LINEAR PARTIAL DIFFERENTIAL EQUATIONS

by

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(Leiden 1993)

PHYSICAL BACKGROUND

1. The wave equation in one dimension

In this section we derive the equations of motion for a vibrating string and a vibrating membrane.

Consider a string which we assume to be described as the graph of a function of x (space) and t (time):

$$y = u(x, t).$$

Vertical external forces acting on a piece of the string between $x = a$ and $x = b$, (a, b) for short, may be described as

$$\int_a^b f(x,t)dx \qquad \text{(in positive } \ y\text{−direction).}$$

Here $f(x,t)$ is the force per unit of lenght, and $u_x = \partial u/\partial x$ is assumed to be small, so that the arc lenght

$$\sqrt{1 + (\frac{\partial u}{\partial x})^2} \ \ dx \approx dx.$$

Now what are the internal forces acting on (a, b)?

In $x = a$ we have a tangential force proportional to the strain,

$$\vec{F}_a = -\sigma(a) \ \frac{1}{\sqrt{1 + u_x(a)^2}} \begin{pmatrix} 1 \\ u_x(a) \end{pmatrix}.$$

Similarly, at $x = b$,

$$\vec{F}_b = \sigma(b) \ \frac{1}{\sqrt{1 + u_x^2(b)}} \begin{pmatrix} 1 \\ u_x(b) \end{pmatrix}.$$

Assuming again that u_x is small, the total internal force acting on (a, b) is given by

$$\vec{F} = \sigma(b) \begin{pmatrix} 1 \\ u_x(b) \end{pmatrix} - \sigma(a) \begin{pmatrix} 1 \\ u_x(a) \end{pmatrix}.$$

Newton's law says that the combined forces determine the change of impuls moment. Ignoring motion in the x-direction, we conclude that $\sigma(a) = \sigma(b)$, and since a, b where arbitrary,

$$\sigma(x) \equiv \sigma \qquad \text{is constant.}$$

Thus the impuls moment of (a, b) has only a y-component, given by

$$\int_a^b \rho(x) \frac{\partial u}{\partial t}(x,t) dx,$$

where $\rho(x)$ is the mass density of the string per unit lenght, so that

$$\frac{d}{dt}\int_a^b \rho(x)\frac{\partial u}{\partial t}(x,t)dx = \sigma(b)\frac{\partial u}{\partial x}(b,t) - \sigma(a)\frac{\partial u}{\partial x}(a,t) + \int_a^b f(x,t)dx,$$

or, assuming also $\rho(x) \equiv \rho$ is a constant,

$$\int_a^b \rho\frac{\partial^2 u}{\partial t^2}(x,t) = \int_a^b \frac{\partial}{\partial x}\sigma\frac{\partial u}{\partial x}(x,t)dx + \int_a^b f(x,t)dx.$$

Again, since a and b are arbitrary, we conclude that

$$\rho\frac{\partial^2 u}{\partial t^2} = \sigma\frac{\partial^2 u}{\partial x^2} + f, \tag{1.1}$$

which is the *one-dimensional inhomogeneous wave equation.*

2. The wave equation in more dimensions

Next we consider a vibrating membrane. We examine where the derivation above has to be adjusted. Instead of $y = u(x,t)$ we have

$$z = u(x,y,t),$$

and instead of (a,b) we take a small open disk D in the (x,y)-plane. The horizontal internal force acting on the piece corresponding to D is given by, again assuming that u_x and u_y are small,

$$\oint_{\partial D} \sigma(x,y)\nu(x,y)dS,$$

Here ν is the outward normal, dS is the arc lenght, ∂D is the boundary of D, and σ is the strain. By the vector valued integral version of the divergence theorem, this equals

$$\int_D \nabla\sigma(x,y)d(x,y),$$

which has to be zero again, because we neglect motion in the horizontal directions. But D is arbitrary so $\nabla\sigma \equiv 0$, i.e. $\sigma(x,y) = \sigma$ is constant. The vertical internal force acting on D is then

$$\sigma\oint_{\partial D} \nabla u(x,y,t)\cdot\nu(x,y)dS =$$

(by the divergence theorem)

$$\sigma\int_D \Delta u(x,y,t)\ d(x,y).$$

Here ∇ and Δ act only on x and y, but not on t. The *inhomogeneous wave equation in two (and in fact any n) dimensions* thus reads

$$\rho\frac{\partial^2 u}{\partial t^2} = \sigma\Delta u + f. \tag{2.1}$$

3. Conservation laws and diffusion

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, i.e. a bounded open connected set. We assume Ω is filled with some sort of *diffusive* material, with concentration given by

$$c = c(x,t) = c(x_1, x_2, x_3, t),$$

where x is space, t is time. Motion is then usually described by the *mass flux*

$$\vec{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \vec{\Phi}(x,t).$$

The direction of $\vec{\Phi}$ coincides with the direction of the motion, and its magnitude says how much mass flows through a plane perpendicular to $\vec{\Phi}$, per unit of surface area.

If we consider any ball B contained in Ω and compute what comes out of B per unit of time, we find

$$\oint_{\partial B} \Phi(x,t)\nu(x)dS(x) = \int_B \operatorname{div}\Phi(x,t)dx =$$

$$\int_B \{\frac{\partial\Phi_1(x,t)}{\partial x_1} + \frac{\partial\Phi_2(x,t)}{\partial x_2} + \frac{\partial\Phi_3(x,t)}{\partial x_3}\}d(x_1,x_2,x_3).$$

Assuming that new material is being produced in Ω, and that per unit of time the production rate in any disk B is given by

$$\int_B q(x,t)dx,$$

we have by the *conservation of mass principle*

$$\frac{d}{dt}\int_B c(x,t)dx = -\int_B \operatorname{div}\Phi(x,t)dx + \int_B q(x,t)dt.$$

Since B was arbitrary, we find

$$\frac{\partial c}{\partial t} = -\mathrm{div}\Phi(x,t) + q(x,t), \tag{3.1}$$

which is commonly called a *conservation law.*

This conservation law has to be combined with some sort of second relation between the concentration c and the flux Φ in order to arrive at a single equation for c. An example of such a relation is the principle of *diffusion* which says that mass flows from higher to lower concentrations, i.e. the flux Φ and the gradient of the concentration, point in opposite directions:

$$\vec{\Phi} = -D\nabla C. \tag{3.2}$$

Here $D > 0$ is the diffusion coefficient, which may depend on space, time, etc. In the simplest case D is a constant. Substituting this second relation in the conservation law we obtain, if D is a constant,

$$\frac{\partial c}{\partial t} = \mathrm{div}\ D\nabla c + q \;\;=\;\; D\Delta c + q. \tag{3.3}$$

Because a similar derivation can be given for the flow of heat in a physical body, this equation is often called the *inhomogeneous heat equation.*

PART 2: THE WAVE EQUATION

4. The Cauchy problem in one space dimension

For $u = u(x,t)$ we consider the equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{4.1}$$

where $c \in \mathbb{R}^+$ is fixed and x and t are real variables. We change variables by setting

$$\xi = x + ct, \quad \eta = x - ct. \tag{4.2}$$

Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta},$$

so that

$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} = c^2\frac{\partial^2}{\partial \xi^2} - 2c^2\,\frac{\partial^2}{\partial \xi \partial \eta} + c^2\frac{\partial^2}{\partial \eta^2}$$

$$-c^2\frac{\partial^2}{\partial \xi^2} - 2c^2\frac{\partial^2}{\partial \xi \partial \eta} - c^2\frac{\partial^2}{\partial \eta^2} = -(2c)^2\frac{\partial^2}{\partial \xi \partial \eta},$$

and (4.1) reduces to

$$u_{\xi\eta} = 0. \tag{4.3}$$

Formally then every function of the form

$$u(x,t) = f(\xi) + g(\eta) = f(x+ct) + g(x-ct), \tag{4.4}$$

is a solution. The lines $\xi =$ *constant* and $\eta =$ *constant* are called *characteristics* .

Next consider the initial value problem

$$(CP) \begin{cases} u_{tt} - c^2 u_{xx} = 0 & x,t \in \mathbb{R}; \\ u(x,0) = \alpha(x) & x \in \mathbb{R}; \\ u_t(x,0) = \beta(x) & x \in \mathbb{R}. \end{cases}$$

This is usually called the *Cauchy problem* for the wave equation in one space dimension. To solve (CP) for given functions α and β we use (4.4). Thus we have to find f and g such that

$$\alpha(x) = u(x,0) = f(x) + g(x) \quad \text{and} \quad \beta(x) = u_t(x,0) = cf'(x) - cg'(x).$$

It is no restriction to assume that $f(0) - g(0) = 0$. Hence

$$f(x) - g(x) = \frac{1}{c}\int_0^x \beta(s)ds \quad \text{and} \quad f(x) + g(x) = \alpha(x).$$

Solving for f and g we obtain

$$f(x) = \frac{1}{2}\alpha(x) + \frac{1}{2c}\int_0^x \beta(s)ds \quad \text{and} \quad g(x) = \frac{1}{2}\alpha(x) - \frac{1}{2c}\int_0^x \beta(s)ds.$$

Here the only restriction on the functions α and β is that the latter one has to be locally integrable. Using (4.2) and (4.4) we conclude that

$$u(x,t) = \frac{1}{2}\big\{\alpha(x+ct) + \alpha(x-ct)\big\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(s)ds. \tag{4.5}$$

Clearly, u defined as such, satisfies $u(x,0) = \alpha(x)$, and if α is differentiable, and β continuous, then

$$u_t(x,t) = \frac{1}{2}\big\{c\alpha'(x+ct) - c\alpha'(x-ct)\big\} + \frac{1}{2c}\big\{c\beta(x+ct) + c\beta(x-ct)\big\},$$

so that $u_t(x,0) = \beta(x)$.

For the (1.1) to be satisfied in a classical way, i.e. for u_{tt} and u_{xx} to be continuous, we need α to be twice and β to be once continuously differentiable. We summarize these results in the following theorem.

4.1 Theorem Let $\alpha \in C^2(\mathbb{R})$ and $\beta \in C^1(\mathbb{R})$. Then problem (CP) has a unique solution $u \in C^2(\mathbb{R} \times \mathbb{R})$, given by

$$u(x,t) = \frac{1}{2}\{\alpha(x+ct) + \alpha(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(s)ds.$$

The right hand side of this expression is defined for all $\alpha : \mathbb{R} \to \mathbb{R}$ and all locally integrable $\beta : \mathbb{R} \to \mathbb{R}$.

Proof The derivation of the formula is correct if u is a twice continuously differentiable solution and it is easy to check that under the hypotheses u as defined in the theorem is indeed such a solution. ∎

4.2 Corollary Suppose supp $\alpha \cup$supp $\beta \subset [A,B]$. Then supp $u \subset [A-ct, B+ct]$, for $t > 0$,

5. The inhomogeneous wave equation in dimension one

Next we consider the Cauchy problem for the inhomogeneous wave equation,

$$(CP_i) \begin{cases} u_{tt} - c^2 u_{xx} = \varphi(x,t) & x,t \in \mathbb{R}; \\ u(x,0) = \alpha(x) & x \in \mathbb{R}; \\ u_t(x,0) = \beta(x) & x \in \mathbb{R}, \end{cases}$$

for given functions α, β, φ. We assume φ is integrable.

We shall derive a representation formula for the solution of (CP_i). To do so, fix x_0 and $t_0 > 0$, and consider the triangle G in $\mathbb{R} \times \mathbb{R}$ bounded by the segments $C_1 = \{x - x_0 = c(t - t_0), 0 < t < t_0\}$, $C_2 = \{x - x_0 = -c(t - t_0), 0 < t < t_0\}$, and $I = \{t = 0, x_0 - ct_0 < x < x_0 + t_0\}$. Assume u is smooth and satisfies

$$u_{tt} - c^2 u_{xx} = \operatorname{div}\begin{pmatrix} -c^2 u_x \\ u_t \end{pmatrix} = \varphi(x,t). \tag{5.1}$$

Here x is the first, and t the second coordinate. Applying the divergence theorem we have

$$\int\int_G \varphi(x,t)dxdt = \oint_{\partial G} \begin{pmatrix} -c^2 u_x \\ u_t \end{pmatrix} \cdot \nu \; dS =$$

(where ν is the outward normal on ∂G)

$$\oint_{\partial G} (-c^2 u_x \; dt - u_t \; dx) = \int_{C_1} + \int_{C_2} + \int_I (-c^2 u_x \; dt - u_t \; dx) =$$

(using $dx = cdt$ along C_1 and $dx = -cdt$ along C_2)

$$\int_{C_1} (-cu_x\ dx - cu_t\ dt) + \int_{C_2} (cu_x dx + cu_t\ dt) + \int_I -u_t\ dx =$$

$$-c\alpha(x_0 - ct_0) + 2cu(x_0, t_0) - c\alpha(x_0 + ct_0) - \int_{x_0-ct_0}^{x_0+ct_0} \beta(s)ds.$$

Thus problem (CP_i) should have as a solution

$$u(x,t) = \frac{1}{2}\{\alpha(x-ct)+\alpha(x+ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct} \beta(s)ds+\frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)d\xi d\tau.$$

We have already investigated for which α and β this makes sense, so consider the new term, which we denote by

$$u_p(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)d\xi d\tau. \tag{5.2}$$

For all locally integrable φ the function u_p is well defined as a function of $x \in \mathbb{R}$ and $t \in \mathbb{R}$, and since φ is integrated over a domain in $\mathbb{R} \times \mathbb{R}$ with continuously varying boundary, it is clear that $u_p \in C(\mathbb{R} \times \mathbb{R})$, and that $u_p(x, 0) = 0$ for all $x \in \mathbb{R}$. Also, the measure of G equals ct^2, so that for locally bounded φ,

$$u_p(x,t) = O(t^2) \quad \text{as} \quad t \to 0,$$

uniformly on bounded x-intervals. In particular,

$$\frac{\partial u_p}{\partial t}(x, 0) = 0,$$

for all $x \in \mathbb{R}$.

Next we give conditions on φ for u_p to be a classical solution of the inhomogeneous wave equation. We assume that $\varphi \in C(\mathbb{R} \times \mathbb{R})$. Then

$$u_p(x,t) = \frac{1}{2c}\int_0^t g(x,t,\tau)d\tau; \quad g(x,t,\tau) = \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)d\xi,$$

so that

$$\frac{\partial g}{\partial x}\ (x,t,\tau) = \varphi(x + c(t-\tau),\tau) - \varphi(x - c(t-\tau),\tau),$$

and

$$\frac{\partial g}{\partial t}(x,t,\tau) = c\varphi(x + c(t-\tau),\tau) + c\varphi(x - c(t-\tau),\tau).$$

Thus g is differentiable with respect to x and t, with partial derivatives continuous in x, t and τ. Hence

$$\frac{\partial u_p}{\partial t}(x,t) = \frac{1}{2c} g(x,t,t) + \frac{1}{2c}\int_0^t \frac{\partial g}{\partial t}(x,t,\tau)d\tau$$

$$= \frac{1}{2}\int_0^t \{\varphi(x+c(t-\tau),\tau) + \varphi(x-c(t-\tau),\tau\} d\tau,$$

which is continuous because φ is. Similarly we find that

$$\frac{\partial u_p}{\partial x}(x,t) = \frac{1}{2c}\int_0^t \frac{\partial g}{\partial x}(x,t,\tau)d\tau =$$

$$\frac{1}{2c}\int_0^t \{\varphi(x+c(t-\tau),\tau) - \varphi(x-c(t-\tau),\tau)\} d\tau$$

is continuous. We conclude that $u_p \in C^1(\mathbb{R}\times\mathbb{R}))$.

If we want u_p to be in $C^2(\mathbb{R}\times\mathbb{R})$, we need more regularity on φ because we have to differentiate once more under the integral sign. This is allowed if φ_x is continuous. Then

$$\frac{\partial^2 u_p}{\partial t^2}(x,t) = \varphi(x,t) + \frac{1}{2}\int_0^t \{c\varphi_x(x+c(t-\tau),\tau) - c\varphi_x(x-c(t-\tau),\tau)\} d\tau,$$

while

$$\frac{\partial^2 u_p}{\partial x^2}(x,t) = \frac{1}{2c}\int_0^t \{\varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)\} d\tau,$$

so that indeed u_p is a solution of the inhomogeneous wave equation.

5.1 Theorem Suppose $\alpha \in C^2(\mathbb{R})$, $\beta \in C^1(\mathbb{R})$, $\varphi \in C(\mathbb{R}\times\mathbb{R})$, and $\varphi_x \in C(\mathbb{R}\times\mathbb{R})$. Then problem (CP_i) has a unique solution $u \in C^2(\mathbb{R}\times\mathbb{R})$, which for $t > 0$ is given by

$$u(x,t) = \frac{1}{2}\{\alpha(x-ct) + \alpha(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(\xi)d\xi + \frac{1}{2c}\int\int_{G(x,t)} \varphi(\xi,\tau)d\xi\, d\tau,$$

where

$$G(x,t) = \{(\xi,\tau), 0 \le \tau \le t,\ |\xi - x| \le c(t-\tau)\}.$$

Proof The derivation above is correct if u is a twice continuously differentiable solution and we have seen that under the hypotheses u as defined in the theorem is indeed such a solution. ∎

6. Initial boundary value problems

We now consider the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + \varphi \tag{6.1}$$

on the strip $\{(x,t) : a < x < b\}$. Initial conditions are again of the form

$$(IC) \quad \begin{cases} u(x,0) = \alpha(x) & x \in (a,b); \\ u_t(x,0) = \beta(x) & x \in (a,b). \end{cases}$$

For (lateral) boundary conditions one can take any of the following four combinations

$$(DD) \quad \begin{cases} u(a,t) = A(t) \\ u(b,t) = B(t) \end{cases} \qquad (DN) \quad \begin{cases} u(a,t) = A(t) \\ u_x(b,t) = B(t) \end{cases}$$

$$(ND) \quad \begin{cases} u_x(a,t) = A(t) \\ u(b,t) = B(t) \end{cases} \qquad (NN) \quad \begin{cases} u_x(a,t) = A(t) \\ u_x(b,t) = B(t) \end{cases}$$

6.1 Theorem For any $T > 0$ there is atmost one solution $u \in C^2([a,b] \times [0,T])$ of (6.1) satisfying the initial conditions (IC) as well as the lateral boundary conditions (DD), (ND), (DN) or (NN).

Proof Assuming the existence of two different solutions we obtain, by subtraction, the existence of a nontrivial solution u with boundary conditions given by $A(t) \equiv B(t) \equiv 0$, and $\alpha(x) \equiv \beta(x) \equiv 0$. Define the "energy" integral

$$E(t) = \frac{1}{2} \int_a^b \{c^2 u_x^2 + u_t^2\} dx.$$

Then for all $t \geq 0$,

$$\frac{dE}{dt}\,(t) = \int_a^b \{c^2 u_x u_{xt} + u_t u_{tt}\} dx = \int_a^b \{c^2 u_x u_{xt} + u_t c^2 u_{xx}\} dx$$

$$= c^2 \int_a^b \frac{\partial}{\partial x}\,(u_x u_t)\ dx = c^2\ [u_x u_t]_{x=a}^{x=b} = 0.$$

Thus $E(t) \equiv E(0) = 0$, so that $u \equiv 0$.
Contradiction, because we assumed u to be nontrivial. ∎

For the construction of solutions we use the following lemma.

6.2 Lemma Let $u \in C^2(\vartheta)$ for some open subset ϑ of $\mathbb{R} \times \mathbb{R}$. Then u is a solution of $u_{tt} = u_{xx}$ in ϑ, if and only if u satisfies the difference equation

$$u(x-k, t-h) + u(x+k, t+h) = \quad u(x-h, t-k) + u(x+h, t+k)$$

for all x, t, k, h such that the rectangle R with vertices $A = (x - k, t - h)$, $B = (x + h, t + k)$, $C = (x + k, t + h)$, and $D = (x - h, t - k)$ is contained in ϑ. (R is called a characteristic rectangle, because its boundary consists of characteristics.)

Proof Suppose u solves $u_{tt} = u_{xx}$. Then u is of the form $u(x, t) = f(x+t)+g(x-t)$. Since

$$f(A) + f(C) = f(x + t - h - k) + f(x + t + h + k) = f(B) + f(D),$$

and

$$g(A) + g(C) = g(x - k - t + h) + g(x + k - t - h) = g(B) + g(D),$$

it follows that $u(A) + u(C) = u(B) + u(D)$.

Conversely, suppose u satisfies the difference equation for all characteristics rectangles $R \subset \vartheta$. Put $h = 0$, then

$$\frac{u\ (x - k, t) - 2u(x, t) + u(x + k, t)}{k^2} = \frac{u\ (x, t - k) - 2u(x, t) + u(x, t + k)}{k^2}.$$

Using Taylor's theorem with respect to the variable k in the numerators, we obtain, as $k \to 0$, that $u_{tt} = u_{xx}$. This completes the proof of the lemma. ∎

With this lemma we can obtain a solution of the inhomogeneous wave equation satisfying initial conditions (IC) and lateral boundary conditions (DD).

6.3 Theorem Let $\alpha \in C^2([a, b])$, $\beta \in C^1([a, b])$, $A, B \in C^2([0, \infty])$, $\varphi, \varphi_x \in C([a, b]) \times [0, \infty])$, and suppose that the following six compatibility conditions are satisfied:

$$A''(0) = c^2\alpha''(a) + \varphi(a, 0)\ ;\ \alpha(a) = A(0)\ ;\ A'(0) = \beta(a)\ ;$$

$$B''(0) = c^2\alpha''(b) + \varphi(b, 0)\ ;\ \alpha(b) = B(0)\ ;\ B'(0) = \beta(b).$$

Then the problem

$$\begin{cases} u_{tt} - c^2u_{xx} = \varphi & a < x < b,\ t > 0; \\ u(a, t) = A(t)\ ;\ u(b, t) = B(t) & t > 0; \\ u(x, 0) = \alpha(x)\ ;\ u_t(x, 0) = \beta(x) & a \le x \le b, \end{cases}$$

has a unique solution $u \in C^2([a, b] \times [0, \infty))$.

Proof It suffices to prove existence. First we reduce the problem to the case $\varphi \equiv 0$. To do so we observe that we may assume that φ and φ_x belong to $C(\mathbb{R} \times [0, \infty))$ by setting

$$\varphi(x, t) = \varphi(b, t) + \varphi_x(b, t)(x - b) \text{ for } x \ge b,$$

and

$$\varphi(x,t) = \varphi(a,t) + \varphi_x(a,t)(x-a) \text{ for } x \le a.$$

We also assume without loss of generality that $c = 1$. Taking the difference between the unknown function $u(x,t)$ and

$$\frac{1}{2}\int\int_{G(x,t)} \varphi(\xi,\tau)d\xi\ d\tau,$$

and renaming this difference u again, we obtain a new problem, with new functions A, B, α and β, and with $\varphi = 0$, satisfying the same regularity and compatibility conditions.

