A}$ is a closed subalgebra of $C(\mathcal{M})$ which contains the constant functions, separates points and is closed under involution. By the Stone-Weierstrass theorem this implies that $\hat{A} = C(\mathcal{M})$. In other words, as C^*-algebra's, A and $C(\mathcal{M})$ may be identified. Summing up, we have:

Theorem 20.19 *Let A be a commutative C^*-algebra with unit. Then the Gelfand transform is an isometric C^*-algebra isomorphism of A onto $C(\mathcal{M})$.*

21 Application to normal operators

In this section we consider the application of Theorem 20.19 to the (commutative) C^*-subalgebra A_x generated by a normal element $a \neq e$ (normal meaning $aa^* = a^*a$) in a noncommutative C^*-algebra A with unit. As an example we have in mind that $A = B(H)$ where H is a complex Hilbert space and A_T is the commutative C^*-subalgebra generated by a normal operator $T \in B(H)$, i.e. $T^*T = TT^*$.

Proposition 21.1 *Let A be a C^*-algebra A with unit. If $x^* = x$ (i.e. x is selfadjoint), then $\sigma(x) \subset \mathbb{R}$. If B is a closed C^*-subalgebra with $e \in B$, then $y \in B$ is invertible in B if and only if it is invertible in A. Consequently $\sigma_A(y) = \sigma_B(y)$*

Theorem 21.2 *Let A be a noncommutative C^*-algebra with unit e and let $a \in A$, $a \neq e$, be normal, i.e. $a^*a = aa^*$. If A_a is the smallest closed subalgebra containing e, a and a^*, then A_a is commutative and thus isometrically C^*-algebra isomorphic to $C(\mathcal{M}_a)$ where $\mathcal{M}_a$ is the carrier space of A_a. Moreover, the Gelfand transform $\hat{a}$ of a is a homeomorphism of $\mathcal{M}_a$ on $\sigma(a)$. Thus A_a is isometrically C^*-algebra isomorphic to $C(\sigma(a))$ through*

$$x \in A_a \to \hat{x} \in C(\mathcal{M}_a) \to \tilde{x} = \hat{x} \circ \hat{a}^{-1} \in C(\sigma(a)).$$

For every $F \in C(\sigma(a))$ we can now define $F(a) \in A_a$ as the inverse image of F under the mapping $x \in A_a \to \tilde{x} \in C(\sigma(a))$.

22 Exercises

Exercise 93 Let $X = C([0,1])$ (endowed with the maximumnorm) and $K : [0,1] \times [0,1] \to \mathbb{R}$ continuous. Prove that $T : X \to X$ defined by

$$(T(f))(x) = \int_0^1 K(x,y)f(y)dy,$$

belongs to $\mathcal{B}(X,X)$. Give an estimate for $||T||$. Same question when X is endowed with the norm $||f||_1 = \int_0^1 |f(x)|dx$.

Exercise 94 Let $X = C([0,1])$ (endowed with the maximumnorm) and $K : \{(x,y) : 0 \leq y \leq x \leq 1\} \to \mathbb{R}$ continuous. Prove that $T : X \to X$ defined by

$$(T(f))(x) = \int_0^x K(x,y)f(y)dy,$$

belongs to $\mathcal{B}(X,X)$. Show that $||T^n|| \leq \frac{C^n}{n!}$. Show that $I - K$ is bijective and $(I-K)^{-1} \in \mathcal{B}(X,X)$.

Exercise 95 Let $X = C([0,1])$ be endowed with the maximumnorm. Show that the following maps belong to X^* and compute their norms (i) $\phi(f) = f(a)$ with $a \in [0,1]$ fixed; (ii) $\phi(f) = \int fg$ with $g : [0,1] \to \mathbb{R}$ Lebesgue measurable and integrable; (iii) $\phi(f) = \sum_{n=1}^{\infty} \xi_n f(a_n)$ with $\xi = (\xi_1, \xi_2, \ldots) \in l^1$ and $a_1, a_2, \ldots \in [0,1]$.

Exercise 96 Let X be a normed space and $f : X \to \mathbb{R}$ a linear map which is not indentically equal to zero. Show that

$$f \in X^* \iff N(f) = \{x \in X : f(x) = 0\} \text{ is closed}$$

Show that the closure of $N(f)$ is X if $f \notin X^*$.

