Exercises: 'Conservation Laws'.

1. Construct the fundamental solution of Burgers equation: i.e. solve the initial value problem

$$\begin{cases} u_t + (\frac{1}{2}u^2)_x = \nu u_{xx} & \text{in} \quad Q \\ u(\cdot,0) = M\delta(\cdot) & \text{on} \quad \mathbb{R}. \end{cases}$$

Hint: Set $u(x,t) = t^{-\frac{1}{2}}\varphi(\eta)$, with $\eta = xt^{-\frac{1}{2}}$, where φ satisfies the boundary value problem

$$\begin{cases} -\varphi - \eta\varphi' + (\varphi^2)' = \nu\varphi'' & \text{on} \quad \mathbb{R} \\ \varphi(\pm\infty) = 0 \\ \int_{\mathbb{R}} \varphi(z)dz = M. \end{cases}$$

2. Let u and v satisfy

$$\begin{cases} (u+v)_t + qu_x = 0 \\ v_t = k\{u^p - v\} & \text{in} \quad Q, \end{cases}$$

where $k, q > 0$ and $0 < p < 1$.

Find travelling wave solutions $\{u, v, c\}$, i.e.

$$u = u(\eta), \quad v = v(\eta) \qquad \text{with} \qquad \eta = x - ct,$$

satisfying

$$\begin{aligned} u(-\infty) = v(-\infty) = 1 \\ u(+\infty) = v(+\infty) = 0. \end{aligned}$$

3. Use the Method of Characteristics to solve the initial value problem

$$u_t + (\frac{1}{2}u^2)_x = 0 \qquad \text{in} \qquad Q$$

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

4. Suppose u is a weak solution of Problem (**P**) (see Lecture notes) in which $u_0 \in C(\mathbb{R})$ and

$$u \in C^1((0,\delta) \times \mathbb{R}) \cap C([0,\delta] \times \mathbb{R})$$

for some $\delta > 0$. Show $u(\cdot, 0) = u_0(\cdot)$ on $\mathbb{R}$

5. Determine the entropy solution of the Riemann problem

$$u_t + (u^p)_x = 0 \qquad \text{in} \qquad Q,$$

$$u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x > 0. \end{cases}$$

Here $0 \le u_l < u_r \le 1$. Discuss the cases $0 < p < 1$ and $p > 1$.

6. Show that any smooth solution of

$$u_t + (f(u))_x = 0 \qquad \text{in} \qquad Q,$$

can be transformed into a solution of Burgers equation.

7. Construct the entropy solution of the Riemann problem

$$\begin{cases} u_t + (f(u))_x = 0 & \text{in} \qquad Q \\ u(x,0) \begin{cases} 1 & x < 0 \\ 0 & x > 0. \end{cases} \end{cases}$$

where $f(s) = \frac{Ms^2}{(1-s)^2 + Ms^2}$ and $M > 0$.

8. A weak rarefaction wave is a function $r : \mathbb{R} \to \mathbb{R}$, which satisfies (for given boundary conditions $u_l, u_r \in \mathbb{R}$):

(i) $r \in C([-\infty, \infty])$;

(ii) $r \in [u_l, u_r]$ and $r(-\infty) = u_l, r(+\infty) = u_r$;

(iii) $\int_{\mathbb{R}} \left\{ (f(r) - \eta r) \frac{d\varphi}{d\eta} - r\varphi \right\} d\eta = 0$ for all $\varphi \in C_0^\infty(\mathbb{R})$.

Show that

$$f(r) - \eta r \in C^1(\mathbb{R})$$

and

$$\frac{d}{d\eta}\left\{ f(r) - \eta r \right\} + r = 0 \qquad \text{on} \qquad \mathbb{R}.$$

9. (Discrete Comparison Principle). Let $M > 0$ and $A := \max_{|<| \le M} |f'(s)|$, when f is a smooth flux function. Further let $h, l > 0$ such that,

$$\frac{Ah}{l} \le 1 \qquad \text{(Stability)}.$$

Consider the Lax-schema

$$\begin{cases} u_n^{k+1} = \dfrac{1}{2}(u_{n-1}^k + u_{n+1}^k) - \dfrac{k}{2l}\left\{ f(u_{n+1}^k) - f(u_{n-1}^k) \right\} \\ u_n^0 = u_{0,n} \quad \text{(given)} \end{cases}$$

for all $k \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$.

Show that

$$-M \leq u_{0,n} \leq v_{0,n} \leq M \qquad \text{for} \qquad n \in \mathbb{Z}$$

implies

$$-M \leq u_n^k \leq v_n^h \leq M \qquad \text{for} \qquad n \in \mathbb{Z} \;\text{ and }\; K \in \mathbb{Z}^+.$$