We construct a solution for $0 < t \le b-a$. The square $[a,b]\times[0,b-a]$ if subdivided by its diagonals into four triangles, which we number counterclockwise starting at the bottom as I, II, III and IV. To compute u in I, we use the formula

$$u(x,t) = \frac{1}{2}\big\{\alpha(x+t)+\alpha(x-t)\big\} + \frac{1}{2}\int_{x-t}^{x+t}\beta(s)ds.$$

We then define u for every (x,t) in II and IV using the difference equation in Lemma 6.2 for characteristic rectangles with two vertices contained in I, one on the lateral boundary, and the last one at (x,t). Then with u being determined for every point in II and IV, we extend u to III using the difference equation again, now applied to characteristic rectangles with one vertex in each triangle. This defines a function u on $[a,b]\times[0,b-a]$.

Repeating the construction on $[a,b]\times[b-a,2(b-a)]$, etc., we obtain the value of $u(x,t)$ for every (x,t) in $(a,b)\times(0,\infty)$. We claim that $u\in C^2([a,b]\times[0,\infty])$, and that $u_{tt} = u_{xx}$. Clearly, because of the previous results it suffices to establish $u\in C^2([a,b]\times[0,\infty])$ This is left as an exercise. ∎

7. The fundamental solution in one space dimension

We have seen that under appropriate conditions on α, β and φ, the solution of

$$(CP_i)\ \begin{cases} u_{tt} - u_{xx} = \varphi(x,t) & x,t\in\mathbb{R}; \\ u(x,0) = \alpha(x) & x\in\mathbb{R}; \\ u_t(x,0) = \beta(x) & x\in\mathbb{R}, \end{cases}$$

is given by

$$u(x,t) = u_\alpha(x,t) + u_\beta(x,t) + u_p(x,t), \tag{7.1}$$

where

$$u_\alpha(x,t) = \frac{1}{2}\alpha(x+t) + \frac{1}{2}\alpha(x-t);\quad u_\beta(x,t) = \frac{1}{2}\int_{x-t}^{x+t}\beta(s)ds;$$

$$u_p(x,t) = \frac{1}{2}\int\int_{G(x,t)} \varphi(\xi,\tau)d\xi\ d\tau;\quad G(x,t) = \{(\xi,\tau), 0 \le \tau \le t,\ |\xi - x| \le t - \tau\}.$$

Note that u_α is the solution of $u_{tt} = u_{xx}$ with $u(x,0) = \alpha(x)$ and $u_t(x,0) \equiv 0$, u_β of $u_{tt} = u_{xx}$ with $u(x,0) \equiv 0$ and $u_t(x,0) = \beta(x)$, and u_p of $u_{tt} = u_{xx} + \varphi$ with $u(x,0) \equiv u_t(x,0) \equiv 0$. In fact these three different functions are constructed by means of one (*fundamental*) solution. To see this we have to make a small detour into the theory of distributions.

As an example we consider first the so-called *Heaviside function*:

$$H(s) = \begin{cases} 0 & s < 0; \\ 1 & s > 0. \end{cases}$$

If we look at H as an element of $L^1_{loc}(\mathbb{R})$, $H(0)$ need not be defined. If we look at H as a *"maximal monotone graph"* , we must set $H(0) = [0,1]$. We cannot differentiate H in the class of functions, but we can in the class of distributions. The "*testfunctionspace*" is defined by

$$D(\mathbb{R}) = \{\psi \in C^\infty(\mathbb{R}); \psi \text{ has compact support}\}.$$

We say that for $\psi_n, n = 1, 2...$, and ψ in $D(\mathbb{R})$,

$$\psi_n \to \psi \quad \text{as} \quad n \to \infty \quad \text{in} \quad D(\mathbb{R}),$$

if the supports of $\psi_n^{(k)}$ are uniformly bounded, and if $\psi_n^{(k)} \to \psi^{(k)}$ uniformly on $\mathbb{R}$ for all $k = 0, 1, 2, ...$

7.1 Definition A linear functional $T : D(\mathbb{R}) \to \mathbb{R}$ is called a distribution if $\psi_n \to \psi$ in $D(\mathbb{R})$ implies that $T\psi_n \to T\psi$.

Every $\varphi \in L^1_{loc}(\mathbb{R})$ defines a distribution

$$T_\varphi(\psi) =< \varphi, \psi >= \int_{-\infty}^{\infty} \varphi\psi. \tag{7.2}$$

Now suppose we take for φ a smooth function. Then

$$T_{\varphi'}(\psi) =< \varphi', \psi >= \int_{-\infty}^{\infty} \varphi'\psi = -\int_{-\infty}^{\infty} \varphi\psi' = - < \varphi, \psi' >= -T_\varphi(\psi'). \tag{7.3}$$

In view of this property, the following definition is natural.

7.2 Definition Let $T : D(\mathbb{R}) \to \mathbb{R}$ be a distribution. Define $T' : D(\mathbb{R}) \to \mathbb{R}$ by $T'(\psi) = -T(\psi')$. Then T' is called the *distributional derivative* of T. Note that T' is again a distribution.

7.3 Example Let H be the Heaviside function. Then

$$T_H(\psi) =< H, \psi >= \int_{-\infty}^{\infty} H(s)\psi(s)ds = \int_0^{\infty} \psi(s)ds,$$

and

$$(T_H)'(\psi) =< H', \psi >= -\int_{-\infty}^{\infty} H(s)\psi'(s)ds = -\int_0^{\infty} \psi'(s)ds = \psi(0).$$

We introduce the *Dirac delta distribution* $\delta = \delta(x)$ by

$$< \delta, \psi >= \int_{-\infty}^{\infty} \delta(x)\psi(x)dx = \psi(0). \tag{7.4}$$

Clearly δ is the distributional derivative of H. Intuitively, δ is a function with

$$\delta(x) = 0 \text{ for } x \neq 0;\ \delta(0) = +\infty;\ \int_{-\infty}^{\infty} \delta(x)dx = 1,$$

but one should always remember that mathematically speaking, δ is not a function. A better and correct way is to say that δ is a measure which assigns the value one to any set containing zero.

Returning to u_β we have that, for $t \geq 0$

$$u_\beta(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \beta(s)ds = \int_{-\infty}^{\infty} \frac{1}{2}H(x+t-s)H(s-x+t)\beta(s)ds =$$

$$\int_{-\infty}^{\infty} E^+(x-s,t)\beta(s)ds,$$

where

$$E^+(x,t) = \frac{1}{2}H(t+x)H(t-x),\ x \in \mathbb{R},\quad t \geq 0.$$

We extend E^+ to the whole of $\mathbb{R}^2$ by setting $E^+(x,t) = 0$ for $t \leq 0$. Note that we can also write

$$E^+(x,t) = \frac{1}{2}H(t)\{H(x+t) - H(x-t)\}, \tag{7.5}$$

and that supp $E^+ \subset \mathbb{R} \times \bar{\mathbb{R}}^+$. Extending the definitions of distributions and their derivatives in the obvious way from $\mathbb{R}$ to $\mathbb{R}^2$, and in particular defining the Dirac distribution in $\mathbb{R} \times \mathbb{R}$ by

$$< \delta, \psi >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,t)\psi(x,t)dx = \psi(0,0), \tag{7.6}$$

we claim that

$$E^+_{tt} - E^+_{xx} = \delta(x,t) = \delta(x)\delta(t) \quad \text{in } \mathbb{R}\times\mathbb{R}. \tag{7.7}$$

To see this, let ψ be any smooth function with compact support in $\mathbb{R}\times\mathbb{R}$, i.e. $\psi \in D(\mathbb{R}\times\mathbb{R})$, and let γ be the boundary of the triangle $\{(x,t):\ -t < x < t,\ 0 < t < T\}$, where T is so large that the support of ψ is contained in $\{t < T\}$. Then

$$< E^+_{tt} - E^+_{xx}, \psi >= \int\int E^+(x,t)(\psi_{tt} - \psi_{xx})dxdt =$$

$$\frac{1}{2}\int\int_{-t\le x\le t}(\psi_{tt}-\psi_{xx})dxdt = \frac{1}{2}\int\int_{-t\le x\le t}\frac{\partial}{\partial x}(-\psi_x) + \frac{\partial}{\partial t}(\psi_t)dxdt$$

$$= \frac{1}{2}\int\int_{-t\le x\le t}\operatorname{div}\begin{pmatrix}-\psi_x\\ \psi_t\end{pmatrix}dxdt = \frac{1}{2}\oint_\gamma \begin{pmatrix}-\psi_x\\ \psi_t\end{pmatrix}\cdot\nu ds =$$

$$-\frac{1}{2}\oint_\gamma \psi_x\ dt + \psi_t dx = \psi(0,0) =< \delta, \psi > .$$

Next we compute, as distributions on $\mathbb{R}$, for $t > 0$,

$$< E^+(\cdot,t), \psi >= \int_{-\infty}^{\infty} E^+(x,t)\psi(x)\ dx = \int_{-t}^{t}\psi(x)dx,$$

for all $\psi \in D(\mathbb{R})$. Clearly, $< E^+(\cdot,t), \psi >\to 0$ as $t \downarrow 0$. In view of the following definition we say that $E^+(\cdot,t) \to 0$ as $t \downarrow 0$ in the class of distributions on $\mathbb{R}$.

7.4 Definition Let $T_n, n = 1, 2, ...,$ and T be distributions on an open set $\Omega \subset \mathbb{R}^n$. We say that $T_n \to T$ if $T_n\psi \to T\psi$ for all $\psi \in D(\Omega)$.

Finally we look at E^+_t. Again let $\psi \in D(\mathbb{R}\times\mathbb{R})$. Then

$$< E^+_t, \psi >= - < E^+, \psi_t >= -\frac{1}{2}\int\int_{-t\le x\le t}\psi_t(x,t)dxdt =$$

$$\frac{1}{2}\int_0^\infty \psi(x,x)\ dx + \frac{1}{2}\int_{-\infty}^0 \psi(x,-x)\ dx = \frac{1}{2}\int_0^\infty (\psi(t,t) + \psi(-t,t))\ dt$$

$$= \int_0^\infty < \frac{1}{2}(\delta(x-t) + \delta(x+t)), \psi(x,t) >\ dt.$$

Here we have used the notation

$$< \delta(\cdot \pm t), \psi >= \int \delta(x\pm t)\psi(x)\ dx = \psi(\mp t).$$

Symbolically we write for $t > 0$,

$$E_t^+(x,t) = \frac{1}{2}\delta(x+t) + \frac{1}{2}\delta(x-t). \tag{7.8}$$

Consequently, for $\psi \in D(\mathbb{R})$,

$$< E_t^+(\cdot,t), \psi >= \frac{1}{2}\psi(-t) + \frac{1}{2}\psi(t) \to \psi(0) =< \delta(x), \psi(x) >$$

as $t \downarrow 0$, i.e. $E_t^+(\cdot,t) \to \delta \quad \text{as} \quad t \downarrow 0$.

7.5 Definition E^+ is called the *fundamental solution* of $u_{tt} = u_{xx}$. Its support, the set $\{|x| \leq t\}$ is called the *forward light cone.*

7.6 Remark The derivation of the formula above for E_t^+ is formal, but can be made mathematically rigourous, if one considers δ as a measure.

7.7 Definition For $f, g : \mathbb{R} \to \mathbb{R}$ the convolution of f and g is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-s)g(s)ds,$$

whenever this integral exists.

Now recall that for $t > 0$

$$u_\beta(x,t) = \int_{-\infty}^{\infty} E^+(x-s,t)\beta(s)ds, \tag{7.9}$$

i.e. $u_\beta(\cdot,t)$ is the convolution of $E^+(\cdot,t)$ and β.

Next we consider u_α. For $t > 0$ we have

$$u_\alpha(x,t) = \frac{1}{2}\alpha(x+t) + \frac{1}{2}\alpha(x-t) = \int_{-\infty}^{\infty} \frac{1}{2}(\delta(x-s+t) + \delta(x-s-t))\alpha(s)ds$$

$$= \int_{-\infty}^{\infty} E_t^+(x-s,t)\alpha(s)ds,$$

so that formally u_α is the convolution of $E_t^+(\cdot,t)$ and α.

Finally we look at u_p. We have for $t > 0$

$$u_p(x,t) = \frac{1}{2}\int\int_{G(x,t)} \psi(\xi,\tau)d\xi\; d\tau = \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} \psi(\xi,\tau)d\xi\; d\tau$$

$$= \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} H\ (\xi - x + t - \tau)\ H\ (x + t - \tau - \xi)\ \psi\ (\xi, \tau)\ d\xi\ d\tau$$

$$= \int_0^t \int_{-\infty}^{\infty} E^+\ (x - \xi, t - \tau)\ \psi\ (\xi, \tau)\ d\xi\ d\tau.$$

Now this is the convolution of E^+ and ψ with respect to both variables in $\mathbb{R} \times \mathbb{R}^+$. Summarizing we have for $t > 0$

$$u_\alpha = E_t^+(\cdot, t) * \alpha \quad \text{and} \quad u_\beta = E^+(\cdot, t) * \beta \quad (\text{convolution in } x);$$

$$u_p = E^+ * \varphi \qquad (\text{convolution in } x \text{ and } t).$$

8. The fundamental solution in three and two space dimensions

For the wave equation in one dimension, we have constructed the fundamental solution

$$E^+(x, t) = \frac{1}{2}\ H(t)\{H(x + t) - H(x - t)\},$$

which was a distributional solution on $\mathbb{R} \times \mathbb{R}$ of $u_{tt} - u_{xx} = \delta(x, t) = \delta(x)\delta(t)$, with support contained in $\mathbb{R} \times [0, \infty)$.

Next we turn to the 3-dimensional case and try to find the analog of E^+. Thus we try to find a distribution in $\mathbb{R}^3 \times \mathbb{R}$ with support contained in $\{t \geq 0\}$, satisfying

$$u_{tt} - \Delta u = \delta(x_1, x_2, x_3, t) = \delta(x_1)\delta(x_2)\delta(x_3)\delta(t). \tag{8.1}$$

We shall first obtain a solution by formal methods, and then give a rigorous proof.

Because of the radial symmetry in this problem, we look for a solution of the form $u = u(r, t)$. For $t > 0$ this implies

$$u_{tt} = u_{rr} + \frac{2}{r} u_r,$$

or (this trick only works for $N = 3$)

$$(ru)_{tt} = (ru)_{rr}.$$

As in the one dimensional case we conclude that

$$ru(r, t) = v(t - r) + w(t + r).$$

Because the second term reflects signals coming inwards, we neglect it. Thus we consider

$$u(r,t) = \frac{v(t-r)}{r}.$$

Tracing "characteristics" of the form $t - r = c$ backwards in time, we conclude that $v(c) = 0$ if $c \neq 0$. These considerations suggest that $v(t-r) = \delta(t-r)$ (up to a constant).

8.1 Theorem The fundamental solution of the wave equation in $\mathbb{R}^3 \times \mathbb{R}$, i.e. the solution of (8.1) with support in $\mathbb{R}^3 \times [0,\infty]$, is given by

$$E^+(x_1,x_2,x_3,t) = \frac{\delta(t-r)}{4\pi r},$$

which we define as a distribution below.

In order to define E^+ as a distribution, we first compute formally what $< E^+, \psi >$ would be for $\psi \in D(\mathbb{R}^3 \times \mathbb{R})$, using the "rule"

$$\int \varphi(s)\ \delta(t-s)ds = \varphi(t).$$

Thus we evaluate $< E^+, \psi >$ using polar coordinates

$$x_1 = r\sin\theta\cos\varphi; \quad x_2 = r\sin\theta\sin\varphi; \quad x_3 = r\cos\theta.$$

Then

$$< E^+, \psi >=$$

$$\int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\delta(t-r)}{4\pi r}\psi(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta, t)\ r^2\sin\theta\ drd\varphi d\theta dt$$

$$= \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{1}{4\pi t}\ \psi(t\sin\theta\cos\varphi, t\sin\theta\sin\varphi, t\cos\theta, t)t^2\sin\theta d\varphi d\theta dt =$$

$$\int_0^{\infty} \frac{1}{4\pi t}\oint_{x_1^2+x_2^2+x_3^2=t^2} \psi(x_1,x_2,x_3,t)\ dS\ dt,$$

and we use this final expression as a definition of E^+.

8.2 Definition We define the distribution E^+ on $\mathbb{R}^3 \times \mathbb{R}$ by

$$< E^+, \psi >= \int_0^{\infty} \frac{1}{4\pi t}\oint_{x_1^2+x_2^2+x_3^2=t^2} \psi(x_1,x_2,x_3,t)\ dS(x_1,x_2,x_3)\ dt$$

for all $\psi \in D(\mathbb{R}^3 \times \mathbb{R})$. We also define $E^+(\cdot,\cdot,\cdot,t)$ as a distribution on $\mathbb{R}^3$ by

$$< E^+(t), \psi >= \frac{1}{4\pi t}\oint_{x_1^2+x_2^2+x_3^2=t^2} \psi(x_1,x_2,x_3)\ dS(x_1,x_2,x_3).$$

Next we prove that E^+ is a fundamental solution.

8.3 Lemma E^+ satisfies $E^+_{tt} - \Delta E^+ = \delta(x_1, x_2, x_3, t)$ in $\mathbb{R}^3 \times \mathbb{R}$.

Proof Let $\psi \in D(\mathbb{R}^3 \times \mathbb{R})$. Since $< \delta(x_1, x_2, x_3, t),\ \psi(x_1, x_3, x_3, t) >= \psi(0,0,0,0)$, and $< E^+_{tt} - \Delta E^+, \psi >=< E^+, \psi_{tt} - \Delta\psi >$, we have to show that $< E^+, \psi_{tt} - \Delta\psi >= \psi(0,0,0,0)$. Again we use polar coordinates. We have

$$\Delta\psi = \frac{1}{r^2}(r^2\psi_r)_r + \frac{1}{r^2\sin\theta}(\sin\theta\psi_\theta)_\theta + \frac{1}{r^2\sin^2\theta}\quad\psi_{\varphi\varphi},$$

so that

$$< E^+, \psi_{tt} - \Delta\psi >=$$

$$\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{1}{4\pi t}\big[\psi_{tt} - \frac{1}{r^2}\{(r^2\psi_r)_r - \frac{1}{\sin\theta}(\sin\theta\psi_\theta)_\theta - \frac{1}{\sin^2\theta}\psi_{\varphi\varphi}\}\big]_{r=t} t^2\sin\theta d\varphi d\theta dt$$

$$= \int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{1}{4\pi}\ \big[(r\psi)_{tt} - (r\psi)_{rr}\big]_{r=t}\sin\theta d\varphi d\theta dt$$

$$-\int_0^\infty\int_0^\pi\int_0^{2\pi}(\sin\theta\ \psi_\theta)_\theta d\varphi d\theta dt - \int_0^\infty\int_0^\pi\int_0^{2\pi}\ \frac{\psi_{\varphi\varphi}}{\sin\theta}\ d\varphi d\theta dt.$$

Obviously, the last two integrals are zero, so if γ is the curve $\{r = t > 0\}$ in the (r,t)- plane (along which we have $dr = dt$), then

$$< E^+_{tt} - \Delta E^+, \psi >= \int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{1}{4\pi}\big[(r\psi)_{tt} - (r\psi)_{rr}\big]_{r=t}\sin\theta d\varphi d\theta dt =$$

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta\int_\gamma\{(r\psi)_{tt} - (r\psi)_{rr}\}\ dt\ d\varphi\ d\theta =$$

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta\int_\gamma\{(r\psi)_{tt}dt + (r\psi)_{tr}dr - (r\psi)_{rr}dr - (r\psi)_{rt}dt\}d\varphi\ d\theta =$$

(since ψ has compact support)

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta\big[(r\psi)_r - (r\psi)_t\big]_{r=t=0}d\varphi\ d\theta = \psi(0,0,0,0).$$

This completes the proof. ■

Formally now, the solution of the equation $u_{tt} - \Delta u = \varphi(x_1, x_2, x_3, t)$ in $\mathbb{R}\times[0,\infty]$ with $u = u_t \equiv 0$ for $t < 0$, should be obtained by taking the convolution of E^+ and φ with respect to all variables, just like in the one-dimensional case. However, here E^+ is no longer a function, so the definition of this convolution is not entirely obvious. We shall restrict ourselves here to the formal computation. Then, with $(x, y, z) = (x_1, x_2, x_3)$, we have for $t > 0$,

$$u(x, y, x, t) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x-\xi, y-\eta, z-\zeta, t-\tau)\varphi(\xi, \eta, \zeta, \tau) d\xi d\eta d\zeta d\tau =$$

(writing $P = (x, y, z), Q = (\xi, \eta, \zeta)$, and $r_{PQ} = \sqrt{(x-\xi)^2 + (y-\eta^2) + (z-\zeta)^2)}$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^t \frac{\delta(t - r - r_{PQ})}{4\pi r_{PQ}} \varphi(\xi, \eta, \zeta, \tau) d\tau \; d\xi d\eta d\zeta =$$

(using the "rule" $\int \varphi(\tau) \; \delta(s-t) ds = \varphi(t)$)

$$\frac{1}{4\pi} \int \int \int_{r_{PQ} \le t} \frac{\varphi(\xi, \eta, \zeta, t - r_{PQ})}{r_{PQ}} d\xi d\eta d\zeta =$$

$$\int \int \int_{G(x,y,z,t)} \frac{\varphi(\xi, \eta, \zeta, t - \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta^2)}}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} d\xi d\eta d\zeta,$$

where

$$G(x, y, z, t) = \{(\xi, \eta, \zeta, \tau) : (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \le t^2\}.$$

Next we treat (only formally) some special cases.

8.4 Example Consider

$$\varphi(x, y, z, t) = \delta(x)\delta(y)\delta(z)f(t).$$

We find that $u(x, y, z, t) =$

$$\int \int \int_{G(x,y,z,t)} \frac{\delta(\xi)\delta(\eta)\delta(\zeta)f(t - \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta^2)})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} d\xi d\eta d\zeta$$

$$= \frac{f(t-r)}{4\pi r}.$$

8.5 Example Consider $\varphi(x, y, z, t) = \delta(x)\delta(y)f(t)$. Then $u(x, y, z, t) =$

$$\int \int \int_{G(x,y,z,t)} \frac{\delta(\xi)\delta(\eta)f(t - \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta^2)})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} d\xi d\eta d\zeta$$

$$= \frac{1}{2\pi} \int_{x^2+y^2+(z-\zeta)^2 \le t^2, \; z-\zeta \ge 0} \frac{f(t - \sqrt{x^2 + y^2 + (z-\zeta^2)})}{\sqrt{x^2 + y^2 + (z-\zeta)^2}} \; d\zeta$$

$$= \frac{1}{2\pi}\int_0^{t-r} \frac{f(\tau)d\tau}{\sqrt{(t-\tau)^2 - r^2}}.$$

(here $r = \sqrt{x^2+y^2}$, $\tau = t - \sqrt{x^2+y^2+(z-\zeta)^2}$, $d\tau = \frac{-\zeta+z}{\sqrt{x^2+y^2+(z-\zeta)^2}}\ d\zeta$, $(z-\zeta)^2 = (t-\tau)^2 - r^2$)

8.6 Example Consider $\varphi(x,y,z,t) = \delta(x)\delta(y)\delta(t)$ (or $f(t) = \delta(t)$ in the last example), then

$$u(x,y,z,t) = \frac{1}{2\pi}\int_0^{t-r} \frac{\delta(\tau)}{\sqrt{(t-\tau)^2 - r^2}}\ dt = \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}$$

Note however that this last expression is independent of t, so we have found the fundamental solution for the wave equation in two dimensions.