Exercise 97 Let $(c_{00}) = \{x = (x_1, x_2, x_3, \ldots) : \exists N \, \forall n \geq N \, x_n = 0\}$ be endowed with the $|| \cdot ||_\infty$ norm. Show that (c_{00}) is a normed vector space which is not Banach. Give a Hamelbasis of (c_{00}).

Exercise 98 Let $T : (c_{00}) \to (c_{00})$ be defined by $T(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$. Show that T is a bounded linear bijective operator (with respect to $|| \cdot ||_\infty$) but that T^{-1} is unbounded.

Exercise 99 Let X be one of the sequence spaces l^p $(1 \leq p \leq \infty)$, (c_0) or (c). Consider the shift operator S defined by $S(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$. Show that $S \in \mathcal{B}(X,X)$, determine $||S||$ and discuss the existence of the limits $\lim_{n\to\infty} S^n(x)$ in X (for fixed $x \in X$) and $\lim_{n\to\infty} S^n$ in $\mathcal{B}(X,X)$.

23 Exercises about compact operators

In what follows X is a real or complex Banach space, $B(X)$ is the Banach algebra of bounded linear operators from X to X, $K(X)$ is the subalgebra of compact operators, $T \in K(X)$, $\lambda \neq 0$ and $n \in \mathbb{N}$.

Exercise 100 Show that $K(X)$ is closed in $B(X)$.

Exercise 101 Show that $K(X)$ is a two-sided ideal in $B(X)$.

Exercise 102 Show that the null spaces $N((T-\lambda)^n)$ have finite dimension.

Exercise 103 Let M be a closed subspace of X such that $M \cap N(T-\lambda) = \{0\}$. Then $T - \lambda : M \to R(T-\lambda)$ has a bounded inverse and $R(T-\lambda)$ is closed.

Exercise 104 Use the previous exercise and show that the ranges $R((T-\lambda)^n)$ are all closed.

Exercise 105 Show that $\lambda \in \sigma(T)$ if and only if λ is an eigenvalue.

Exercise 106 Show that $\lambda \in \rho(T)$ if and only if $R(T-\lambda) = X$.

Exercise 107 Show that $R(T-\lambda) = X$ if and only if $N(T-\lambda) = \{0\}$. With $\lambda = 1$ this is called the Fredholm alternative which says that either the inhomogeneous equation $x - Tx = y$ is uniquely solvable for every $y \in X$ or there exists a nontrivial solution to the homogeneous equation $x - Tx = 0$.

The ascent $\alpha(T-\lambda)$ is by definition the smallest n such that $N((T-\lambda)^n) = N((T-\lambda)^{n+1})$. The descent $\delta(T-\lambda)$ is by definition the smallest n such that $R((T-\lambda)^n) = R((T-\lambda)^{n+1})$.

Exercise 108 Show that $\alpha(T-\lambda) = \delta(T-\lambda) < \infty$ and that, writing $p = \alpha(T-\lambda)$, $X = N((T-\lambda)^p) \oplus R((T-\lambda)^p)$.

24 Exercises about algebra's

Exercise 109 Let X be a compact Hausdorff space and $A = C(X)$. Show that every nonzero multiplicative linear functional ϕ is of the form $\phi(f) = f(x)$ for all $f \in A$ with $x \in X$ uniquely determined by ϕ.

Exercise 110 Let W be the algebra of absolutely convergent Fourier series $f = \sum_{n=-\infty}^{\infty} a_n e_n$ where $e_n(x) = \exp(inx)$. Show that W is a commutative Banach algebra with unit if we endow W with the obvious algebra structure and with the norm $||f|| = \sum_{n=-\infty}^{\infty} |a_n|$. By definition W is isometrically isomorphic to $L^1(\mathbf{Z})$. What is the multiplication in the latter algebra?

Exercise 111 Show that every nonzero multiplicative linear functional ϕ on W is of the form $\phi(f) = f(x)$ for all $f \in W$ with $x \in [0, 2\pi)$ uniquely determined by ϕ.

Exercise 112 Show that if $f \in W$ has no real zero's then $1/f$ is also in W.

Exercise 113 Relate the Fourier transform on $L^1(\mathbf{Z})$ to the Gelfand transform.

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