8.7 Proposition Let $E^+(x,y,t)$be given by

$$E^+(x,y,t) = \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}$$

Then E^+ is the fundamental solution of the wave equation in two dimensions, i.e. E^+ has support in $\{t \geq 0\}$ and satisfies $E^+_{tt} - E^+_{xx} - E^+_{yy} = \delta(x,y,t) = \delta(x,y,t)$ on $\mathbb{R}^2 \times \mathbb{R}$ in the sense of distributions.

Proof First note that E^+ is now a function. We have to show that $< E^+_{tt} - E^+_{xx} - E^+_{yy}, \psi > = < E^+, \psi_{tt} - \psi_{xx} - \psi_{yy} > = \psi(0,0,0)$ for all $\psi \in D(\mathbb{R}^2 \times \mathbb{R})$. To do so we introduce polar coordinates on $\mathbb{R}^2$, $x = r\cos\varphi$; $y = r\sin\varphi$. Then

$$\Delta\psi = \psi_{xx} + \psi_{yy} = \frac{1}{r}(r\psi_r)_r + \frac{1}{r^2}\psi_{\varphi\varphi}.$$

Thus

$$< E^+, \psi_{tt} - \Delta\psi > = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}\ \ (\psi_{tt} - \Delta\psi)\ \ dx\ dy\ dt$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_0^{t} \frac{\psi_{tt} - r^{-1}(r\psi r)_r - r^{-2}\psi_{\varphi\varphi}}{2\pi\sqrt{t^2-r^2}}\ r\ dr\ d\varphi\ dt$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_0^{t} \frac{r\psi_{tt} - (t\psi_r)_r}{2\pi\sqrt{t^2-r^2}}\ dr\ d\varphi\ dt = \frac{1}{2\pi}\int_0^{2\pi} J(\varphi)\ d\varphi,$$

where

$$J(\varphi) = \int_0^{\infty}\int_0^{t} \frac{r\psi_{tt} - (r\psi_r)_r}{\sqrt{t^2-r^2}}\ dr\ dt =$$

(if supp$\psi \subset \{t \leq T\}$)

$$\int_0^T \int_0^t \frac{r\psi_{rr} - (r\psi_r)_r}{\sqrt{t^2-r^2}}\ dr\ dt = \lim_{\varepsilon\downarrow 0} \int_\varepsilon^T \int_\varepsilon^t \frac{r\psi_{rr} - (r\psi_r)_r}{\sqrt{t^2-r^2}}\ dr\ dt =$$

(using the transformation $x = r$, $y = t/r$)

$$\lim_{\varepsilon\downarrow 0} \int_\varepsilon^T \int_1^{T/x} \{\frac{-(\psi_y\sqrt{y^2-1})_y}{x} - \frac{(x\psi_x)_x}{\sqrt{y^2-1}} + \frac{2y\psi_{xy}}{\sqrt{y^2-1}}\}\ dy\ dx$$

(here we have used

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x} - \frac{y}{x}\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{1}{x}\ \frac{\partial}{\partial y},$$

to transform the derivatives, and $drdt = xdxdy$)

$$= \lim_{\varepsilon\downarrow 0}\{\int_\varepsilon^T [-\frac{\psi_y\sqrt{y^2-1}}{x}]_{y=1}^{y=T/x} dx + \int_1^{T/\varepsilon} [\frac{2y\psi_y - x\psi_x}{\sqrt{y^2-1}}]_{x=\varepsilon}^{x=T/y} dy\}$$

$$= \lim_{\varepsilon\downarrow 0} \int_1^{T/\varepsilon} \frac{x\psi_x - 2y\psi_y}{\sqrt{y^2-1}} \Big|_{x=\varepsilon}\ dy =$$

(transforming the x- and y-derivatives back to r- and t-derivatives)

$$\lim_{\varepsilon\downarrow 0} \int_1^{T/\varepsilon} \frac{\varepsilon\psi_r - \varepsilon y\psi_t}{\sqrt{y^2-1}}\Big|_{x=\varepsilon} dy =$$

(writing $\psi(r, \varphi, t)$)

$$\lim_{\varepsilon\downarrow 0} \int_1^{T/\varepsilon} \frac{\varepsilon\psi_r(\varepsilon,\varphi,\varepsilon y) - \varepsilon y\psi_t(\varepsilon,\varphi,t)}{\sqrt{y^2-1}}\ dy =$$

(substituting $t = \varepsilon y$)

$$\lim_{\varepsilon\downarrow 0} \int_\varepsilon^T \frac{\varepsilon\psi_r(\varepsilon,\varphi,t) - t\psi_t(\varepsilon,\varphi,t)}{\sqrt{t^2-\varepsilon^2}}\ dt =$$

$$\lim_{\varepsilon\downarrow 0} \int_0^{T-\varepsilon} \{\frac{\varepsilon\psi_r(\varepsilon,\varphi,t+\varepsilon)}{\sqrt{(t+2\varepsilon)t}} - \frac{t+\varepsilon}{\sqrt{(t+2\varepsilon)t}}\ \psi_t(\varepsilon,\varphi,t+\varepsilon)\}\ dt$$

$$= -\int_0^T \psi_t(0,\varphi,t)dt = \psi(0,\varphi,0),$$

which completes the proof. ∎

PART 3: THE HEAT EQUATION

9. The fundamental solution of the heat equation in dimension one

As a first example we consider the problem

$$(P)\begin{cases} u_t = u_{xx} & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = H(x) & x \in \mathbb{R}, \end{cases}$$

where H is the Heaviside function. Now observe that if $u(x,t)$ is a solution of (P), then $u_a(x,t) = u(ax, a^2t)$ is also a solution of (P). Since we expect the solution to be unique, we should have

$$u(ax, a^2t) = u(x,t), \tag{9.1}$$

for all $a > 0$, $x \in \mathbb{R}$, $t > 0$. Thus if we put $a = 1/\sqrt{t}$, we obtain

$$u(x,t) = u(\frac{x}{\sqrt{t}}, 1) = U(\eta);\quad \eta = \frac{x}{\sqrt{t}}. \tag{9.2}$$

Here η is called the *similarity variable*. From (9.2) it follows that

$$u_t = U'(\eta)\frac{\partial \eta}{\partial t} = U'(\eta)\frac{x}{2t\sqrt{t}} = -\frac{\eta U'(\eta)}{2t};\quad u_{xx} = \frac{U''}{t},$$

so that $u(x,t) = U(\eta)$ is a solution of the heat equation if

$$U''(\eta) + \eta U'(\eta)/2 = 0, \tag{9.3}$$

or

$$(e^{\eta^2/4}U'(\eta))' = 0.$$

Thus

$$e^{\eta^2/4}U'(\eta) = \text{ constant } = A,$$

and

$$U(\eta) = B + A\int_{-\infty}^{\eta} e^{-s^2/4}\ ds = B + 2A\int_{-\infty}^{\eta/2} e^{-y^2}\, dy.$$

Since for $x < 0$,

$$0 = u(x,0) = \lim_{t\downarrow 0} U(\frac{x}{\sqrt{t}}) = U(-\infty) = B,$$

and for $x > 0$,

$$1 = u(x,0) = \lim_{t\downarrow 0} U(\frac{x}{\sqrt{t}}) = U(+\infty) = B + 2A\int_{-\infty}^{\infty} e^{-\eta^2}\, d\eta = 2A\sqrt{\pi},$$

we find that

$$U(\eta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\eta/2} e^{-s^2} ds; u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/2\sqrt{t}} e^{-s^2} ds. \tag{9.4}$$

We make the following observations.

(i) $u(x,t)$ is *smooth* for $t > 0$, but not at $t = 0$,

(ii) $\lim_{t\downarrow 0} u(x,t) = \begin{cases} 0 & \text{x} < 0 \\ 1/2 & \text{x} = 0 \\ 1 & \text{x} > 0 \end{cases},$

(iii) $0 = \min_{x\in\mathbb{R}}\ u(x,0) < u(x,t) < \max_{x\in\mathbb{R}}\ u(x,0) = 1$ (i.e a *strong comparison principle* seems to hold),

(iv) The positivity of u on $\mathbb{R}^+$ for $t = 0$ causes u to become positive immediately for $t > 0$ on the whole of $\mathbb{R}$ (*infinite speed of propagation*, in sharp constrast with the finite speed of propagation for the wave equation),

(v) $u(x,t) = U(x/\sqrt{t})$ is a self similar solution (or *similarity solution*).

Next we compute the solution of the heat equation with the initial value

$$u(x,0) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}.$$

Naturally we obtain

$$u_a(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{(x-a)/2\sqrt{t}} e^{-s^2} ds,$$

so that the solution with initial conditions

$$w(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < a \\ 0 & x > a \end{cases},$$

is given by

$$w(x,t) = u(x,t) - u_a(x,t) = \frac{1}{\sqrt{\pi}} \int_{(x-a)/2\sqrt{t}}^{x/2\sqrt{t}} e^{-s^2} ds,$$

which obviously satisfies

$$|w(x,t)| = |u(x,t) - u_a(x,t)| < \frac{a}{2\sqrt{\pi t}}.$$

Thus $w(x,t) \to 0$ as $t \to \infty$, the *decay order* being $1/\sqrt{t}$. Note that $w(x,0)$ is a bounded integrable function.

Going back to the solution with $u(x,0) = H(x)$, which is given by (9.4), we differentiate it with respect to t, to obtain a new solution

$$E^+(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}. \tag{9.5}$$

Obviously E^+ satisfies $E^+_t = E^+_{xx}$ for $t > 0$, and it is in fact the *fundamental solution for the heat equation*, that is, extending E^+ by $E^+(x,t) = 0$ for $t < 0$, we have

9.1 Proposition The function E^+ satisfies the fundamental equation $E^+_t - E^+_{xx} = \delta(x,t) = \delta(x)\delta(t)$ in $\mathbb{R}^2$.

Proof To check that E^+ is indeed a fundamental solution, we let $\psi \in D(\mathbb{R} \times \mathbb{R})$ and compute

$$< E^+_t - E^+_{xx}, \psi >= - < E^+, \psi_t + \psi_{xx} >= - \int\int_{\mathbb{R}\times\mathbb{R}^+} E^+(\psi_t + \psi_{xx}) d(x,t) =$$

$$- \lim_{\varepsilon \downarrow 0} \int\int_{\mathbb{R}\times(\varepsilon,\infty)} E^+(\psi_t + \psi_{xx}) d(x,t) =$$

$$- \lim_{\varepsilon \downarrow 0} \{ \int_{-\infty}^{\infty}\int_{\varepsilon}^{\infty} E^+ \psi_t dt dx + \int_{\varepsilon}^{\infty}\int_{-\infty}^{\infty} E^+ \psi_{xx} dx dt \} =$$

$$- \lim_{\varepsilon \downarrow 0} \{ \int_{-\infty}^{\infty} [E^+\psi]_{t=\varepsilon}^{t=\infty} dx - \int_{-\infty}^{\infty}\int_{\varepsilon}^{\infty} E^+_t \psi dt dx + \int_{\varepsilon}^{\infty}\int_{-\infty}^{\infty} E^+_{xx} \psi dx dt \} =$$

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} E^+(x,\varepsilon)\psi(x,\varepsilon) dx = \lim_{t \downarrow 0} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \psi(x,t) dx.$$

To complete the proof we have to show that this limit equals $\psi(0,0)$.

For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x| < \delta$ and $t < \delta$ then $|\psi(x,t) - \psi(0,0)| < \varepsilon$. Thus

$$\Big| \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \psi(x,t) dx - \psi(0,0) \Big| = \Big| \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} (\psi(x,t) - \psi(0,0)) dx \Big|$$

$$\leq \varepsilon \int_{-\delta}^{\delta} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} + 2 \sup |\psi| \int_{|x|\geq\delta} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} dx \leq$$

$$\varepsilon + \frac{\sup|\psi|}{\sqrt{\pi}} \int_{|s|\geq\delta/\sqrt{t}} e^{-s^2/4} ds \to \varepsilon \quad \text{as} \quad t \downarrow 0.$$

Since $\varepsilon > 0$ was arbitrary this completes the proof. ■

10. The Cauchy problem in one dimension

For a given function $u_0 : \mathbb{R} \to \mathbb{R}$ we consider the problem

$$(CP) \begin{cases} u_t = u_{xx} & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

Our experience with the wave equation suggests to consider the convolution

$$u(x,t) = (E^+(t) * u_0)(x) = \int_{-\infty}^{\infty} E^+(x-\xi,t)u_0(\xi)d\xi$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-(x-\xi)^2/4t} u_0(\xi)d\xi. \tag{10.1}$$

10.1 Notation Let $Q = \mathbb{R} \times \mathbb{R}^+$. Then

$$C^{2,1}(Q) = \{u : Q \to \mathbb{R};\ u, u_t, u_x, u_{xx} \in C(Q)\}.$$

10.2 Theorem Suppose $u_0 \in C(\mathbb{R})$ is bounded. Then (CP) has a unique bounded classical solution $u \in C^{2,1}(Q) \cap C(\overline{Q})$, given by the convolution (10.1).

Proof of existence Clearly $E^+(\cdot,t) * u_0$ is well defined and bounded for all $(x,t) \in \mathbf{Q}$, because u_0 is bounded and $E^+(x,t)$ decays exponentially fast to zero as $|x| \to \infty$. Since the same holds for all partial derivatives of $E^+(x,t)$, we can differentiate under the integral with respect to x and t. Thus for any $n, l \in \mathbb{N} \cup \{0\}$,

$$\Big(\frac{\partial}{\partial t}\Big)^n \Big(\frac{\partial}{\partial x}\Big)^l u(x,t) = \Big(\frac{\partial}{\partial t}\Big)^n \Big(\frac{\partial}{\partial x}\Big)^l \int_{\infty}^{\infty} E^+(x-\xi,t)u_0(\xi)d\xi$$

$$= \int_{-\infty}^{\infty} \frac{\partial^{n+l}E^+(x-\xi,t)}{\partial t^n \partial x^l} u_0(\xi)d\xi = \Big(\frac{\partial^{n+l}E^+}{\partial t^n \partial x^l}(\cdot,t) * u_0\Big)(x),$$

and in particular

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \Big(\frac{\partial E^+}{\partial t} - \frac{\partial^2 E^+}{\partial x^2}\Big) * u_0 = 0.$$

Hence $u \in C^\infty(Q)$ satisfies $u_t = u_{xx}$ in Q.

It remains to show that for every $x_0 \in \mathbb{R}$

$$\lim_{\substack{x \to x_0 \\ t \downarrow 0}} u(x,t) = u_0(x).$$

The argument is similar to the proof that E^+ satisfies the fundamental equation.

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|u_0(x) - u_0(x_0)| < \varepsilon$ for $|x - x_0| < \delta$. For $|x - x_0| < \frac{1}{2}\delta$ we have

$$|u(x,t) - u_0(x_0)| = \Big| \int_{-\infty}^{\infty} E^+(x-\xi,t)(u_0(\xi) - u_0(x_0))d\xi \Big| \le$$

$$\int_{|x-\xi|<\frac{1}{2}\delta} E^+(x-\xi,t)\big|u_0(\xi) - u_0(x_0)\big|d\xi + \int_{|x-\xi|>\frac{1}{2}\delta} E^+(x-\xi,t)\big|u_0(\xi) - u_0(x_0)\big|d\xi$$

(since $|x - \xi| < \frac{1}{2}\delta$ together with $|x - x_0| < \frac{1}{2}\delta$ implies $|\xi - x_0| < \delta$)

$$\le \varepsilon \int_{|x-\xi|<\frac{1}{2}\delta} E^+(x-\xi,t)d\xi + 2\sup|u_0| \int_{|x-\xi|>\frac{1}{2}\delta} E^+(x-\xi,t)d\xi$$

$$\le \varepsilon + 2\sup|u_0| \int_{|\xi|>\frac{1}{2}\delta} E^+(\xi,t)d\xi \to \varepsilon \quad \text{as} \quad t \downarrow 0$$

as before. Since $\varepsilon > 0$ was arbitrary, this completes the proof of the existence of a solution. Note that for the continuity of $u(x,t)$ at $(x,t) = (x_0, 0)$ we have only used the continuity of $u_0(x)$ at $x = x_0$. ∎

Proof of uniqueness Suppose there exist two different solutions of (CP) in $C^{2,1}(Q) \cap C(\overline{Q})$. Then the difference is a nontrivial classical solution u of

$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = 0 & x \in \mathbb{R}. \end{cases}$$

This is impossible in view of a *maximum principle* which we state and prove below.

10.3 Lemma Suppose $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$, where $Q_T = \mathbb{R} \times (0,T]$ $(T > 0)$, satisfies

$$u_t \le u_{xx} \text{ in } Q_T.$$

If (i) $u(x,0) \le 0$ for all $x \in \mathbb{R}$;
(ii) $u(x,t) \le Ae^{Bx^2}$ for all $(x,t) \in Q_T$,
where $A > 0$ and B are fixed constants, then

$$u \le 0 \text{ in } \overline{Q}_T.$$

10.4 Lemma For $-\infty < a < b < \infty$ and $T > 0$ let $Q_T^{a,b} = (a,b) \times (0,T]$, and $\Gamma_T^{a,b} = \overline{Q}_T^{a,b} \backslash Q_T^{a,b}$. $\Gamma_T^{a,b}$ is called the *parabolic boundary* of $Q_T^{a,b}$. Suppose $u \in C^{2,1}(Q_T^{a,b}) \cap C(\overline{Q}_T^{a,b})$ satisfies

$$u_t \leq u_{xx} \text{ in } Q_T^{a,b}.$$

Then

$$\sup_{Q_T^{a,b}} u = \max_{\Gamma_T^{a,b}} u.$$

Proof of Lemma 10.4 First observe that if $u_t < u_{xx}$ in $Q_T^{a,b}$, then u cannot have a (local or global) maximum in $Q_T^{a,b}$. Indeed, if this maximum would be situated at (x_0,t_0) with $a < x_0 < b$ and $0 < t_0 < T$, then at $(x,t) = (x_0,t_0)$ one has $u_{xx} > u_t = u_x = 0$, contradiction. Also a maximum at (x_0,T) is impossible because then $u_{xx} > u_t \geq 0$, again a contradiction.

Next we reduce the case $u_t \leq u_{xx}$ to $u_t < u_{xx}$. Let

$$u_n(x,t) = u(x,t) + \frac{x^2}{2n}.$$

Then obviously

$$u_{nt} = u_t \leq u_{xx} < u_{xx} + \frac{1}{n} = u_{nxx},$$

so that

$$\sup_{Q_T^{a,b}} u_n = \max_{\Gamma_T^{a,b}} u_n.$$

Taking the limit $n \to \infty$ the lemma follows. ∎

Proof of Lemma 10.3 It is sufficient to prove the statement for one fixed $T > 0$. For $\alpha, \beta, \gamma > 0$ let

$$h(x,t) = \exp(\frac{\alpha x^2}{1-\beta t} + \gamma t) \qquad x \in \mathbb{R},\ 0 \leq t < \frac{1}{\beta}.$$

Define $u(x,t)$ by $u = hv$. Then

$$0 \geq u_t - u_{xx} = (hv)_t - (hv)_{xx} = hv_t + h_t v - hv_{xx} - 2h_x v_x - h_{xx} v =$$

$$h(v_t - v_{xx} - v_x \frac{2h_x}{h} + v\frac{h_t - h_{xx}}{h} =$$

$$h\big(v_t - v_{xx} - v_x \frac{4\alpha x}{1-\beta t} + v(\frac{\alpha\beta x^2}{(1-\beta t)^2} + \gamma - (\frac{2\alpha x}{1-\beta t})^2 - \frac{2\alpha}{1-\beta t})\big) =$$

$$h\big(v_t - v_{xx} - \frac{4\alpha x}{1-\beta t}v_x + v(\gamma - \frac{(4\alpha-\beta)\alpha x^2}{(1-\beta t)^2} - \frac{2\alpha}{1-\beta t})\big). \tag{9.6}$$

Choosing $\beta > 4\alpha$ and $\gamma > 4\alpha$ the coefficient of v is positive for $x \in \mathbb{R}$ and $0 \leq t \leq 1/2\beta$. We then also have

$$v(x,t) = u(x,t)\exp(-\frac{\alpha x^2}{1-\beta t} - \gamma t) \leq Ae^{(B-\alpha)x^2},$$

so that, choosing $\alpha > B$,

$$\limsup_{|x|\to\infty} v(x,t) \leq 0 \quad \text{uniformly on} \quad [0, \frac{1}{2\beta}]. \tag{9.7}$$

Now suppose the lemma is false for $T = 1/2\beta$. Then u and v achieve positive values on $Q_{1/2\beta}$. In view (9.7) this implies that v must have a positive maximum in $Q_{1/2\beta}$. By the inequality for $u_t - u_{xx}$ and the choice of α, β, γ this implies $v_t < v_{xx}$ at this maximum. But in the proof of Lemma 10.4 we have seen that this is impossible, contradiction. ∎

10.5 Exercise Finish the uniqueness proof. ∎

10.6 Exercise For $u_0 \in C(\mathbb{R})$ satisfying

$$|u_0(x)| \leq Ae^{Bx^2}$$

for all $x \in \mathbb{R}$, prove that (CP) has a classical solution $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ for all $T < 1/4B$, and give a growth condition which determines the solution uniquely.

Next we consider the equation

$$u_t = u_{xx} + \varphi \qquad x \in \mathbb{R}, \quad 0 < t \leq T,$$

where $\varphi : \mathbb{R} \times (0,T) \to \mathbb{R}$. If φ is measurable and bounded, we can try as a particular solution

$$u_p(x,t) = \int_0^t \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)\varphi(\xi,\tau)d\xi d\tau. \tag{9.8}$$

Clearly, u_p is well defined, because the integral is dominated by

$$\int_0^t \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau) \sup_{Q_T} |\varphi| \; d\xi d\tau \leq t \sup_{Q_T} |\varphi|,$$

so that in particular $u_p(x,t) \to 0$ uniformly in x as $t \downarrow 0$.

One would like to have $u_p \in C^{2,1}(Q_T)$, which is however rather technical to establish and unfortunately requires more than just the continuity of φ. Here we just restrict ourselves to

10.7 Proposition Let $\varphi \in L^\infty(Q_T)$. Then

$$u_p(x,t) = \int_0^t \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)\varphi(\xi,\tau)d\xi d\tau$$

defines a bounded function which is a solution of $u_t = u_{xx} + \varphi$ in the sense of distributions on $\mathbb{R} \times (0,T)$, and tends to zero uniformly on $\mathbb{R}$ as $t \downarrow 0$.

Proof If we set $E^+(x,t) \equiv \varphi(x,t) \equiv 0$ for all $t < 0$, then

$$u_p(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)\varphi(\xi,\tau)d\xi d\tau.$$

Let $\psi \in D(\mathbb{R} \times (0,T))$, and extend ψ to $\mathbb{R} \times \mathbb{R}$ by $\psi(x,t) \equiv 0$ for $t \leq 0$ and $t \geq T$. Then

$$< \frac{\partial u_p}{\partial t} - \frac{\partial^2 u_p}{\partial x^2}, \psi >= - < u_p, \psi_t + \psi_{xx} >=$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E^+(x-\xi,t-\tau)\varphi(\xi,\tau)(\psi_t(x,t)+\psi_{xx}(x,t))d\xi d\tau dx dt =$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E^+(x-\xi,t-\tau)(\psi_t(x,t)+\psi_{xx}(x,t))dx dt\}\varphi(\xi,\tau)d\xi d\tau =$$

(as in the proof that E^+ is a fundamental solution)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\tau)\psi(\xi,\tau)d\xi d\tau =< \varphi, \psi >,$$

so that u_p is a solution of the inhomogeneous heat equation in the sense of distributions. ∎

We can now write down the solution of

$$(CP_i) \; \begin{cases} u_t = u_{xx} + \varphi & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

as

$$u(x,t) = \int_{-\infty}^{\infty} E^+(x-\xi,t)u_0(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} E^+(x-\xi,t-\tau)\varphi(\xi,\tau)d\xi d\tau,$$

but we do not give the precise hypothesis here on u_0 and φ that guarantee that this formula defines a *classical* solution, i.e.

$$u \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+) \cap C(\overline{\mathbb{R}} \times \overline{\mathbb{R}}^+).$$

10.8 Exercise Show that (CP_i) has at most one bounded classical solution.

11. Initial boundary value problems

First we indicate how one can generalize results for

$$(CP) \begin{cases} u_t = u_{xx} & x \in \mathbb{R} \ , \ t > 0; \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

to

$$(CD) \begin{cases} u_t = u_{xx} & x > 0 \ , \ t > 0; \\ u(0,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & x \geq 0, \end{cases}$$

and

$$(CN) \begin{cases} u_t = u_{xx} & x > 0 \ , \ t > 0; \\ u_x(0,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & x \geq 0. \end{cases}$$

For (CD) and (CN) we consider (CP) with odd and even initial data respectively.

We begin with (CD). Extending u_0 to the whole of $\mathbb{R}$ by $u_0(-x) = -u_0(x)$, the integral representation of solutions gives

$$u(x,t) = -\int_{-\infty}^{0} E^+(x-\xi,t)u_0(-\xi)d\xi + \int_0^\infty E^+(x-\xi,t)u_0(\xi)d\xi =$$

$$\int_0^\infty \{E^+(x-\xi,t) - E^+(x+\xi,t)\}u_0(\xi)d\xi = \int_0^\infty G_1(x,\xi,t)u_0(\xi)d\xi, \qquad (11.1)$$

where

$$G_1(x,\xi,t) = E^+(x-\xi,t) - E^+(x+\xi,t) \qquad (11.2)$$

is called the *Green's function of the first kind.*

For (CN) we extend u_0 by $u_0(-x) = u_0(x)$, and thus

$$u(x,t) = \int_{-\infty}^{0} E^+(x-\xi,t)u_0(-\xi)d\xi + \int_0^\infty E^+(x-\xi,t)u_0(\xi)d\xi =$$

$$\int_0^\infty \{E^+(x-\xi,t) + E^+(x+\xi,t)\}u_0(\xi)d\xi = \int_0^\infty G_2(x,\xi,t)u_0(\xi)d\xi, \qquad (11.3)$$

where

$$G_2(x,\xi,t) = E^+(x-\xi,t) + E^+(x+\xi,t) \qquad (11.3)$$

is called the *Green's function of the second kind.*

11.1 Exercise Let $u_0 \in C(\overline{\mathbb{R}}^+)$ be bounded, and let $u_0(0) = 0$. Prove that (CD) has a unique bounded solution $u \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^+) \cap C(\overline{\mathbb{R}}^+ \times \overline{\mathbb{R}}^+)$.

11.2 Exercise Let $u_0 \in C(\overline{\mathbb{R}}^+)$ be bounded. Prove that (CN) has a unique bounded solution $u \in C^{2,1}(\overline{\mathbb{R}}^+ \times \mathbb{R}^+) \cap C(\overline{\mathbb{R}}^+ \times \overline{\mathbb{R}}^+)$.

11.3 Exercise Derive formal integral representations for the solutions of

$$(CD_i) \begin{cases} u_t = u_{xx} + \varphi & x > 0,\ t > 0; \\ u(0,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & x \geq 0, \end{cases}$$

and

$$(CN_i) \begin{cases} u_t = u_{xx} + \varphi & x > 0\ ,\ t > 0; \\ u_x(0,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & x \geq 0. \end{cases}$$

Next we consider what is usually called the *Dirichlet problem* for the heat equation on $(0,1)$:

$$(D) \begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0; \\ u(0,t) = u(1,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & 0 \leq x \leq 1. \end{cases}$$

To find an integral representation for the solution of (D) we extend u_0 to a 2-periodic function $\tilde{u}_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{u}_0 \equiv u_0 \text{ on } (0,1);\ \tilde{u}_0(x) = -\tilde{u}_0(-x);\ \tilde{u}_0(1+x) = -\tilde{u}_0(1-x).$$

For the Cauchy problem with initial dat $\tilde{u}_0$ we then have

$$u(x,t) = \int_{-\infty}^{\infty} E^+(x-\xi,t)\tilde{u}_0(\xi)d\xi = \sum_{k=-\infty}^{\infty} \int_k^{k+1} E^+(x-\xi,t)\tilde{u}_0(\xi)d\xi =$$

$$\sum_{k=-\infty}^{\infty} \int_0^1 E^+(x-\xi-k,t)\tilde{u}_0(\xi+k)d\xi =$$

$$\sum_{n=-\infty}^{\infty} \{\int_0^1 E^+(x-\xi-2n,t)\tilde{u}_0(\xi+2n)d\xi + \int_0^1 E^+(x-\xi-2n-1,t)\tilde{u}_0(\xi+2n+1)d\xi\} =$$

$$\sum_{n=-\infty}^{\infty} \{\int_0^1 E^+(x-\xi-2n,t)\tilde{u}_0(\xi)d\xi + \int_0^1 E^+(x-\xi-2n-1,t)\tilde{u}_0(\xi+1)d\xi\} =$$

$$\sum_{n=-\infty}^{\infty} \{\int_0^1 E^+(x-\xi-2n,t)\tilde{u}_0(\xi)d\xi - \int_0^1 E(x-\xi-2n-1,t)\tilde{u}_0(1-\xi)d\xi\} =$$

$$\sum_{n=-\infty}^{\infty}\{\int_0^1 E^+(x-\xi-2n,t)\tilde{u}_0(\xi)d\xi - \int_0^1 E^+(x+\xi-1-2n-1,t)\tilde{u}_0(\xi)d\xi\}$$

$$= \int_0^1 \sum_{n=-\infty}^{\infty}\big\{E^+(x-\xi-2n,t) - \ E^+(x+\xi-2n,t)\big\}\tilde{u}_0(\xi)$$

$$= \int_0^1 G_D(x,\xi,t)\tilde{u}_0(\xi)d\xi, \tag{11.4}$$

where

$$G_D(x,\xi,t) = \sum_{n=-\infty}^{\infty}\big\{E^+(x-\xi-2n,t) - E^+(x+\xi-2n,t)\big\}. \tag{11.5}$$

(Note that this sum is absolutely convergent for $t > 0$, uniformly in x.)

11.4 Theorem Let $u_0 \in C([0,1]), u_0(0) = u_0(1) = 0$, and let $Q_T = (0,1)\times(0,T]$. Then for every $T > 0$ there exists a unique bounded solution $u \in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ of (D), given by

$$u(x,t) = \int_0^1 G_D(x,\xi,t)u_0(\xi)d\xi.$$

Proof Exercise, for the uniqueness part, the maximum principle has to be used again. ■

G_D is called the *Green's function for the Dirichletproblem.*

For the *Neumannproblem*, that is

$$(N)\begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0; \\ u_x(0,t) = u_x(1,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & 0 \le x \le 1, \end{cases}$$

we extend u_0 to a 2-periodic function $\tilde{u}_0 : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{u}_0 \equiv u_0 \text{ on } [0,1];\ \tilde{u}_0(x) = \tilde{u}_0(-x);\ \tilde{u}_0(1+x) = \tilde{u}_0(1-x).$$

We now obtain

$$u(x,t) = \int_0^1 G_N(x,\xi,t)u_0(\xi)d\xi, \tag{11.6}$$

where

$$G_N(x,\xi,t) = \sum_{n=-\infty}^{\infty}\big\{E^+(x-\xi-2n,t) + E^+(x+\xi-2n,t)\big\} \tag{11.7}$$

is *Green's function for the Neumannproblem.*

11.5 Theorem Let $u_0 \in C([0,1])$. Then for every $T > 0$ there exists a unique bounded classical solution of (N), given by

$$u(x,t) = \int_0^1 G_N(x,\xi,t)u_0(\xi)d\xi.$$

11.6 Exercise Give a suitable definition of a classical solution of (N) and prove this theorem.

11.7 Exercise Derive a representation formula for solutions of the mixed problem

$$(DN)\begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0; \\ u(0,t) = u_x(1,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & 0 \le x \le 1, \end{cases}$$

and formulate and prove a uniqueness/existence theorem.

11.8 Exercise Give formal derivations for integral representations of solutions to the problems above with $u_t = u_{xx}$ replaced by the inhomogeneous equation $u_t = u_{xx} + \varphi$.

PART 4: FUNCTIONAL ANALYSIS

12. Banach spaces

12.1 Definition A real vector space X is called a *real normed space* if there exists a map

$$\|\cdot\| : X \to \overline{\mathbb{R}}^+,$$

such that, for all $\lambda \in \mathbb{R}$ and $x, y \in X$, (i) $\|x\| = 0 \Leftrightarrow x = 0$; (ii) $\|\lambda x\| = |\lambda|\,\|x\|$; (iii) $\|x+y\| \le \|x\| + \|y\|$. The map $\|\cdot\|$ is called the *norm.*

12.2 Definition Suppose X is a real normed space with norm $\|\cdot\|$, and that $|||\cdot|||$ is also a norm on X. Then $\|\cdot\|$ and $|||\cdot|||$ are called equivalent if there exist $A, B > 0$ such that for all $x \in X$

$$A\|x\| \le |||x||| \le B\|x\|.$$

12.3 Notation $B_R(y) = \{x \in X : \|x-y\| < R\}.$

12.4 Definition Let S be a subset of a normed space X. S is called *open* ($\Leftrightarrow X\backslash S$ is closed) if for every $y \in S$ there exists $R > 0$ such that $B_R(y) \subset S$. The open sets form a *topology* on X, i.e. (i) $\emptyset$ and X are open; (ii) unions of open sets are open; (iii) finite intersections of open sets are open.

12.5 Remark Equivalent norms define the same topology.

12.6 Definition Let X be a normed space, and $(x_n)_{n=1}^\infty \subset X$ a sequence. Then $(x_n)_{n=1}^\infty$ is called *convergent* with limit $\overline{x} \in X$ (notation $x_n \to \overline{x}$) if $\|x_n - \overline{x}\| \to 0$ as $n \to \infty$. If $\|x_n - x_m\| \to 0$ as $m, n \to \infty$, then $(x_n)_{n=1}^\infty$ is called a *Cauchy sequence*.

12.7 Definition A normed space X is called a *Banach space* if every Cauchy sequence in X is convergent.

12.8 Theorem (Banach contraction theorem) Let X be a Banach space and $T : X \to X$ a contraction, i.e. a map satisfying

$$\|Tx - Ty\| \leq \theta \|x - y\| \quad \forall x, y \in X,$$

for some fixed $\theta \in [0, 1)$. Then T has a unique fixed point $\overline{x} \in X$. Moreover, if $x_0 \in X$ is arbitrary, and $(x_n)_{n=1}^\infty$ is defined by

$$x_n = Tx_{n-1} \qquad \forall n \in \mathbf{N},$$

then $x_n \to \overline{x}$ as $n \to \infty$.

12.9 Definition Let X and Y be normed spaces, and $T : X \to Y$ a linear map, i.e.

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty \quad \text{for all} \quad \lambda, \mu \in \mathbb{R} \quad \text{and} \quad x, y \in X.$$

Then T is called *bounded* if

$$\|T\| = \sup_{0 \neq x \in X} \frac{\|Tx\|_Y}{\|x\|_X} < \infty.$$

The map $T \to \|T\|$ defines a norm on the vector space $\mathcal{B}(X, Y)$ of bounded linear maps $T : X \to Y$, so that $\mathcal{B}(X, Y)$ is a normed space. In the case that $Y = \mathbb{R}$, the space $X^* = \mathcal{B}(X, \mathbb{R})$ is called the *dual space* of X.

12.10 Theorem Let X be a normed space and Y a Banach space. Then $\mathcal{B}(X, Y)$ is also a Banach space. In particular every dual space is a Banach space.

Many problems in linear partial differential equations boil down to the question as to whether a given linear map $T : X \to Y$ is invertible.

12.11 Theorem Let X and Y be Banach spaces, and let $T \in \mathcal{B}(X,Y)$ a bijection. Then $T^{-1} \in \mathcal{B}(Y,X)$.

12.12 Theorem (method of continuity) Let X be a Banach space and Y a normed space, and T_0 and $T_1 \in B(X,Y)$. For $t \in [0,1]$ let $T_t \in \mathcal{B}(X,Y)$ be defined by

$$T_t x = (1-t)T_0 x + tT_1 x.$$

Suppose there exists $C > 0$ such that

$$\|x\|_X \leq C\|T_t x\|_Y \qquad \forall x \in X.$$

Then T_0 is surjective if and only if T_1 is surjective, in which case all T_t are invertible with

$$T_t^{-1} \in \mathcal{B}(Y,X) \quad \text{and} \quad \|T_t^{-1}\| \leq C.$$

12.13 Definition Let X, Y be normed spaces, and $T \in \mathcal{B}(X,Y)$. The *adjoint* T^* of T is defined by

$$T^* f = f \circ T \qquad \forall f \in Y^*,$$

i.e.

$$(T^* f)(x) = f(Tx) \quad \forall x \in X.$$

12.14 Remark Observe that if $X = Y$ and I is the identity on X, then I^* is the identity on X^*.

12.15 Theorem Let X, Y be Banach spaces and $T \in \mathcal{B}(X,Y)$. Then $T^* \in \mathcal{B}(Y^*, X^*)$ and

$$\|T\|_{B(X,Y)} = \|T^*\|_{\mathcal{B}(Y^*,X^*)}.$$

12.15 Definition Let X, Y be normed spaces and $T \in \mathcal{B}(X,Y)$. Then T is called *compact* if, for every bounded sequence $(x_n)_{n=1}^{\infty} \subset X$, the sequence $(Tx_n)_{n=1}^{\infty}$ contains a convergent (in Y) subsequence. The linear subspace of compact bounded linear maps is denoted by $\mathcal{K}(X,Y)$.

12.16 Theorem Let X be a normed space and Y a Banach space. Then $\mathcal{K}(X,Y)$ is a closed linear subspace of $\mathcal{B}(X,Y)$. Furthermore: $T \in \mathcal{K}(X,Y) \Leftrightarrow T^* \in \mathcal{K}(Y^*, X^*)$.

In all practical cases one can only verify that $T \in \mathcal{K}(X,Y)$ if Y is Banach, because then it suffices to extract a Cauchy sequence from $(Tx_n)_{n=1}^{\infty}$.

12.17 Definition Let X be a normed space and $M \subset X$. Then

$$M^{\perp} = \{f \in X^* : \quad f(x) = 0 \quad \forall x \in M\}.$$

12.18 Definition Let X, Y be vector spaces, and $T : X \to Y$ linear. Then

$$N(T) = \{x \in X : Tx = 0\} \quad \text{(kernel of } T\text{)},$$

$$R(T) = \{y \in Y : \exists x \in X \ \text{ with } \ Y = Tx\} \quad \text{(range of } T\text{)}.$$

Clearly these are linear subspaces of X and Y respectively.

12.18 Definition Let X be a normed space and $M \subset X^*$. Then

$$^{\perp}M = \{x \in X : f(x) = 0 \qquad \forall f \in M\}.$$

12.19 Theorem (Fredholm alternative) Let X be a Banach space and $T \in \mathcal{K}(X) = \mathcal{K}(X, X)$. Let $I \in \mathcal{B}(X) = \mathcal{B}(X, X)$ denote the identity. Then

(i) $\dim N(I - T) = \dim N(I^* - T^*) < \infty$;

(ii) $R(I - T) =^{\perp} N(I^* - T^*)$ is closed;

(iii) $N(I - T) = \{0\} \ \Leftrightarrow \ R(I - T) = X$.

Thus $I - T$ has properties resembling those of *matrices*.

12.20 Definition Let X be a Banach space and $T \in \mathcal{B}(X)$. Then

$$\rho(t) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is a bijection}\}$$

is called the *resolvent set* of T, and $\sigma(T) = \mathbb{R} \backslash \rho(T)$ the *spectrum* of T. A subset of the spectrum is

$$\sigma_E(T) = \{\lambda \in \sigma(T) : \lambda \text{ is an eigenvalue of } T\} = \{\lambda \in \mathbb{R} : N(T - \lambda I) \neq \{0\}\}.$$

12.21 Theorem Let X be a Banach space and $T \in \mathcal{K}(X)$. Then

(i) $\sigma(T) \subset [-\|T\|, \|T\|]$ is compact;

(ii) $\dim X = \infty \ \Rightarrow \ 0 \in \sigma(T)$;

(iii) $\sigma(T) \backslash \{0\} \subset \sigma_E(T)$;

(iv) either $\sigma(T) \backslash \{0\}$ is finite or $\sigma(T) \backslash \{0\}$ consists of a sequence converging to zero.

12.22 Definition Let X be a vector space. A *convex cone* in X is a set $C \subset X$ with

$$\lambda x + \mu y \in C \ \ \forall \lambda, \mu \in \overline{\mathbb{R}}^+ \ \forall x, y \in C.$$

12.23 Theorem (Krein-Rutman) Let X be a Banach space and $C \subset X$ a closed convex cone with

$$\text{int}C \neq \emptyset \ \text{ and } \ C \cap (-C) = \{0\}.$$

Suppose $T \in \mathcal{K}(X)$ satisfies

$$T(C \backslash \{0\}) \subset \text{int}C.$$

Then $\overline{\lambda} = \sup \sigma(T)$ is the only eigenvalue with an eigenvector in C, and its multiplicity is one.

12.24 Theorem Let X, Y, X be Banach spaces, and $T \in \mathcal{B}(X, Y)$, $S \in \mathcal{B}(Y, Z)$. If $T \in \mathcal{K}(X, Y)$ or $S \in \mathcal{K}(Y, Z)$, then $S \circ T \in \mathcal{K}(X, Z)$.

12.25 Definition Let X be a normed space. The *weak topology* on X is the smallest topology on X for which every $f \in X^*$ is a continuous function from X to $\mathbb{R}$ (with respect to this topology).

The weak topology is weaker then the norm topology, i.e. every norm open set is also weakly open. If X is finite dimensional, the converse also holds, but never if $\dim X = \infty$.

12.26 Notation In every topology one can define the concept of convergence. For x_n converging to x in the weak topology we use the notation $x_n \rightharpoonup x$.

12.27 Proposition Let X be a Banach space, and $(x_n)_{n=1}^{\infty}$ a sequence in X. Then

(i) $x_n \rightharpoonup x \ \Leftrightarrow \ f(x_n) \to f(x) \ \ \forall f \in X^*$;

(ii) $x_n \rightharpoonup x \Rightarrow \|x\|$ is bounded and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

12.28 Theorem Let X be a Banach space, and $K \subset X$ a convex set, i.e. $\lambda x + (1-\lambda)y \in K \ \forall x, y \in K \ \forall \lambda \in [0,1]$. Then K is weakly closed if and only if K is norm closed.

12.29 Notation Let X be a normed space, $x \in X$ and $f \in X^*$. Then we shall frequently write $< f, x >= f(x)$. Thus $< \cdot, \cdot >: X^* \times X \to \mathbb{R}$. Note that for every fixed $x \in X$ this expression defines a function from X^* to $\mathbb{R}$.

12.30 Definition The *weak** topology on X^* is the smallest topology for which all $x \in X$ considered as functions from X^* to $\mathbb{R}$ are continuous.

12.31 Notation For convergence in the weak* topology we write $f_n \overset{*}{\rightharpoonup} f$, and again this is equivalent to $< f_n, x > \to < f, x >$ for all $x \in X$. The importance of the weak* topology lies in

12.32 Theorem (Alaoglu) Let X be a Banach space. Then the closed unit ball in X^* is compact in the *weak** topology.

12.33 Definition A Banach space X is called *reflexive* if every $\varphi \in (X^*)^*$ is of the form

$$\varphi(f) = f(x) =< f, x > \qquad \forall f \in X^*$$

for some $x \in X$.

12.34 Theorem Let X be a separable reflexive space. Then every bounded sequence in X has a weakly convergent subsequence.

13. Hilbert spaces

13.1 Definition Let H be a (real) vector space. A function

$$(\cdot, \cdot) : H \times H \to \mathbb{R}$$

is called an *inner product* if, for all $u, v, w \in H$ and for all $\lambda, \mu \in \mathbb{R}$, (i) $(u, u) \geq 0$, and $(u, u) = 0 \Leftrightarrow u = 0$; (ii) $(u, v) = (v, u)$; (iii) $(\lambda u + \mu v, w) = \lambda(u, w) + \mu(v, w)$.

13.2 Remark Any inner product satisfies

$$|(u, v)| \leq \sqrt{(u, u)(v, v)} \qquad \forall u, v \in H, \quad \text{(Schwartz)}$$

and also

$$\sqrt{(u + v, u + v)} \leq \sqrt{(u, u)} + \sqrt{(v, v)} \qquad \forall u, v \in H.$$

Consequently, $\|u\| = \sqrt{(u, u)}$ defines a norm on H, called the *inner product norm*.

13.3 Definition If H is a Banach space with respect to this inner product norm, then H is called a *Hilbert space*.

13.4 Theorem For every closed convex subset K of a Hilbert space H, and for every $f \in H$, there exists a unique $u \in K$ such that

$$\|f - u\| = \min_{v \in K} \|f - v\|,$$

or, equivalently,

$$(f - u, v - u) \leq 0 \qquad \forall v \in K.$$

Moreover the map $P_K : f \in H \to u \in K$ is contractive in the sense that

$$\|P_K f_1 - P_K f_2\| \leq \|f_1 - f_2\|.$$

13.5 Theorem (Riesz) For fixed $f \in H$ define $\varphi \in H^*$ by

$$\varphi(v) = (f, v) \qquad \forall v \in H.$$

Then the map $f \to \varphi$ defines an *isometry* between H and H^*, which allows one to identify H and H^*.

13.6 Corollary Every Hilbert space is reflexive. In particular bounded sequences of Hilbert spaces have weakly convergent subsequences.

13.7 Theorem Let H be a Hilbert space, and $M \subset H$ a closed subspace. Let

$$M^\perp = \{u \in H : \quad (u, v) = 0 \quad \forall v \in M\}.$$

Then $H = M \oplus M^\perp$, i.e. every $w \in H$ can be uniquely written as

$$w = u + v, \quad u \in M, \quad v \in M^\perp.$$

13.8 Definition Let H be a Hilbert space. A *bilinear form* $A : H \times H \to \mathbb{R}$ is called *bounded* if, for some $C > 0$,

$$|A(u, v)| \le C\|u\|\,\|v\| \quad \forall u, v \in H,$$

coercive if, for some $\alpha > 0$,

$$A(u, u) \ge \alpha\|u\|^2 \qquad \forall u \in H,$$

and *symmetric* if

$$A(u, v) = A(v, u) \qquad \forall u, v \in H.$$

13.9 Remark A symmetric bounded coercive bilinear form on H defines an equivalent inner product on H.

13.10 Theorem (Stampacchia) Let K be a closed convex subset of a Hilbert space H, and $A : H \times H \to \mathbb{R}$ bounded coercive bilinear form. Let $\varphi \in H^*$. Then there exists a unique $u \in K$ such that

$$A(u, v - u) \ge \varphi(v - u) \qquad \forall v \in K.$$

Moreover, if A is also symmetric, then u is uniquely determined by

$$\frac{1}{2}A(u, u) - \varphi(u) = \min_{v \in K}\{\frac{1}{2}A(v, v) - \varphi(v)\}.$$

13.11 Corollary (Lax -Hilgram) Under the same conditions there exists a unique $u \in H$ such that

$$A(u,v) = \varphi(v) \qquad \forall v \in H.$$

Moreover, if A is symmetric, then u is uniquely determined by

$$\frac{1}{2}A(u,u) - \varphi(u) = \min_{v \in H}\{\frac{1}{2}A(v,v) - \varphi(v)\}.$$

Proof of Theorem 13.10 Let (Riesz) φ correspond to $f \in H$. Fix $u \in H$. Then the map $v \to A(u,v)$ belongs to H^*. Thus, again by the Riesz Theorem, there exists a unique element in H, denoted by $\hat{A}u$, such that

$$A(u,v) = (\hat{A}u, v).$$

Clearly $\|\hat{A}u\| \leq C\|u\|$, and $(\hat{A}u, u) \geq \alpha\|u\|^2$ for all $u \in H$. We want to find $u \in K$ such that

$$A(u, v-u) = (\hat{A}u, v-u) \geq (f, v-u) \qquad \forall v \in K.$$

For $\rho > 0$ to be fixed later, this is equivalent to

$$(\rho f - \rho\hat{A}u + u - u,\ v - u) \leq 0 \qquad \forall v \in K,$$

i.e.

$$u = P_K(\rho f - \rho\hat{A}u + u).$$

Thus we have to find a fixed point of the map S defined by

$$S:\ u \to P_K(\rho f - \rho\hat{A}u + u),$$

so it suffices to show that S is a strict contraction. We have,

$$\begin{aligned}\|Su_1 - Su_2\| &= \|P_K(\rho f - \rho\hat{A}u_1 + u_1) - P_K(\rho f - \rho\hat{A}u_2 + u_2)\\ &\leq \|(u_1 - u_2) - \rho(\hat{A}u_1 - \hat{A}u_2)\|,\end{aligned}$$

so that

$$\begin{aligned}\|Su_1 - Su_2\|^2 &\leq \|u_1 - u_2\|^2 - 2\rho(\hat{A}u_1 - \hat{A}u_2, u_1 - u_2) + \rho^2\|\hat{A}u_1 - \hat{A}u_2\|^2\\ &\leq \|u_1 - u_2\|^2(1 - 2\rho\alpha + \rho^2 C^2).\end{aligned}$$

Thus for $\rho > 0$ sufficiently small, S is a strict contraction, and has a unique fixed point. This completes the first part of the theorem.

Next, if A is symmetric, then by Remark 13.9 above and Riesz' theorem, there is a unique $g \in H$ such that

$$\varphi(v) = A(g,v) \qquad \forall v \in H.$$

So we must find u such that $A(g-u,v-u) \leq 0 \quad \forall v \in K$, i.e. $u = P_K g$, if we replace the scalar product by $A(\cdot,\cdot)$, or, equivalently, $u \in K$ is the minimizer for

$$\min_{v\in K} A(g-v,g-v)^{\frac{1}{2}} = (\min_{v\in K} A(g,g) + A(v,v) - 2A(g,v))^{\frac{1}{2}}. \blacksquare$$

13.12 Definition Let H be a Hilbert space. Then $T \in \mathcal{B}(H)$ is called *symmetric* if

$$(Tx,y) = (x,Ty) \qquad \forall x,y \in H.$$

13.13 Definition A Hilbert space H is called *separable* if there exists a countable subset $S \subset H$ such that for every $x \in H$ there exists a sequence $(x_n)_{n=1}^{\infty} \subset S$ with $x_n \to x$.

13.14 Theorem Every separable Hilbert space has a *orthonormal Schauderbasis* or *Hilbert basis*, i.e. a countable set $\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$ such that

(i) $(\varphi_i, \varphi_j) = \delta_{ij}$;

(ii) every $x \in H$ can be written uniquely as

$$x = x_1\varphi_1 + x_2\varphi_2 + x_3\varphi_3 + \ldots,$$

where $x_1, x_2, x_3, \ldots \in \mathbb{R}$. Moreover,

$$\|x\|^2 = x_1^2 + x_2^2 + x_3^2 + \ldots \ ,$$

and $x_i = (x, \varphi_i)$.

13.14 Theorem Let H be a Hilbert space, and $T \in \mathcal{K}(H)$ symmetric. Then H has a Hilbert basis $\{\varphi_1, \varphi_2, ...\}$ consisting of eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots \in \mathbb{R}$ with

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \downarrow 0,$$

and such that

$$|\lambda_1| = \sup_{0\neq x\in H} |\frac{(Tx,x)}{(x,x)}| = |\frac{(T\varphi_1,\varphi_1)}{(\varphi_1,\varphi_1)}|;$$

$$|\lambda_2| = \sup_{\substack{0\neq x\in H \\ (x,\varphi_1)=0}} |\frac{(Tx,x)}{(x,x)}| = |\frac{(T\varphi_2,\varphi_2)}{(\varphi_2,\varphi_2)}|;$$

$$|\lambda_3| = \sup_{\substack{0\neq x\in H \\ (x,\varphi)=(x,\varphi_2)=0}} |\frac{(Tx,x)}{(x,x)}| = |\frac{(T\varphi_3,\varphi_3)}{(\varphi_3,\varphi_3)}|,$$

etcetera. Moreover, if $\psi \in H$ satisfies $(\psi, \varphi_1) = (\psi, \varphi_2) = \ldots = (\psi, \varphi_n) = 0$, and $(T\psi, \psi) = \lambda_{n+1}(\psi, \psi)$, then ψ is an eigenvector for λ_{n+1}.

PART 5: POISSON'S EQUATION

14. The weak solution approach in one space dimension

Instead of the Dirichlet problem for Poisson's equation,

$$\begin{cases} -\Delta u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we first consider the one-dimensional version of

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is, for given f in say $C([a,b])$, we look for a function u satisfying

$$(P) \quad \begin{cases} -u'' + u = f & \text{in } (a,b); \\ u(a) = u(b) = 0 \end{cases}$$

Of course we can treat (P) as a linear second order inhomogeneous equation, and construct a solution by means of ordinary differential equation techniques, but that is not the point here. We use (P) to introduce a method that also works in more space dimensions. Let $\psi \in C^1([a,b])$ with $\psi(a) = \psi(b) = 0$ and suppose that $u \in C^2([a,b])$ is a classical solution of (P). Then

$$\int_a^b (-u'' + u)\psi = \int_a^b f\psi,$$

so that integrating by parts, and because $\psi(a) = \psi(b) = 0$,

$$\int_a^b (u'\psi' + u\psi) = \int_a^b f\psi.$$

For the moment we say that $u \in C^1([a,b])$ is a *weak solution* of (P) if

$$\forall \psi \in C^1([a,b]) \text{ with } \psi(a) = \psi(b) = 0 : \quad \int_a^b (u'\psi' + u\psi) = \int_a^b f\psi. \tag{14.1}$$

A *classical solution* is a function $u \in C^2([a,b])$ which satisfies (P).

The program to solve (P) is as follows.

A. Adjust the definition of a weak solution so that we can work with functions on a suitable Hilbert space.

B. Obtain the unique existence of a weak solution u by means of Riesz' Theorem or the Lax-Milgram Theorem.

C. Show that $u \in C^2([a,b])$ and $u(a) = u(b) = 0$, under appropriate conditions on f.

D. Show that a weak solution which is in $C^2([a,b])$ is also a classical solution.

Step D is easy, for if $u \in C^2([a,b])$ with $u(a) = u(b) = 0$ is a weak solution, then

$$\int_a^b (-u'' + u - f)\psi = \int_a^b (u'\psi' + u\psi - f\psi) = 0$$

for all $\psi \in C^1([a,b])$ with $\psi(a) = \psi(b) = 0$, and this implies $-u'' + u = f$ on $[a,b]$, so u is a classical solution of (P).

For step A we introduce the *Sobolev spaces* $W^{1,p}$.

14.1 Definition Let $\emptyset \neq I = (a,b) \subset \mathbb{R}, 1 \leq p \leq \infty$. Recall that $D(I)$ is the set of all smooth functions with compact support in I. Then $W^{1,p}(I)$ consists of all $u \in L^p(I)$ such that the distributional derivative of u can be represented by a function in $v \in L^p(I)$, i.e.

$$\int_I v\psi = -\int_I u\psi' \quad \forall \psi \in D(I).$$

We write $u' = v$.

14.2 Exercise Show that u' is unique.

14.3 Remark For I bounded it is immediate that

$$C^1(\overline{I}) \subset W^{1,p}(I) \;\; \forall p \in [1,\infty].$$

14.4 Exercise Show that $W^{1,p}(I) \not\subset C^1(I)$.

14.5 Definition$H^1(I) = W^{1,2}(I)$.

14.6 Theorem $W^{1,p}(I)$ is a Banach space with respect to the norm

$$\|u\|_{1,p} = |u|_p + |u'|_p,$$

where $|\cdot|_p$ denotes the L^p-norm.

14.7 Theorem $H^1(I)$ is a Hilbert space with respect to the inner product

$$(u,v)_1 = (u,v) + (u',v') = \int_I (uv + u'v').$$

The inner product norm is equivalent to the $W^{1,2}$-norm.

14.8 Theorem $W^{1,p}(I)$ is reflexive for $1 < p < \infty$.

14.9 Theorem $W^{1,p}(I)$ is separable for $1 \le p < \infty$. In particular $H^1(I)$ is separable.

14.10 Theorem For $1 \le p \le \infty$ and $x, y \in I$ we have

$$u(x) - u(y) = \int_y^x u'(s)ds,$$

for every $u \in W^{1,p}(I)$, possibly after redefining u on a set of Lebesque measure zero.

14.11 Remark In particular we have by Hölders inequality $(1/p + 1/q = 1)$,

$$|u(x) - u(y)| = \Big| \int_I \chi_{[x,y]}(s)u'(s)ds \Big| \le |\chi_{[x,y]}|_q |u'|_p \le$$

$$|x-y|^{1/q} \|u\|_{1,p} = \|u\|_{1,p} |x-y|^{\frac{p-1}{p}} \quad \text{if} \quad p > 1.$$

14.12 Definition For $0 < \alpha \le 1$, I bounded, and $f \in C(\overline{I})$, let the Hölder seminorm be defined by

$$[u]_\alpha = \sup_{\substack{x,y \in I \\ x \ne y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

Then

$$C^\alpha(\overline{I}) = \{u \in C(\overline{I}) : [u]_\alpha < \infty\}$$

is called the class of *uniformly Hölder continuous* functions with *exponent* α.

14.13 Theorem $C^\alpha(\overline{I})$ is a Banach space with respect to the norm

$$\|u\| = |u|_\infty + [u]_\alpha,$$

and for $1 < p \leq \infty$, and I bounded, $W^{1,p}(I) \subset C^{1-1/p}(\overline{I})$.

14.14 Corollary For $1 < p \leq \infty$, and I bounded, the injection $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is compact.

14.15 Theorem Let $u \in W^{1,p}(\mathbb{R})$, $1 \leq p < \infty$. Then there exists a sequence $(u_n)_{n=1}^{\infty} \subset D(\mathbb{R})$ with $\|u_n - u\|_{1,p} \to 0$. In other words, $D(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$.

14.16 Corollary Let $u, v \in W^{1,p}(I)$, $1 \leq p \leq \infty$. Then $uv \in W^{1,p}(I)$ and $(uv)' = uv' + u'v$. Moreover, for all $x, y \in I$

$$\int_x^y u'v = [uv]_x^y - \int_x^y uv'.$$

14.17 Definition Let $1 \leq p < \infty$. Then the space $W_0^{1,p}(I)$ is defined as the closure of $D(I)$ in $W^{1,p}(I)$.

14.18 Theorem Let $1 \leq p < \infty$ and I bounded. Then

$$W_0^{1,p}(I) = \{u \in W^{1,p}(I) : u = 0 \quad \text{on} \quad \partial I\}$$

14.19 Remark For $1 \leq p < \infty$, $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$.

14.20 Proposition (Poincaré) Let $1 \leq p < \infty$, and I bounded. Then there exists a $C > 0$, depending on I, such that for all $u \in W_0^{1,p}(I)$:

$$\|u\|_{1,p} \leq C|u'|_p.$$

Proof We have

$$|u(x)| = |u(x) - u(a)| = |\int_a^x u'(s)ds| \leq \int_a^b |u'(s)|ds \leq |u'|_p|1|_q,$$

implying

$$|u|_p \leq (b-a)^{\frac{1}{p}}|1|_q|u'|_p. \blacksquare$$

14.21 Corollary Let $1 \leq p < \infty$ and I bounded. Then $\|u\|_{1,p} = \|u'\|_p$ defines an equivalent norm on $W_0^{1,p}(I)$. Also

$$((u,v)) = \int_I u'v'$$

defines an equivalent inner product on $H_0^1(I) = W_0^{1,2}(I)$. (Two inner products are called equivalent if their inner product norms are equivalent.)

We now focus on the spaces $H_0^1(I)$ and $L^2(I)$ with I bounded. We have established the (compact) embedding $H_0^1(I) \hookrightarrow L^2(I)$. Thus every bounded linear functional on $L^2(I)$ is automatically also a bounded linear functional on $H_0^1(I)$, if we consider $H_0^1(I)$ as being contained in $L^2(I)$, but having a stronger topology. On the other hand, not every bounded functional on $H_0^1(I)$ can be extended to L^2, e.g. if $\psi \in L^2 \backslash H^1$, then $\varphi(f) = \int_I \psi f'$ defines a bounded functional on $H_0^1(I)$ which cannot be extended. This implies that if we want to consider $H_0^1(I)$ as being contained in $L^2(I)$, we *cannot simultaneously apply* Riesz' Theorem to both spaces and identify them with their dual spaces. If we identify $L^2(I)$ and $L^2(I)^*$, we obtain the triplet

$$H_0^1(I) \overset{i}{\hookrightarrow} L^2(I) = L^2(I)^* \overset{i^*}{\hookrightarrow} H_0^1(I)^*.$$

Here i is the natural embedding, and i^* its adjoint. One usually writes

$$H_0^1(I)^* = H^{-1}(I).$$

Then

$$H_0^1(I) \hookrightarrow L^2(I) \hookrightarrow H^{-1}(I). \tag{14.2}$$

The action of H^{-1} on H_0^1 is made precise by

14.22 Theorem Suppose $F \in H^{-1}(I)$. Then there exist $f_0, f_1 \in L^2(I)$ such that

$$F(v) = \int_I f_0 v - \int_I f_1 v' \qquad \forall v \in H_0^1(I).$$

Thus $H^{-1}(I)$ consists of L^2 functions and their first order distributional derivatives. Note however that this characterization depends on the (standard) identification of L^2 and its Hilbert space dual. Also F does not determine f_0 and f_1 uniquely (e.g. $f_0 \equiv 0, f_1 \equiv 1$ gives $F(v) = 0 \quad \forall v \in H_0^1(I)$).

14.23 Remark Still identifying L^2 and L^{2^*} we have for $1 < p < \infty$, writing $W_0^{1,p}(I)^* = W^{-1,p}(I)$,

$$W_0^{1,p}(I) \hookrightarrow L^2(I) \hookrightarrow W^{-1,p}(I), \tag{14.3}$$

and Theorem 14.22 remains true but now with f_0 and f_1 in $L^q(I)$, where $1/p + 1/q = 1$.

We return to

$$(P) \begin{cases} -u'' + u = f & \text{in } I = (a,b); \\ u = 0 \text{ on } \delta I = \{a,b\}. \end{cases}$$

14.24 Definition A *weak solution* of (P) is a function $u \in H_0^1(I)$ such that

$$\int_I (u'v' + uv) = \int_I fv \qquad \forall v \in H_0^1(I). \tag{14.4}$$

Since $D(I)$ is dense in $H_0^1(I)$ it suffices to check this integral identity for all $\psi \in D(I)$. Thus a weak solution is in fact a function $u \in H_0^1(I)$ which satisfies $-u'' + u = f$ *in the sense of distributions*. Note that the boundary condition $u = 0$ on ∂I follows from the fact that $u \in H_0^1(I)$.

14.25 Theorem Let $f \in L^2(I)$. Then (P) has a unique weak solution $u \in H_0^1(I)$, and

$$\frac{1}{2}\int_I (u'^2 + u^2) - \int_I fu = \min_{v \in H_0^1(I)} \left\{\frac{1}{2}\int_I (v'^2 + v^2) - \int_I fv\right\}.$$

Proof The left hand side of (14.4) is the inner product on $H_0^1(I)$. The right hand side defines a bounded linear functional

$$\varphi(v) = \int_I fv$$

on $L^2(I)$, and since $H_0^1(I) \hookrightarrow L^2(I)$, φ is also a bounded linear functional on $H_0^1(I)$. Thus the unique existence of u follows immediately from Riesz' Theorem. It also follows from Lax-Milgram applied with

$$A(u, v) = \int_I (u'v' + uv),$$

and then A being symmetric, the minimum formula is also immediate. ∎

How regular is this solution? We have $u \in H_0^1(I)$, so that $u' \in L^2(I)$ and also $u'' = u - f \in L^2(I)$. Thus

$$u \in H^2(I) = \{u \in H^1(I) : \quad u' \in H^1(I)\}. \tag{14.5}$$

Clearly if $f \in C(\overline{I})$, then $u'' \in C(\overline{I})$.

14.26 Corollary Let $f \in C(\overline{I})$. Then (P) has a unique classical solution $u \in C^2(\overline{I})$.

14.27 Exercise Let $\alpha, \beta \in \mathbb{R}$. Use Stampachia's Theorem applied to $K = \{u \in H^1(I) : u(0) = \alpha,\ u(1) = \beta\}$ with $A(u, v) = ((u, v))$ and $\varphi(v) = \int fv$ to generalize the method above to

$$(P') \quad \begin{cases} -u'' + u = f & \text{in } (0,1); \\ u(0) = \alpha; \quad u(1) = \beta. \end{cases}$$

Next we consider the Sturm-Liouville problem

$$(SL) \quad \begin{cases} -(pu')' + qu = f & \text{in } (0,1); \\ u(0) = u(1) = 0, \end{cases}$$

where $p, q \in C([0,1])$, $p, q > 0$, and $f \in L^2(0,1)$.

14.28 Definition $u \in H_0^1(0,1)$ is a weak solution of (SL) if

$$A(u,v) = \int_0^1 (pu'v' + quv) = \int_0^1 fv \qquad \forall v \in H_0^1(0,1). \tag{14.6}$$

14.29 Exercise Prove that (SL) has a unique weak solution.

Finally we consider the Neumann problem

$$(N) \quad \begin{cases} -u'' + u = f & \text{in } (0,1); \\ u'(0) = u'(1) = 0. \end{cases}$$

14.30 Definition $u \in H^1(0,1)$ is a weak solution of (N) if

$$\int_0^1 (u'v' + uv) = \int_0^1 fv \qquad \forall v \in H^1(0,1).$$

14.31 Exercise Explain the difference between Definitions 14.24 and 14.30, and prove that for $f \in L^2(0,1)$, (N) has a unique weak solution. Show that if $f \in C([0,1])$ there exists a unique classical solution $u \in C^2([0,1])$. (Don't forget the boundary conditions.)

15. Eigenfunctions for the Sturm-Liouville problem

Recall that (SL) was formulated weakly as

$$A(u,v) = \varphi(v) \qquad \forall v \in H_0^1(0,1),$$

where

$$A(u,v) = \int_0^1 pu'v' + quv \quad \text{and} \quad \varphi(v) = \int_0^1 fv.$$

For $p,q \in C([0,1])$, $p,q > 0$, $A(\cdot,\cdot)$ defines an equivalent inner product on $H_0^1(0,1)$, and for $f \in L^2(0,1)$ (in fact $f \in H^{-1}(0,1)$ is sufficient), φ belongs to the dual of $H_0^1(0,1)$.

15.1 Exercise Define $T : L^2(0,1) \to L^2(0,1)$ by $Tf = u$, where u is the (weak) solution of (SL) corresponding to f. Show that T is linear, compact and symmetric.

15.2 Theorem Let $p,q \in C([0,1])$, $p,q > 0$. Then there exists a Hilbert basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2(I)$, such that φ_n is a weak solution of

$$\begin{cases} -(pu')' + qu = \lambda_n u & \text{in } (0,1); \\ u(0) = u(1) = 0, \end{cases}$$

where $(\lambda_n)_{n=1}^{\infty}$ is a nondecreasing unbounded sequence of positive numbers.

15.3 Exercise Prove this theorem (to show that $\lambda_n > 0$ use φ_n as testfunction).

15.4 Remark By (the Krein-Rutman) Theorem 12.23, $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, and φ_1 can be chosen positive. In fact all λ_i are simple. This follows from the theory of ordinary differential equations, see e.g. [CL].

15.5 Exercise Show that $T : H_0^1(0,1) \to H_0^1(0,1)$ is also symmetric with respect to the inner product $A(\cdot,\cdot)$. Derive, using the eigenvalue formulas in Theorem 13.14 for compact symmetric operators, that

$$\lambda_1 = \min_{0 \neq u \in H_0^1(0,1)} \frac{\int_0^1 pu'^2 + qu^2}{\int_0^1 u^2}, \quad \lambda_2 = \min_{(u,\varphi_1)=0} \frac{\int_0^1 pu'^2 + qu^2}{\int u^2},$$

etcetera.

16. Generalization to more dimensions

Throughout this section $\Omega \subset \mathbb{R}^n$ is an open connected set.

16.1 Definition Let $1 \leq p \leq \infty$. Then the *Sobolev space* $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \in L^p(\Omega)\}.$$

16.2 Theorem With respect to the norm

$$\|u\|_{1,p} = |u|_p + \sum_{i=1}^{n} |\frac{\partial u}{\partial x_i}|_p$$

$W^{1,p}(\Omega)$ is a Banach space, which is reflexive for $1 < p < \infty$, and separable for $1 \le p < \infty$.

16.3 Proposition Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \qquad (i = 1, \ldots, n).$$

16.4 Theorem (Sobolev embedding) Let $1 \le p < n$. Define p^* by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Then $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, and the embedding is continuous, i.e.

$$|u|_{p^*} \le C \; \|u\|_{1,p} \quad \forall u \in W^{1,p}(\Omega).$$

To get some feeling for the relation between p and p^*, we consider the scaling $u_\lambda(x) = u(\lambda x)$. This scaling implies that on $\mathbb{R}^n$ an estimate of the form

$$|u|_q \le C(n,p)|\nabla u|_p$$

for all sufficiently smooth functions $u : \mathbb{R}^n \to \mathbb{R}$, implies necessarily that $q = p^*$, and indeed, for this value of q, this (Sobolev) inequality can be proved.

16.5 Theorem Let Ω be bounded with $\partial\Omega \in C^1$. Then

$$p < n \quad \Rightarrow \quad W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in (1, p^*);$$
$$p = n \quad \Rightarrow \quad W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in (1, \infty);$$
$$p > n \quad \Rightarrow \quad W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}),$$

and all three injections are compact.

16.6 Definition Let $1 \le p < \infty$. Then $W_0^{1,p}(\Omega)$ is the closure of $D(\Omega)$ in the $\|\cdot\|_{1,p}$-norm. Here $D(\Omega)$ is the space of all smooth functions with compact support in Ω.

16.7 Theorem Suppose $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $\partial\Omega \in C^1$, where $1 \le p < \infty$. Then

$$u = 0 \text{ on } \partial\Omega \;\text{ if and only if }\; u \in W_0^{1,p}(\Omega).$$

16.8 Theorem (Poincaré inequality) For all bounded $\Omega \subset \mathbb{R}^n$ there exists $C = C(p, \Omega)$, such that for all $1 \le p < \infty$,

$$|u|_p \le C|\nabla u|_p \qquad \forall u \in W_0^{1,p}(\Omega).$$

Thus $|||u|||_p = |\nabla u|_p$ is an equivalent norm on $W_0^{1,p}(\Omega)$.

Next we turn our attention to the Hilbert space case $p = 2$. We write $H^1(\Omega) = W^{1,2}(\Omega)$ and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

16.9 Proposition $H^1(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v) = \int_\Omega uv + \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i},$$

and $H_0^1(\Omega) \subset H^1(\Omega)$ is a closed subspace, and has

$$((u, v)) = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}$$

as an equivalent inner product.

16.10 Corollary For any bounded domain $\Omega \subset \mathbb{R}^n$, the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

From now on we assume that Ω is bounded. Consider for $f \in L^2(\Omega)$ the problem

$$(D) \begin{cases} -\Delta u = f & \text{in } \Omega; \\ u = 0 \quad \text{in} & \partial\Omega. \end{cases}$$

16.11 Definition $u \in H_0^1(\Omega)$ is called a weak solution of (D) if

$$\int_\Omega \nabla u \nabla v = \int_\Omega fv \qquad \forall v \in H_0^1(\Omega).$$

Note that, defining $\varphi \in (H_0^1(\Omega))^*$ by

$$\varphi(v) = \int_\Omega fv,$$

this inequality is equivalent to

$$((u, v)) = \varphi(v) \qquad \forall v \in H_0^1(\Omega).$$

As in the one-dimensional case we have from Riesz's Theorem 13.5 (or Lax-Milgram):

16.12 Theorem Let Ω be a bounded domain, $f \in L^2(\Omega)$. The (D) has a unique weak solution $u \in H_0^1(\Omega)$, and the function $E : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv,$$

attains its minimum in u.

16.13 Theorem (regularity) Suppose $\partial\Omega \in C^\infty$ and $f \in C^\infty(\overline{\Omega})$. Then $u \in C^\infty(\overline{\Omega})$.

Next we consider the operator $T : L^2(\Omega) \to H_0^1(\Omega)$ defined by $Tf = u$. Clearly T is a bounded linear operator. Since we may also consider T as $T : L^2(\Omega) \to L^2(\Omega)$ or $T : H_0^1(\Omega) \to H_0^1(\Omega)$, and since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we have

16.14 Proposition $T : H_0^1(\Omega) \to H_0^1(\Omega)$ is compact, and also $T : L^2(\Omega) \to L^2(\Omega)$ is compact.

16.15 Theorem (i) $T : L^2(\Omega) \to L^2(\Omega)$ is symmetric with respect to the standard inner product in $L^2(\Omega)$; (ii) $T : H_0^1(\Omega) \to H_0^1(\Omega)$ is symmetric with respect to the inner product $((\cdot,\cdot))$.

Proof (i) $(Tf,g) = (g,Tf) = ((Tg,Tf)) = ((Tf,Tg)) = (f,Tg)$. (ii) $((Tf,g)) = (f,g) = (g,f) = ((Tg,f)) = ((f,Tg))$. ∎

16.16 Theorem Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exists a sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 \le \ldots \uparrow \infty,$$

and a Hilbert basis of eigenfunctions

$$\varphi_1, \varphi_2, \varphi_3, \ldots$$

of $L^2(\Omega)$, such that

$$(E_\lambda)\ \begin{cases} -\Delta u = \lambda u & \text{in } \Omega; \\ u = 0 \quad \text{on} & \partial\Omega, \end{cases}$$

has nontrivial weak solutions if and only if $\lambda = \lambda_i$ for some $i \in \mathbf{N}$. Moreover φ_i is a weak solution of (E_{λ_i}), and

$$\lambda_1 = \min_{0 \ne \varphi \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} \qquad \text{(attained in } \varphi = \varphi_1\text{)};$$

$$\lambda_{m+1} = \min_{\substack{0 \ne \varphi \in H_0^1(\Omega) \\ (\varphi,\varphi_1) = \ldots = (\varphi,\varphi_m) = 0}} \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} \qquad \text{(attained in } \varphi = \varphi_{m+1}\text{)}.$$

The function $\varphi_i \in C^\infty(\Omega)$ satisfies the partial differential equation in a classical way, and if $\partial\Omega \in C^\infty$, then also $\varphi_i \in C^\infty(\overline{\Omega})$, for all $i \in \mathbf{N}$. Finally $\varphi_1 > 0$ in Ω.

Proof Applying the spectral decomposition theorem to $T : L^2(\Omega) \to L^2(\Omega)$ we obtain a Hilbert basis $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \ldots\}$ corresponding to eigenvalues (counted with multiplicity) $\mu_1, \mu_2, \mu_3, \mu_4, \ldots$ of $T : L^2(\Omega) \to L^2(\Omega)$, with $|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq \ldots \downarrow 0$. Since φ_i satisfies

$$\int_\Omega \nabla \mu_i \varphi_i \nabla v = \int_\Omega \varphi_i v \qquad \forall v \in H_0^1(\Omega),$$

we have, putting $v = \varphi_i$,

$$\mu_i \int_\Omega |\nabla \varphi_i|^2 = \int_\Omega {\varphi_i}^2,$$

so that clearly $\mu_i > 0$. Thus setting $\lambda_i = 1/\mu_i$, we obtain $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \uparrow \infty$ as desired. The eigenvalue formulas for λ_i follow from the eigenvalue formulas for μ_i applied to $T : H_0^1(\Omega) \to H_0^1(\Omega)$, since

$$\frac{((T\varphi, \varphi))}{((\varphi, \varphi))} = \frac{(\varphi, \varphi)}{((\varphi, \varphi))} = \frac{\int \varphi^2}{\int |\nabla \varphi|^2}.$$

We have completed the proof except for regularity, and the fact that $\lambda_1 < \lambda_2$, and $\varphi_1 > 0$ in Ω. Both these properties follow from the Krein-Rutman theorem applied to the cone C of nonnegative functions. Unfortunately $H_0^1(\Omega)$ is not the space for which T satisfies the conditions of this theorem with C as above, so one has to choose another space, with 'smoother' functions. Here we do not go into the details of this argument, and restrict ourselves to the following observation.

16.17 Proposition Let u be the weak solution of (D). Then

$$f \geq 0 \;\; a.e. \text{ in } \Omega \;\Rightarrow\; u \geq 0 \;\; a.e. \text{ in } \Omega.$$

Proof We use the following fact. Let $u \in H^1(\Omega)$. Define

$$u^+(x) = \begin{cases} u(x) & \text{if } \; u(x) > 0; \\ 0 & \text{if } \; u(x) \leq 0, \end{cases}$$

and

$$u^-(x) = \begin{cases} -\,u(x) & \text{if } \; u(x) < 0; \\ 0 & \text{if } \; u(x) \geq 0. \end{cases}$$

Then $u^+, u^- \in H^1(\Omega)$, and

$$\nabla u^+(x) = \begin{cases} \nabla u(x) & \text{if } \; u(x) > 0; \\ 0 & \text{if } \; u(x) \leq 0, \end{cases}$$

and

$$\nabla u^-(x) = \begin{cases} -\nabla u(x) & \text{if} \quad u(x) < 0 \\ 0 & \text{if} \quad u(x) \geq 0. \end{cases}$$

Now taking u^- as test function we obtain

$$0 \leq \int_\Omega |\nabla u^-|^2 = \int_\Omega \nabla u^- \nabla u^- = -\int_\Omega \nabla u \nabla u^- = -\int f u^- \leq 0,$$

implying $u^- \equiv 0$. ■

We conclude this section with some remarks.

Consider the problem

$$(D_1) \quad \begin{cases} -\Delta u + u = f & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

This problem can be dealt with in the same manner as (D) with $((u,v)) = \int \nabla u \nabla v$ replaced by

$$(u,v) = \int_\Omega (uv + \nabla u \nabla v).$$

Consider the Neumann problem

$$(N_1) \quad \begin{cases} -\Delta u + u = f & \text{in} \quad \Omega; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where ν is the outward normal on $\partial\Omega$. Here the approach is as in $N = 1$.

16.18 Definition $u \in H^1(\Omega)$ is a weak solution of (N_1) if

$$\int_\Omega \nabla u \nabla v + uv = \int fv \quad \forall u \in H^1(\Omega).$$

Unique existence of a weak solution follows as before.

Consider the problem

$$(N) \quad \begin{cases} -\Delta u = f & \text{in} \quad \Omega; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Here we have a complication. By Gauss' theorem, a solution should satisfy

$$0 = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_\Omega \Delta u = -\int_\Omega f,$$

so we must restrict to f with $\int f = 0$. Also, if u is a solution, then so is $u + C$ for any constant C. Therefore we introduce the spaces

$$\tilde{L}^2(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f = 0\} \quad \text{and} \quad \tilde{H}^1(\Omega) = \{u \in H^1(\Omega) : \int_\Omega u = 0\}.$$

16.19 Definition Let $f \in \tilde{L}^2(\Omega)$. Then $u \in \tilde{H}^1(\Omega)$ is called a weak solution if

$$\int_\Omega \nabla u \nabla v = \int f v \qquad \forall v \in \tilde{H}^1(\Omega).$$

Observe that if this relation holds for all $v \in \tilde{H}^1(\Omega)$, it also holds for all $v \in H^1(\Omega)$.

16.20 Proposition The brackets $((\cdot, \cdot))$ also define an inner product on $\tilde{H}^1(\Omega)$, and $((\cdot, \cdot))$ is equivalent (on $\tilde{H}^1(\Omega)$) to the standard inner product.

16.21 Corollary For all $f \in \tilde{L}^2(\Omega)$, there exists a unique weak solution in $\tilde{H}^1(\Omega)$ of (N).

In the problems above, as well as in the methods, we can replace $-\Delta$ by any linear second order operator in divergence form

$$-\mathrm{div}(A\,\nabla) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}),$$

where

$$A(x) = (a_{ij}(x))_{i,j=1,\ldots,n}$$

is a symmetric x-dependent matrix with eigenvalues

$$0 < \delta < \lambda_1(x) \le \lambda_2(x) \le \ldots \le \lambda_n(x) < M < \infty,$$

for all $x \in \Omega$, and $a_{ij} \in C(\overline{\Omega})$, $i.j = 1, \ldots, n$. In all the statements and proofs $\int \nabla u \nabla v$ then has to be replaced by

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

17. Harmonic functions

Throughout this section, $\Omega \subset \mathbb{R}^n$ is a bounded domain.

17.1 Definition A function $u \in C^2(\Omega)$ is called *subharmonic* if $\Delta u \geq 0$ in Ω, *harmonic* if $\Delta u \equiv 0$ in Ω, and *superharmonic* if $\Delta u \leq 0$ in Ω.

17.2 Notation The measure of the unit ball in $\mathbb{R}^n$ is

$$\omega_n = |B_1| = |\{x \in \mathbb{R}^n : x_1^2 + ... + x_n^2 \leq 1\}| = \int_{B_1} dx = \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

The $(n-1)$-dimensional measure of the boundary ∂B_1 of B_1 is equal to $n\omega_n$.

17.3 Mean Value Theorem Let $u \in C^2(\Omega)$ be subharmonic, and

$$\overline{B_R(y)} = \{x \in \mathbb{R}^n : |x-y| \leq R\} \subset \Omega.$$

Then

$$u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x) dS(x),$$

where dS is the $(n-1)$-dimensional surface element on $\partial B_R(y)$. Also

$$u(y) \leq \frac{1}{\omega_n R^n} \int_{B_R(y)} u(x) dx.$$

Equalities hold if u is harmonic.

Proof We may assume $y = 0$. Let $\rho \in (0, R)$. Then

$$0 \leq \int_{B_\rho} \Delta u(x) dx = \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}(x) dS(x) = \int_{\partial B_\rho} \frac{\partial u}{\partial r}(x) dS(x) =$$

(substituting $x = \rho\omega$)

$$\int_{\partial B_1} \frac{\partial u}{\partial r}(\rho\omega)\rho^{n-1} dS(\omega) = \rho^{n-1} \int_{\partial B_1} (\frac{\partial}{\partial \rho} u(\rho\omega)) dS(\omega) = \rho^{n-1} \frac{d}{d\rho} \int_{\partial B_1} u(\rho\omega) dS(\omega)$$

(substituting $\omega = x/\rho$)

$$= \rho^{n-1} \frac{d}{d\rho} \frac{1}{\rho^{n-1}} \int_{\partial B_\rho} u(x) dS(x),$$

which implies, writing

$$f(\rho) = \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho} u(x) dS(x),$$

that $f'(\rho) \geq 0$. Hence

$$u(0) = \lim_{\rho \downarrow 0} f(\rho) \leq f(R) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u(x) dS(x),$$

which proves the first inequality. The second one follows from

$$\int_{B_R} u(x) dx = \int_0^R \{ \int_{\partial B_\rho} u(x) dS(x) \} d\rho \geq \int_0^R n\omega_n \rho^{n-1} u(0) d\rho = \omega_n R^n u(0).$$

This completes the proof. ∎

17.4 Corollary (Strong maximum principle for subharmonic functions) Let $u \in C^2(\Omega)$ be bounded and subharmonic. If for some $y \in \Omega$, $u(y) = \sup_\Omega u$, then $u \equiv u(y)$.

Proof Exercise (hint: apply the mean value theorem to the function $\tilde{u}(x) = u(x) - u(y)$, and show that the set $\{x \in \Omega : \tilde{u}(x) = 0\}$ is open). ∎

17.5 Corollary (weak maximum principle for subharmonic functions) Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is subharmonic. Then

$$\sup_\Omega u = \max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof Exercise. ∎

17.6 Corollary Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be subharmonic. If $u \equiv 0$ on $\partial\Omega$, then $u < 0$ on Ω, unless $u \equiv 0$ on $\overline{\Omega}$.

Proof Exercise. ∎

17.7 Corollary Let $\varphi \in C(\partial\Omega)$. Then there exists at most one function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $u = \varphi$ on $\partial\Omega$.

Proof Exercise. ∎

17.8 Theorem (Harnack inequality) Let $\Omega' \subset\subset \Omega$ (i.e. $\Omega' \subset \overline{\Omega'} \subset \Omega$) be a subdomain. Then there exists a constant C which only depends on Ω' and Ω, such that for all harmonic nonnegative functions $u \in C^2(\Omega)$,

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

Proof Suppose that $\overline{B_{4R}(y)} \subset \Omega$. Then for any $x_1, x_2 \in B_R(y)$ we have

$$B_R(x_1) \subset B_{3R}(x_2) \subset B_{4R}(y) \subset \Omega,$$

so that by the mean value theorem,

$$u(x_1) = \frac{1}{\omega_n R^n} \int_{B_R(x_1)} u(x)dx \le \frac{3^n}{\omega_n (3R)^n} \int_{B_{3R}(x_2)} u(x)dx = 3^n u(x_2).$$

Hence, x_1, x_2 being arbitrary, we conclude that

$$\sup_{B_R(y)} u \le 3^n \inf_{B_R(y)} u.$$

Thus we have shown that for $\Omega' = B_R(y)$, with $B_{4R}(y) \subset \Omega$, the constant in the inequality can be taken to be 3^n. Since any $\Omega' \subset\subset \Omega$ can be covered with finitely many of such balls, say

$$\Omega' \subset B_{R_1}(y_1) \cup B_{R_2}(y_2) \cup \ldots \cup B_{R_N}(y_N),$$

we obtain for Ω' that $C = 3^{nN}$. ∎

Next we turn our attention to radially symmetric harmonic functions. Let $u(x)$ be a function of $r = |x|$ alone, i.e. $u(x) = U(r)$. Then u is harmonic if and only if

$$0 = \Delta u(x) = \sum_{i=1}^{n} (\frac{\partial}{\partial x_i})^2 u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (U'(r) \frac{\partial r}{\partial x_i}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\frac{x_i}{r} U'(r))$$

$$= \sum_{i=i}^{n} (\frac{1}{r} U'(r) + \frac{x_i}{r} U''(r) \frac{\partial r}{\partial x_i} - x_i U'(r) \frac{\partial}{\partial x_i} (\frac{1}{r}))$$

$$= \frac{n}{r} U'(r) + \sum_{i=1}^{n} \frac{x_i^2}{r} U''(r) - \sum_{i=1}^{n} x_i U'(r) \frac{x_i}{r^3} =$$

$$U''(r) + \frac{n-1}{r} U'(r) = \frac{1}{r^{n-1}} (r^{n-1} U'(r))',$$

implying

$$r^{n-1} U'(r) = C_1,$$

so that

$$U(r) = \begin{cases} C_1 r + C_2 & n = 1; \\ C_1 \log r + C_2 & n = 2; \\ \dfrac{C_1}{2-n} \dfrac{1}{r^{n-2}} + C_2 & n > 2. \end{cases} \tag{17.1}$$

We define the *fundamental solution* by

$$\Gamma(x) = \begin{cases} \dfrac{1}{2} |x| & n = 1 \\ \dfrac{1}{2\pi} \log |x| & n = 2 \\ \dfrac{1}{n\omega_n (2-n)} \dfrac{1}{|x|^{n-2}} & n > 2, \end{cases} \tag{17.2}$$

i.e. $C_1 = 1/n\omega_n$ and $C_2 = 0$ in (17.1). Whenever convenient we write $\Gamma(x) = \Gamma(|x|) = \Gamma(r)$.

17.9 Theorem The fundamental solution Γ is a solution of the equation $\Delta\Gamma = \delta$ in the sense of distributions, i.e.

$$\int_{\mathbb{R}^n} \Gamma(x)\Delta\psi(x)dx = \psi(0) \qquad \forall\psi \in D(\mathbb{R}^n).$$

Proof First observe that for all $R > 0$, we have $\Gamma \in L^\infty(B_R)$ if $n = 1$, $\Gamma \in L^P(B_R)$ for all $1 \le p < \infty$ if $n = 2$, and $\Gamma \in L^P(B_R)$ for all $1 \le p < \frac{n}{n-2}$ if $n > 2$, so for all ψ in $D(\mathbb{R}^n)$, choosing R large enough, we can compute

$$\int_{\mathbb{R}^n} \Gamma(x)\Delta\psi(x)dx = \int_{B_R} \Gamma(x)\Delta\psi(x)dx = \lim_{\rho\downarrow 0}\int_{A_{R,\rho}} \Gamma(x)\Delta\psi(x)dx =$$

(here $A_{R,\rho} = \{x \in B_R : |x| > \rho\}$)

$$\lim_{\rho\downarrow 0}\{\int_{\partial A_{R,\rho}} \Gamma\frac{\partial\psi}{\partial\nu} - \int_{A_{R,\rho}} \nabla\Gamma\nabla\psi\} = \lim_{\rho\downarrow 0}\{\int_{\partial A_{R,\rho}} (\Gamma\frac{\partial\psi}{\partial\nu} - \frac{\partial\Gamma}{\partial\nu}\psi) + \int_{A_{R,\rho}} \psi\Delta\Gamma\} =$$

$$\lim_{\rho\downarrow 0}\int_{\partial B_\rho}\{\frac{-\partial\psi/\partial\nu}{n\omega_n(2-n)\rho^{n-2}} + \frac{\psi}{n\omega_n\rho^{n-1}}\} = \psi(0).$$

For $n = 1, 2$ the proof is similar. ∎

Closely related to this theorem we have

17.10 Theorem (Green's representation formula) Let $u \in C^2(\overline{\Omega})$ and suppose $\partial\Omega \in C^1$. Then, if ν is the outward normal on $\partial\Omega$, we have

$$u(y) = \int_{\partial\Omega}\{u(x)\frac{\partial}{\partial\nu}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial\nu}(x)\}dS(x) + \int_\Omega \Gamma(x-y)\Delta u(x)dx.$$

Here the derivatives are taken with respect to the x-variable.

Proof Exercise (Hint: take $y = 0$, let $\Omega_\rho = \{x \in \Omega : |x| > \rho\}$, and imitate the previous proof). ∎

If we want to solve $\Delta u = f$ on Ω for a given function f, this representation formula strongly suggests to consider the convolution

$$\int_\Omega \Gamma(x-y)f(x)dx$$

as a function of y, or equivalently,

$$(\Gamma * f)(x) = \int_\Omega \Gamma(x-y)f(y)dy \tag{17.3}$$

as a function of x. This convolution is called the *Newtonpotential* of f.

For any harmonic function $h \in C^2(\overline{\Omega})$ we have

$$\int_\Omega h\Delta u = \int_{\partial\Omega} (h\frac{\partial u}{\partial \nu} - u\frac{\partial h}{\partial \nu}),$$

so that, combining with Green's representation formula,

$$u(y) = \int_{\partial\Omega} \{u\frac{\partial G}{\partial \nu} - G\frac{\partial u}{\partial \nu}\} + \int_\Omega G\Delta u, \tag{17.4}$$

where $G = \Gamma(x-y) + h(x)$. The trick is now to take instead of a function $h(x)$ a function $h(x,y)$ of two variables $x, y \in \overline{\Omega}$, such that h is harmonic in x, and for every $y \in \Omega$,

$$G(x,y) = \Gamma(x-y) + h(x,y) = 0 \qquad \forall x \in \partial\Omega.$$

This will then give us the solution formula

$$u(y) = \int_{\partial\Omega} u\frac{\partial G}{\partial \nu} + \int_\Omega G\Delta u.$$

In particular, if $u \in C^2(\overline{\Omega})$ is a solution of

$$(D) \quad \begin{cases} \Delta u = f & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

then

$$u(y) = \int_{\partial\Omega} \varphi(x)\frac{\partial G(x,y)}{\partial \nu} + \int_\Omega G(x,y)f(x)dx. \tag{17.5}$$

The function $G(x,y) = \Gamma(x-y) + h(x,y)$ is called the *Green's function* for the Dirichletproblem. Of course $h(x,y)$ is by no means trivial to find. The function h is called the *regular part* of the Green's function. If we want to solve

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

(17.5) reduces to

$$u(y) = \int_{\partial\Omega} \varphi(x)\frac{\partial G(x,y)}{\partial \nu} dS(x). \tag{17.6}$$

We shall evaluate (17.6) in the case that $\Omega = B = B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$. Define the *reflection* in ∂B by

$$S(y) = \frac{y}{|y|^2} \quad \text{if} \quad y \neq 0; \quad S(0) = \infty; \quad S(\infty) = 0. \tag{17.7}$$

Here ∞ is the point that has to be added to $\mathbb{R}^n$ in order to construct the one-point compactification of $\mathbb{R}^n$. If $0 \neq y \in B$, then $\overline{y} = S(y)$ is uniquely determined by asking that 0, y and $\overline{y}$ lie (in that order) on one line l, and that the boundary ∂B of B is tangent to the cone C with top $\overline{y}$ spanned by the circle obtained from intersecting the ball B with the plane perpundicular to l going through y (you'd better draw a picture here). Indeed if $\overline{x}$ lies on this circle, then the triangles $0y\overline{x}$ and $0\overline{xy}$ are congruent and

$$|y| = \frac{|y-0|}{|\overline{x}-0|} = \frac{|\overline{x}-0|}{|\overline{y}-0|} = \frac{1}{|\overline{y}|},$$

so that $\overline{y} = S(y)$. It is also easily checked that

$$\partial B = \{x \in \mathbb{R}^n;\ |x-y| = |y||x-\overline{y}|\}.$$

But then the construction of $h(x,y)$ is obvious. We simply take

$$h(x,y) = -\Gamma(|y|(x-\overline{y})),$$

so that

$$G(x,y) = \Gamma(x-y) - \Gamma(|y|(x-\overline{y})). \tag{17.8}$$

Note that since $|y||\overline{y}| = 1$, and since $y \to 0$ implies $\overline{y} \to \infty$, we have, with a slight abuse of notation, that $G(x,0) = \Gamma(x) - \Gamma(1)$. It is convenient to rewrite $G(x,y)$ as

$$\begin{aligned} G(x,y) &= \Gamma(\sqrt{|x|^2+|y|^2-2xy}) - \Gamma(\sqrt{|x|^2|y|^2+|y|^2|\overline{y}|^2-2|y|^2x\overline{y}}) \\ &= \Gamma(\sqrt{|x|^2+|y|^2-2xy}) - \Gamma(\sqrt{|x|^2|y|^2+1-2xy}), \end{aligned}$$

which shows that G is symmetric in x and y. In particular G is also harmonic in the y variables.

Next we compute $\partial G/\partial \nu$ on ∂B. We write

$$r = |x-y|;\ \overline{r} = |x-\overline{y}|;\ \frac{\partial}{\partial \nu} = \nu \cdot \nabla = \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i};\ \frac{\partial r}{\partial x_i} = \frac{x_i - y_i}{r};\ \frac{\partial \overline{r}}{\partial x_i} = \frac{x_i - \overline{y}_i}{\overline{r}},$$

so that since $G = \Gamma(r) - \Gamma(|y|\overline{r})$,

$$\frac{\partial \Gamma(r)}{\partial \nu} = \Gamma'(r)\frac{\partial r}{\partial \nu} = \frac{1}{n\omega_n r^{n-1}} \sum_{i=1}^n x_i \frac{x_i - y_i}{r} = \frac{1 - xy}{n\omega_n r^n},$$

and

$$\frac{\partial\Gamma(|y|\overline{r}|)}{\partial\nu} = \Gamma'(|y|\overline{r})|y|\frac{\partial\overline{r}}{\partial\nu} = \frac{1}{n\omega_n|y|^{n-2}\overline{r}^{n-1}}\frac{\partial\overline{r}}{\partial\nu} = \frac{1}{n\omega_n|y|^{n-2}\overline{r}^{n-1}}\sum_{i=1}^{n} x_i\frac{x_i-\overline{y}_i}{\overline{r}}$$

$$= \frac{1}{n\omega_n}|y|^{2-n}\overline{r}^{-n}\sum_{i=1}^{n} x_i(x_i-\overline{y}_i) = \quad (\text{substituting} \quad r = |y|\overline{r})$$

$$\frac{1}{n\omega_n}|y|^{2-n}(\frac{|y|}{r})^n\sum_{i=1}^{n}(x_i^2-x_i\overline{y}_i) = \frac{1}{n\omega_n r^n}\{|y|^2-xy\},$$

whence

$$\frac{\partial G(x,y)}{\partial\nu(x)} = \frac{1}{n\omega_n r^n}(1-|y|^2) = \frac{1-|y|^2}{n\omega_n|x-y|^n}. \tag{17.9}$$

17.11 Theorem (Poisson integration formula) Let $\varphi \in C(\partial B)$. Define $u(y)$ for $y \in B$ by

$$u(y) = \frac{1-|y|^2}{n\omega_n}\int_{\partial B}\frac{\varphi(x)}{|x-y|^n}dS(x),$$

and for $y \in \partial B$, by $u(y) = \varphi(y)$. Then $u \in C^2(B) \cup C(\overline{B})$, and $\Delta u = 0$ in B.

Proof First we show that $u \in C^\infty(B)$ and that $\Delta u = 0$ in B. We have

$$u(y) = \frac{1-|y|^2}{n\omega_n}\int_{\partial B}\frac{\varphi(x)}{|x-y|^n}dS(x) = \int_{\partial B}K(x,y)\varphi(x)dS(x),$$

where the integrand is smooth in $y \in B$, and $K(x,y)$ is positive, and can be written as

$$K(x,y) = \frac{\partial G(x,y)}{\partial\nu(x)} = \sum_{i=1}^{n} x_i\frac{\partial G(x,y)}{\partial x_i}.$$

Thus $u \in C^\infty(B)$ and

$$\Delta u(y) = \sum_{j=1}^{n}\frac{\partial^2 u}{\partial y_j} = \sum_{j=1}^{n}(\frac{\partial}{\partial y_j})^2\int_{\partial B}\sum_{i=1}^{n} x_i\frac{\partial G(x,y)}{\partial x_i}\varphi(x)dS(x)$$

$$= \int_{\partial B}\{\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}\sum_{j=1}^{n}\frac{\partial^2 G(x,y)}{\partial y_j^2}\}\varphi(x)dS(x) = 0.$$

Next we show that $u \in C(\overline{B})$. Observe that

$$\int_{\partial B}K(x,y)dS(x) = 1,$$

because $\tilde{u} \equiv 1$ is the unique harmonic function with $\tilde{u} \equiv 1$ on the boundary. We have to show that for all $x_0 \in \delta B$

$$\lim_{\substack{y \to x_0 \\ y \in B}} u(y) = \varphi(x_0) = u(x_0),$$

so we look at

$$u(y) - u(x_0) = \int_{\partial B} K(x,y)(\varphi(x) - \varphi(x_0))dS(x).$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(x_0)| < \varepsilon \quad \text{for all } x \in \delta B \text{ with } \; |x - x_0| < \delta.$$

Thus we have, with $M = \max_{\partial B} |\varphi|$, that

$$|u(y) - u(x_0)| \leq \int_{x \in \partial B,\ |x - x_0| < \delta} K(x,y)|\varphi(x) - \varphi(x_0)|dS(x)$$

$$+ \int_{x \in \partial B,\ |x - x_0| \geq \delta} K(x,y)|\varphi(x) - \varphi(x_0)|dS(x) \leq$$

$$\int_{\partial B} K(x,y)\varepsilon dS(x) + \int_{x \in \partial B,\ |x - x_0| \geq \delta} K(x,y)2M dS(x) =$$

$$\varepsilon + 2M \int_{x \in \partial B,\ |x - x_0| \geq \delta} K(x,y)dS(x) \leq \quad (\text{choosing } \; y \in B \text{ with } \; |y - x_0| < \frac{\delta}{2}\ )$$

$$\varepsilon + 2M \int_{x \in \partial B,\ |x - y| \geq \frac{\delta}{2}} \frac{1 - |y|^2}{n\omega_n |x - y|^n} dS(x) \leq \varepsilon + 2M \frac{1 - |y|^2}{n\omega_n} \int_{\partial B} \left(\frac{2}{\delta}\right)^n dS(x) =$$

$$\varepsilon + 2M \left(\frac{2}{\delta}\right)^n (1 - |y|^2) \to \varepsilon \quad \text{as} \quad y \to x_0.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. ∎

17.12 Remark On the ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ the Poisson formula reads

$$u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{\varphi(x)}{|x - y|^n} dS(x).$$

17.13 Corollary A function $u \in C(\Omega)$ is harmonic if and only if

$$u(y) = \frac{1}{\omega_n R^n} \int_{B_R(y)} u(x)dx$$

for all $B_R(y) \subset\subset \Omega$.

Proof Exercise (hint: use Poisson's formula in combination with the weak maximum principle which was proved using the mean value (in-)equalities). ■

17.14 Corollary Uniform limits of harmonic functions are harmonic.

Proof Exercise. ■

17.15 Corollary (Harnack convergence theorem) For a nondecreasing sequence of harmonic functions $u_n : \Omega \to \mathbb{R}$ to converge to a harmonic limit function u, uniformly on compact subsets, it is sufficient that the sequence $(u_n(y))_{n=1}^{\infty}$ is bounded for just one point $y \in \Omega$.

Proof Exercise (hint: use Harnack's inequality to establish convergence). ■

17.16 Corollary If $u : \Omega \to \mathbb{R}$ is harmonic, and $\Omega' \subset\subset \Omega$, $d =$ distance $(\Omega', \partial\Omega)$, then

$$\sup_{\Omega'} |\nabla u| \leq \frac{n}{d} \sup_{\Omega} |u|.$$

Proof Since $\Delta\nabla u = \nabla\Delta u = 0$, we have by the mean value theorem for $y \in \Omega'$

$$\big|\nabla u(y)\big| = \big|\frac{1}{\omega_n d^n} \int_{B_d(y)} \nabla u(x) dx\big| =$$

(by the vector valued version of Gauss' Theorem)

$$\big|\frac{1}{\omega_n d^n} \int_{\partial B_d(y)} u(x)\nu(x) dS(x)\big| \leq \frac{1}{\omega_n d^n} n\omega_n d^{n-1} \sup_{B_d(y)} |u(x)||\nu(x)| = \frac{n}{d} \sup_{B_d(y)} |u(x)|,$$

since ν is the unit normal. ■

17.17 Corollary (Liouville) If $u : \mathbb{R}^n \to \mathbb{R}^+$ is harmonic, then $u \equiv$ constant.

Proof We have

$$\big|\nabla u(y)\big| = \big|\frac{1}{w_n R^n} \int_{\partial B_R(y)} u(x)\nu(x) dS(x)\big| \leq \frac{1}{w_n R^n} \int_{\partial B_R(y)} |u(x)\nu(x)| dS(x)$$

$$= \frac{n}{R} \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x) dS(x) = \frac{n}{R} u(y)$$

for all $R > 0$. Thus $\nabla u(y) = 0$ for all $y \in \mathbb{R}^n$, so that $u \equiv$ constant. ■

We have generalized a number of properties of harmonic functions on domains in $\mathbb{R}^2$, which follow from the following theorem for harmonic functions of two real variables.

17.18 Theorem Let $\Omega \subset \mathbb{R}^2$ be simply connnected. Suppose $u \in C(\Omega)$ is harmonic. Then there exists $v : \Omega \to \mathbb{R}$ such that

$$F(x+iy) = u(x,y) + iv(x,y)$$

is an analytic function on Ω. In particular $u, v \in C^\infty(\Omega)$ and $\Delta u = \Delta v = 0$ in Ω.

17.19 Exercise Let $u : \Omega \to \mathbb{R}$ be harmonic. Show that the function $v = |\nabla u|^2$ is subharmonic in Ω.

18. Perron's method

18.1 Theorem Let Ω be bounded and suppose that the exterior ball condition is satisfied at every point of $\partial\Omega$, i.e. for every point $x_0 \in \partial\Omega$ there exists a ball B such that $\overline{B} \cap \overline{\Omega} = \{x_0\}$. Then there exists for every $\varphi \in C(\partial\Omega)$ exactly one harmonic function $u \in C(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$.

For the proof of Theorem 18.1 we need to extend the definition of sub- and superharmonic to continuous functions.

18.2 Definition A function $u \in C(\Omega)$ is called *subharmonic* if $u \leq h$ on B for every ball $B \subset\subset \Omega$ and every $h \in C(\overline{B})$ harmonic with $u \leq h$ on ∂B. The definition of superharmonic is likewise.

Clearly this is an extension of Definition 17.1, that is, every $u \in C^2(\Omega)$ with $\Delta u \geq 0$ is subharmonic in the sense of Definition 18.2. See also the exercises at the end of this section.

18.3 Theorem Suppose $u \in C(\overline{\Omega})$ is subharmonic, and $v \in C(\overline{\Omega})$ is superharmonic. If $u \leq v$ on $\partial\Omega$, then $u < v$ on Ω, unless $u \equiv v$.

Proof First we prove that $u \leq v$ in Ω. If not, then the function $u - v$ must have a maximum $M > 0$ achieved in some interior point x_0 in Ω. Since $u \leq v$ on $\partial\Omega$ and $M > 0$, we can choose a ball $B \subset\subset \Omega$ centered in x_0, such that $u - v$ is not identical to M on ∂B. Because of the Poisson Integral Formula, there exist harmonic functions $\overline{u}, \overline{v} \in C(\overline{B})$ with $\overline{u} = u$ and $\overline{v} = v$ on ∂B. By definition, $\overline{u} \geq u$ and $\overline{v} \leq v$. Hence $\overline{u}(x_0) - \overline{v}(x_0) \geq M$, while on ∂B we have $\overline{u} - \overline{v} = u - v \leq M$. Because $\overline{u}$ and $\overline{v}$ are harmonic it follows that $\overline{u} - \overline{v} \equiv M$ on B, and therefore the same holds for $u - v$ on ∂B, a contradiction.

Next we show that also $u < v$ on Ω, unless $u \equiv v$. If not, then the function $u - v$ must have a zero maximum achieved in some interior point x_0 in Ω, and, unless $u \equiv v$, we can choose x_0 and B exactly as above, reading zero for M. Again this gives a contradiction. ∎

Using again the Poisson Integral Formula we now introduce

18.4 Definition Let $u \in C(\Omega)$ be subharmonic, and let $B \subset\subset \Omega$ be a ball. The unique function $U \in C(\Omega)$ defined by

(i) $U = u$ for $\Omega \backslash B$;

(ii) U is harmonic on B,

is called the *harmonic lifting* of u in B.

18.5 Proposition The harmonic lifting U on B in Definition 18.4 is also subharmonic in Ω.

Proof Let $B' \subset\subset \Omega$ be an arbitrary closed ball, and suppose that $h \in C(\overline{B}')$ is harmonic in B', and $U \leq h$ on $\partial B'$. We have to show that also $U \leq h$ on B'. First observe that since u is subharmonic $U \geq u$ so that certainly $u \leq h$ on $\partial B'$, and hence $u \leq h$ on B'. Thus $U \leq h$ on $B' \backslash B$, and also on the boundary $\partial \Omega'$ of $\Omega' = B' \cap B$. But both U and h are harmonic in $\Omega' = B' \cap B$, so by the maximum principle for harmonic functions, $U \leq h$ on $\Omega' = B' \cap B$, and hence on the whole of B'. ∎

18.6 Proposition If $u_1, u_2 \in C(\Omega)$ are subharmonic, then $u = \max(u_1, u_2) \in C(\Omega)$ is also subharmonic.

Proof Exercise. ∎

18.7 Definition A function $u \in C(\overline{\Omega})$ is called a *subsolution* for $\varphi : \partial\Omega \to \mathbb{R}$ if u is subharmonic in Ω and $u \leq \varphi$ in $\partial\Omega$. The definition of a supersolution is likewise.

18.8 Theorem For $\varphi : \partial\Omega \to \mathbb{R}$ bounded let S_φ be the collection of all subsolutions, and let

$$u(x) = \sup_{v \in S_\varphi} v(x), \quad x \in \Omega.$$

Then $u \in C(\Omega)$ is harmonic in Ω.

Proof Every subsolution is smaller then or equal to every supersolution. Since $\sup_{\partial\Omega} \varphi$ is a supersolution, it follows that u is well defined. Now fix $y \in \Omega$ and choose a sequence of functions $v_1, v_2, v_3, \ldots \in S_\varphi$ such that $v_n(y) \to u(y)$ as $n \to$

∞. Because of Proposition 18.6 we may take this sequence to be nondecreasing in $C(\Omega)$, and larger then or equal to $\inf_{\partial\Omega}\varphi$. Let $B \subset\subset \Omega$ be a ball with center y, and let V_n be the harmonic lifting of v_n on B. Then $v_n \le V_n \le u$ in Ω, and V_n is also nondecreasing in $C(\Omega)$. By the Harnack Convergence Theorem, the sequence V_n converges on every ball $B' \subset\subset B$ uniformly to a harmonic function $v \in C(B)$. Clearly $v(y) = u(y)$ and $v \le u$ in B. The proof will be complete if we show that $v \equiv u$ on B for then it follows that u is harmonic in a neighbourhood of every point y in Ω. So suppose $v \not\equiv u$ on B. Then there exists $z \in B$ such that $u(z) > v(z)$, and hence we can find $\overline{u} \in S_\varphi$ such that $v(z) < \overline{u}(z) \le u(z)$. Define $w_n = \max(v_n, \overline{u})$ and let W_n be the harmonic lifting of w_n on B. Again it follows that the sequence W_n converges on every ball $B' \subset\subset B$ uniformly to a harmonic function $w \in C(B)$, and clearly $v \le w \le u$ in B, so $v(y) = w(y) = u(y)$. But v and w are both harmonic, so by the strong maximum principle for harmonic functions they have to coincide. However, the construction above implies that $v(z) < \overline{u}(z) \le w(z)$, a contradiction. ∎

Next we look at the behaviour of the harmonic function u in Theorem 18.8 near the boundary.

18.9 Definition Let $x_0 \in \partial\Omega$. A function $w \in C(\overline{\Omega})$ with $w(x_0) = 0$ is called a barrier function in x_0 if w is superharmonic in Ω and $w > 0$ in $\overline{\Omega}\backslash\{x_0\}$.

18.10 Proposition Let u be as in Theorem 18.8, and let $x_0 \in \partial\Omega$, and suppose there exists a barrier function w in x_0. If φ is continuous in x_0, then $u(x) \to \varphi(x_0)$ if $x \to x_0$.

Proof The idea is to find a sub- and a supersolution of the form $u^\pm = \varphi(x_0) \pm \epsilon \pm kw(x)$. Fix $\epsilon > 0$ and let $M = \sup_{\partial\Omega}|\varphi|$. We first choose $\delta > 0$ such that $|\varphi(x) - \varphi(x_0)| < \epsilon$ for all $x \in \partial\Omega$ with $|x - x_0| < \delta$, and then $k > 0$ such that $kw > 2M$ on $\overline{\Omega}\backslash B_\delta(x_0)$. Clearly then u^- is a sub- and u^+ is a supersolution, so that $\varphi(x_0) - \epsilon - kw(x) \le u(x) \le \varphi(x_0) + \epsilon + kw(x)$ for all $x \in \Omega$. Since $\epsilon > 0$ was arbitrary, this completes the proof. ∎

18.11 Exercise Finish the proof of Theorem 18.1, and prove that the map

$$\varphi \in C(\partial\Omega) \to u \in C(\overline{\Omega})$$

is continuous with respect to the supremum norms in $C(\partial\Omega)$ and $C(\overline{\Omega})$.

18.12 Exercise Show that for a function $u \in C(\Omega)$ the following three statements are equivalent:

(i) u is subharmonic in the sense of Definition 18.2;

(ii) for every nonnegative compactly supported function $\phi \in C^2(\Omega)$ the inequality

$$\int_\Omega u\Delta\phi \geq 0$$

holds;

(iii) u satisfies the conclusion of the Mean Value Theorem, i.e. for every $B_R(y) \subset\subset \Omega$ the inequality

$$u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x)dS(x)$$

holds.

Hint: In order to deal with (ii) show that it is equivalent to the existence of a sequence $(\Omega_n)_{n=1}^\infty$ of strictly increasing domains, and a corresponding sequence of subharmonic functions $(u_n)_{n=1}^\infty \in C^\infty(\Omega_n)$, with the property that for every compact $K \subset \Omega$ there exists an integer N such that $K \subset \Omega_N$, and moreover, the sequence $(u_n)_{n=N}^\infty$ converges uniformly to u on K.

Finally we formulate an optimal version of Theorem 18.1.

18.13 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Then there exists for every $\varphi \in C(\partial\Omega)$ exactly one harmonic function $u \in C(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$.

19. Potential theory

We recall that the fundamental solution of Laplace's equation is given by

$$\Gamma(x) = \Gamma(|x|) = \begin{cases} \dfrac{1}{2\pi}\log(|x|) & \text{if } n = 2; \\ \dfrac{1}{n(2-n)\omega_n}|x|^{2-n} & \text{if } n > 2, \end{cases}$$

and that the Newton potential of a bounded function $f : \Omega \to \mathbb{R}$ is defined by

$$w(x) = \int_\Omega \Gamma(x-y)f(y)dy.$$

Note that we have interchanged the role x and y in the previous section.

When $n = 3$, one can view $w(x)$ as the gravitational potential of a body Ω with density function f, that is, the gravitational field is proportional to $-\nabla w(x)$. This gradient is well defined because of the following theorem.

19.1 Theorem Let $f \in L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open and bounded, and let $w(x)$ be the Newton potential of f. Then $w \in C^1(\mathbb{R}^n)$ and

$$\frac{\partial w(x)}{\partial x_i} = \int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i} f(y)dy.$$

Proof First observe that

$$\frac{\partial \Gamma(x-y)}{\partial x_i} = \frac{x_i - y_i}{n\omega_n |x-y|^n} \quad \text{so that} \quad \left|\frac{\partial \Gamma(x-y)}{\partial x_i}\right| \le \frac{1}{n\omega_n |x-y|^{n-1}}.$$

Hence

$$\int_{B_R(y)} \left|\frac{\partial \Gamma(x-y)}{\partial x_i}\right| dx \le \int_{B_R(y)} \frac{dx}{n\omega_n |x-y|^{n-1}} = \int_{B_R(0)} \frac{dx}{n\omega_n |x|^{n-1}}$$

$$= \int_0^R \frac{1}{r^{n-1}} r^{n-1} dr = R < \infty,$$

and

$$\frac{\partial \Gamma(x-y)}{\partial x_i} \in L^1(B_R(y)) \quad \text{for all} \quad R > 0.$$

Thus the function

$$v_i(x) = \int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i} f(y)dy$$

is well defined for all $x \in \mathbb{R}^n$.

Now let $\eta \in C^\infty([0,\infty))$ satisfy

$$\begin{cases} \eta(s) = 0 \quad \text{for} \quad 0 \le s \le 1; \\ 0 \le \eta'(s) \le 2 \quad \text{for} \quad 1 \le s \le 2; \\ \eta(s) = 1 \quad \text{for} \quad s \ge 2, \end{cases}$$

and define

$$w_\varepsilon(x) = \int_\Omega \Gamma(x-y)\eta\left(\frac{|x-y|}{\varepsilon}\right) f(y)dy.$$

Then the integrand is smooth in x, and its partial derivates of any order with respect to x are also in $L^\infty(\Omega)$. Thus $w_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$\frac{\partial w_\varepsilon(x)}{\partial x_i} = \int_\Omega \frac{\partial}{\partial x_i}\Big(\Gamma(x-y)\eta\left(\frac{|x-y|}{\varepsilon}\right) f(y)\Big)dy =$$

$$\int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i}\eta\left(\frac{|x-y|}{\varepsilon}\right) f(y)dy + \int_\Omega \Gamma(x-y)\eta'\left(\frac{|x-y|}{\varepsilon}\right)\frac{|x_i-y_i|}{\varepsilon|x-y|} f(y)dy.$$

We have for $n > 2$, and for all $x \in \mathbb{R}^n$, that

$$\Big|\frac{\partial w_\varepsilon(x)}{\partial x_i} - v_i(x)\Big| = \Big|\int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i}\big(\eta(\frac{|x-y|}{\varepsilon}) - 1\big) f(y)dy$$

$$+ \int_\Omega \Gamma(x-y)\eta'(\frac{|x-y|}{\varepsilon})\frac{x_i - y_i}{\varepsilon|x_i - y_i|} f(y)dy\Big| \leq$$

$$\|f\|_\infty\Big\{\int_{|x-y|\leq 2\varepsilon} \frac{1}{n\omega_n|x-y|^{n-1}}dy + \int_{|x-y|\leq 2\varepsilon} \frac{1}{n(n-2)\omega_n|x-y|^{n-2}}\frac{2}{\varepsilon}dy\Big\}$$

$$= \|f\|_\infty\Big\{\int_0^{2\varepsilon} \frac{1}{r^{n-1}}r^{n-1}dr + \int_0^{2\varepsilon}\frac{1}{(n-2)r^{n-2}}\frac{2}{\varepsilon}r^{n-1}dr\Big\}$$

$$= \|f\|_\infty\Big\{2\varepsilon + \frac{1}{n-2}\frac{1}{\varepsilon}(2\varepsilon)^2\Big\} = \|f\|_\infty\big(2 + \frac{4}{n-2}\big)\varepsilon,$$

so that

$$\frac{\partial w_\varepsilon}{\partial x_i} \to v_i \quad \text{uniformly in } \mathbb{R}^n \quad \text{as} \quad \varepsilon \downarrow 0.$$

Similarly one has

$$|w_\varepsilon(x) - w(x)| = \Big|\int_\Omega \Gamma(x-y)\big(\eta(\frac{|x-y|}{\epsilon}) - 1\big) f(y)dy\Big|$$

$$\leq \|f\|_\infty \int_0^{2\varepsilon} \frac{1}{(n-2)r^{n-2}}r^{n-1}dr = \|f\|_\infty \frac{\varepsilon^2}{2(n-2)},$$

so that also $w_\varepsilon \to w$ uniformly on $\mathbb{R}^n$ as $\varepsilon \downarrow 0$. This proves that $\omega, v_i \in C(\mathbb{R}^n)$, and that $v_i = \partial\omega/\partial x_i$. The proof for $n = 2$ is left as an exercise. ■

The next step would be to show that for $f \in C(\Omega)$, $w \in C^2(\Omega)$ and $\Delta\omega = f$. Unfortunately this is not quite true in general. For a counterexample see Exercise 4.9 in [GT]. To establish $w \in C^2(\Omega)$ we introduce the concept of *Dini continuity*.

19.2 Definition $f : \Omega \to \mathbb{R}$ is called (locally) Dini continuous in Ω, if for every $\Omega' \subset\subset \Omega$ there exists a measurable function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\int_0^R \frac{\varphi(r)}{r}dr < \infty \qquad \text{for all } R > 0,$$

such that

$$|f(x) - f(y)| \leq \varphi(|x-y|)$$

for all x, y in Ω'. If the function φ can be chosen independent of Ω', then f is called uniformly Dini continuous in Ω.

19.3 Definition $f : \Omega \to \mathbb{R}$ is called (uniformly) Hölder continuous with exponent $\alpha \in (0,1]$ if f is (uniformly) Dini continuous with $\varphi(r) = r^\alpha$.

19.4 Theorem Let Ω be open and bounded, and let $f \in L^\infty(\Omega)$ be Dini continuous. Then $w \in C^2(\Omega)$, $\Delta w = f$ in Ω, and for every bounded open set $\Omega_0 \supset \Omega$ with smooth boundary $\partial\Omega_0$,

$$\frac{\partial^2 \omega(x)}{\partial x_i \partial x_j} = \int_{\Omega_0} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}(f(y) - f(x))dy - f(x)\int_{\partial\Omega_0} \frac{\partial \Gamma(x-y)}{\partial x_i}\nu_j(y)dS(y),$$

where $\nu = (\nu_1, ..., \nu_n)$ is the outward normal on $\partial\Omega_0$, and f is assumed to be zero on the complement of Ω.

Proof We give the proof for $n \geq 3$. Note that

$$\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j} = \frac{1}{n\omega_n}\frac{|x-y|^2\partial_{ij} - n(x_i - y_i)(x_j - y_j)}{|x-y|^{n+2}},$$

so that

$$|\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}| \leq \frac{1}{\omega_n}\frac{1}{|x-y|^n},$$

which is insufficient to establish integrability near the singularity $y = x$. Let

$$u_{ij}(x) = \int_{\Omega_0} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}(f(y) - f(x))dy - f(x)\int_{\partial\Omega_0} \frac{\partial \Gamma(x-y)}{\partial x_i}\nu_j(y)dS(y).$$

Since f is Dini continuous, it is easy to see that $u_{ij}(x)$ is well defined for every $x \in \Omega$, because the first integrand is dominated by

$$\frac{1}{\omega_n}\frac{\varphi(r)}{r^n}$$

and the second integrand is smooth. Now let

$$v_{i\varepsilon}(x) = \int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i}\eta(\frac{|x-y|}{\varepsilon})f(y)dy.$$

Then

$$\begin{aligned}\big|v_{i\varepsilon}(x) - \frac{\partial \omega(x)}{\partial x_i}\big| &= \big|\int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i}\{\eta(\frac{|x-y|}{\varepsilon}) - 1\}f(y)dy\big| \\ &\leq \|f\|_\infty n\omega_n \int_0^{2\varepsilon} \frac{1}{n\omega_n r^{n-1}} r^{n-1}dr = 2\|f\|_\infty \varepsilon,\end{aligned}$$

so that $v_{i\varepsilon} \to \partial w/\partial x_i$ uniformly in $\mathbb{R}^n$ as $\varepsilon \downarrow 0$. Extending f to Ω_0 by $f \equiv 0$ in Ω^c, we find for $x \in \Omega$, using the smoothness of $(\partial\Gamma/\partial x_i)\eta f$, that

$$\frac{\partial v_{i\varepsilon}(x)}{\partial x_j} = \int_{\Omega_0} \frac{\partial}{\partial x_j}\frac{\partial \Gamma(x-y)}{\partial x_i}\eta(\frac{|x-y|}{\varepsilon})f(y)dy =$$

$$\int_{\Omega_0}\{f(y)-f(x)\}\frac{\partial}{\partial x_j}\frac{\partial\Gamma(x-y)}{\partial x_i}\eta\Big(\frac{|x-y|}{\varepsilon}\Big)dy+$$

$$f(x)\int_{\Omega_0}\frac{\partial}{\partial x_j}\frac{\partial\Gamma(x-y)}{\partial x_i}\eta\Big(\frac{|x-y|}{\varepsilon}\Big)dy=$$

$$\int_{\Omega_0}\{f(y)-f(x)\}\frac{\partial}{\partial x_j}\frac{\partial\Gamma(x-y)}{\partial x_i}\eta\Big(\frac{|x-y|}{\varepsilon}\Big)dx-f(x)\int_{\partial\Omega_0}\frac{\partial\Gamma(x-y)}{\partial x_i}\nu_j(y)dS(y),$$

provided $2\varepsilon<d(x,\partial\Omega)$, so that

$$\Big|u_{ij}(x)-\frac{\partial v_{i\varepsilon}(x)}{\partial x_j}\Big|=\Big|\int_{\Omega_0}\{f(y)-f(x)\}\frac{\partial}{\partial x_j}\Big(1-\eta\Big(\frac{|x-y|}{\varepsilon}\Big)\Big)\frac{\partial\Gamma(x-y)}{\partial x_i}dy\Big|=$$

$$\Big|\int_{\Omega_0}\Big\{\frac{\partial^2\Gamma(x-y)}{\partial x_i\partial x_j}\Big(1-\eta\Big(\frac{|x-y|}{\varepsilon}\Big)\Big)-\eta'\Big(\frac{|x-y|}{\varepsilon}\Big)\frac{x_j-y_j}{\varepsilon|x-y|}\frac{\partial\Gamma(x-y)}{\partial x_i}\Big\}\times$$

$$\{f(y)-f(x)\}dy\Big|\leq\int_{|x-y|\leq2\varepsilon}\Big\{\frac{1}{\omega_n|x-y|^n}+\frac{2}{\varepsilon n\omega_n|x-y|^{n-1}}\Big\}\varphi(|x-y|)dy\leq$$

$$\int_0^{2\varepsilon}\Big(\frac{n}{r^n}+\frac{2}{\varepsilon r^{n-1}}\Big)\varphi(r)r^{n-1}dr\leq$$

$$n\int_0^{2\varepsilon}\frac{\varphi(r)}{r}dr+2\int_0^{2\varepsilon}\frac{r}{\varepsilon}\frac{\varphi(r)}{r}dr\leq(n+2)\int_0^{2\varepsilon}\frac{\varphi(r)}{r}dr,$$

implying

$$\frac{\partial v_{i\varepsilon}}{\partial x_j}\to u_{ij}\quad\text{as}\quad\varepsilon\downarrow0,$$

uniformly on compact subsets of Ω. This gives $v_i\in C^1(\Omega)$ and

$$u_{ij}(x)=\frac{\partial v_i(x)}{\partial x_j}=\frac{\partial^2w(x)}{\partial x_i\partial x_j}.$$

It remains to show that $\Delta w=f$. Fix $x\in\Omega$ and let $\Omega_0=B_R(x)\supset\Omega$. Then

$$\Delta w(x)=\sum_{i=1}^n u_{ii}(x)=\sum_{i=1}^n\frac{\partial^2w(x)}{\partial x_i^2}=-f(x)\sum_{i=1}^n\int_{\partial B_R(x)}\frac{\partial\Gamma(x-y)}{\partial x_i}\nu_i(y)dS(y)=$$

$$f(x)\int_{\partial B_R(x)}\sum_{i=1}^n\frac{\partial\Gamma(x-y)}{\partial y_i}\nu_i(y)dS(y)=f(x)\int_{\partial B_R(0)}\frac{\partial\Gamma}{\partial\nu}dS=$$

$$f(x)n\omega_nR^{n-1}\frac{1}{n\omega_nR^{n-1}}=f(x),$$

and this completes the proof. ∎

19.5 Theorem Let B be a bounded open ball in $\mathbb{R}^n$. Then

$$(D)\ \begin{cases} \Delta u = f & \text{in } B; \\ u = \varphi & \text{on } \partial B, \end{cases}$$

has a unique classical solution $u \in C^2(B) \cap C(\overline{B})$ for every bounded Dini continuous $f : B \to \mathbb{R}$ and every $\varphi \in C(\partial B)$.

Proof Exercise (hint: write $u = \tilde{u} + w$, where w is the Newton potential of$f$). ∎

19.6 Exercise Generalize Poisson's formula to an integral formula for the solution of (D).

19.7 Definition Let f be locally integrable on Ω. A function $u \in C(\Omega)$ is called a weak C_0-solution of $\Delta u = f$ in Ω if, for every compactly supported $\psi \in C^2(\Omega)$, the equality

$$\int_\Omega u\Delta\psi = \int_\Omega \psi f$$

holds.

19.8 Exercise Let $f \in C(\overline{\Omega})$, and let $w \in C^1(\mathbb{R}^n)$ be the Newton potential of f. Show that w is a weak C_0-solution of $\Delta u = f$ in Ω, and that the map

$$f \in C(\overline{\Omega}) \to w \in C(\overline{\Omega})$$

is compact with respect to the supremum norm in $C(\overline{\Omega})$.

19.9 Exercise Formulate and prove Theorem 19.5 for more general domains.

20. Stationary incompressible two-dimensional flows

In this section we apply the theory of harmonic functions to stationary incompressible two-dimensional flows. Let $q : \mathbb{R}^3 \to \mathbb{R}^3$ be the velocity field of a stationary flow. Assuming that the flow is really only 2-dimensional, we have $q(x,y,z) = q(x,y) = (q_1(x,y), q_2(x,y), 0)$, and write $q = (q_1, q_2) : \mathbb{R}^2 \to \mathbb{R}^2$.

If the density is constant (i.e. the flow is incompressible), the velocity is equal to the flux, and conservation of mass implies

$$0 = \int_{\partial D} q \cdot \nu ds = \int_D \operatorname{div} q,$$

for all bounded D with ∂D smooth (ν is de outer normal on ∂D). Thus the equation we arrive at for $q = (q_1, q_2) = (q_1(x,y), q_2(x,y)$, is

$$\operatorname{div} q = \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0. \tag{20.1}$$

Next consider any smooth curve γ joining $(0,0)$ to an arbitrary point $P \in \mathbb{R}^2$. We compute the total amount of fluid that flows per unit of time (and per unit of the invisible third space coordinat z) through γ. This depends on the choice of the normal vector ν on γ. We choose ν and the unit tangent τ along γ in such a way, that the determinant of the matrix (ν, τ) is plus one. Then what we find is

$$\int_\gamma q \cdot \nu = \int_\gamma (q_1 dy - q_2 dx) = \int_\gamma -q_2 dx + q_1 dy = \int_\gamma \begin{pmatrix} -q_2 \\ q_1 \end{pmatrix} \cdot \tau.$$

Since

$$\nabla \times \begin{pmatrix} -q_2 \\ q_1 \end{pmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -q_2 & q_1 \end{vmatrix} = \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = \operatorname{div} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0,$$

this integral only depends on P, because of Stokes' theorem. Writing $P = (x,y)$ we define

$$\psi(x,y) = \int_\gamma q \cdot \nu = \int_\gamma \begin{pmatrix} -q_2 \\ q_1 \end{pmatrix} \cdot \tau. \tag{20.2}$$

Clearly,

$$\nabla \psi = \begin{pmatrix} -q_2 \\ q_1 \end{pmatrix}. \tag{20.3}$$

This function ψ is called the *stream function*, and level curves of ψ are called *stream lines*, because they are tangential to the flow velocity field. The total flow between two stream lines is constant. Thus, if two streamlines get closer to one another, the velocity has to increase.

If the flow is also *irrotational*, i.e. if

$$\oint_\gamma q \cdot \tau = 0, \tag{20.4}$$

for every closed smooth curve γ, then again by Stokes' theorem, we must have

$$0 = \nabla \times q = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ q_1 & q_2 \end{pmatrix} = -\frac{\partial q_2}{\partial x} + \frac{\partial q_1}{\partial y} = -\frac{\partial}{\partial x}(-\psi_x) + \frac{\partial}{\partial y}\psi_y = \psi_{xx} + \psi_{yy},$$

so that for ψ we obtain the equation

$$\Delta\psi = \psi_{xx} + \psi_{yy} = 0. \tag{20.5}$$

Thus the streamfunction is *harmonic*. Also, since $\nabla \times q = 0$, we can define a function $\varphi(x, y)$ by

$$\varphi(x, y) = \int_\gamma q \cdot \tau, \tag{20.6}$$

where γ is the same smooth curve from $(0, 0)$ to (x, y) that we used before. Again this integral depends only on (x, y), and

$$\nabla\varphi = q, \tag{20.7}$$

so that also

$$\Delta\varphi = \operatorname{div} q = 0. \tag{20.8}$$

The harmonic function φ is called the *velocity potential*. Since

$$\nabla\varphi = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \text{ and } \nabla\psi = \begin{pmatrix} -q_2 \\ q_1 \end{pmatrix},$$

the level curves of the stream function and the velocity potential intersect oneanother perpendicularly. Moreover, writing $z = x + iy$, the *complex stream function*

$$F(z) = F(x + iy) = \varphi(x, y) + i\psi(x, y)$$

is analytic if φ and ψ are differentiable, because the Cauchy-Riemann equations hold:

$$\varphi_x = q_1 = \psi_y \text{ and } \varphi_y = q_2 = -\psi_x.$$

For the derivative of $F(z)$ we find

$$F'(z) = \frac{\partial}{\partial x}F(x + iy) = \varphi_x(x, y) + i\psi_x(x, y) = \varphi_x(x, y) - i\psi_y(x, y),$$

so that in complex notation the velocity field is given by

$$q = q_1 + iq_2 = \overline{F'(z)},$$

where the bar denotes complex conjugation.

Now suppose we have a stationary 2-dimensional irrotational flow in a simply connected domain Ω . Then the derivations above remain valid. We look for an harmonic (stream) function $\psi : \Omega \to \mathbb{R}$ to describe the flow. If Ω is insulated, i.e. if there is no flow going through $\partial\Omega$, then $\partial\Omega$ must consist of streamlines, i.e lines where ψ is constant. For a bounded simply connected domain this implies immediately that ψ is constant in Ω, because of the maximium principle. In other words, no flow at all. Hence to find nontrivial 2-dimensional irrotational flows we must look at domains Ω with $\partial\Omega$ disconnected, or at unbounded domains.

20.1 Example Let $\Omega_1 = \{(x,y) : y > 0\} = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$. A flow in Ω_1 which at infinity has a velocity equal to one parallel to the x-axis, is given by $F(z) = z$, $\varphi(x,y) = x$, $\psi(x,y) = y$, and, in complex notation, $q = 1$.

20.2 Example Flow in a half space past a vertical barrier. Here we take

$$\Omega_2 = \{(x,y) : \; y > 0; \; y > 1 \text{ if } x = 0\}.$$

How do we find a solution? We look for an analytic function $f(z)$, with

$$f(\Omega_2) = \Omega_1 \text{ and } f(\partial\Omega_2) = \partial\Omega_1.$$

Note that here the boundary of a domain has a meaning slightly different from the usual definition. In particular the vertical barrier has to be counted twice, because its left and right hand sides are physically separated. A way to do this would be not to consider the boundary as a set but as a curve.

For the function $f(z)$ we find

$$f(z) = \sqrt{z^2+1},$$

which describes a flow in Ω_2 with the same properties at infinity as the flow in the previous example. This "boundary condition at infinity" can in fact be shown to determine the flow uniquely. The stream function is $\psi(x,y) = \mathrm{Im}\sqrt{z^2+1}$, and

$$q = q_1 + iq_2 = \overline{\left(\frac{z}{\sqrt{z^2+1}}\right)}.$$

Observe that in the corners, i.e. both sides of the barrier near $z = 0$, the velocity is small and tend to zero as $z \to 0$, while near the tip $z = 1$ of the barrier, the velocity is unbounded.

PART 6: MAXIMUM PRINCIPLES

21. Classical maximum principles for elliptic equations

In this section we replace the Laplacian Δ by the operator L, defined by

$$Lu = \sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} + cu. \tag{21.1}$$

Here the coefficients $a(x), b(x)$ and $c(x)$ are continuous functions of $x \in \Omega$, and u is taken in $C^2(\Omega)$. It is no restriction to assume that $a_{ij}(x) = a_{ji}(x)$ for all $x \in \Omega$. The matrix

$$A(x) = \big(a_{ij}(x)\big)_{i,j=1,\dots,N} = \begin{pmatrix} a_{11}(x) & \cdots & a_{1N}(x) \\ \vdots & & \vdots \\ a_{N1}(x) & \cdots & a_{NN}(x) \end{pmatrix}$$

is symmetric and defines a quadratic form on $\mathbb{R}^n$ for every $x \in \Omega$. Denoting the elements of $\mathbb{R}^n$ by $\xi = (\xi_1, ..., \xi_n)$, this form is given by

$$(A(x)\xi, \xi) = \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j. \tag{21.2}$$

Note that $A(x)$, being a symmetric matrix, has exactly N real eigenvalues $\lambda_1(x) \le ... \le \lambda_N(x)$ (counted with multiplicity), corresponding to an orthonormal basis of eigenvectors.

21.1 Definition The operator L is called *elliptic* in $x_0 \in \Omega$ if

$$(A(x_0)\xi, \xi) > 0 \quad \forall\, \xi \in \mathbb{R}^N, \xi \neq 0,$$

i.e. if the quadratic form is positive definite in x_0.

If L is elliptic at x_0, there exist numbers $0 < \lambda(x_0) \le \Lambda(x_0)$ such that $\lambda(x_0)|\xi|^2 \le (A(x_0))\xi, \xi) \le \Lambda(x_0)|\xi|^2$ for all $\xi \in \mathbb{R}^N$, and it is easy to see that $0 < \lambda(x_0) = \lambda_1(x_0) \le \lambda_N(x_0) = \Lambda(x_0)$.

21.2 Definition The operator L is called *uniformly elliptic* in Ω if there exist numbers $0 < \lambda \le \Lambda < \infty$, independent of $x \in \Omega$, such that

$$\lambda|\xi|^2 \le (A(x)\xi, \xi) \le \Lambda|\xi|^2 \quad \forall x \in \Omega \quad \forall \xi \in \mathbb{R}^N.$$

To check the uniform elipticity of a given operator L of the form (21.1), it is sufficient to check that all the eigenvalues of the matrix $A(x)$ are positive, and bounded away from zero and infinity uniformly for $x \in \Omega$. Throughout this section we shall assume that this is always so, and that for some fixed number $b_0 > 0$,

$$|b_i(x)| \leq b_0 \quad \forall x \in \Omega, \quad \forall i = 1, ..., N. \tag{21.3}$$

In many ways uniformly elliptic operators resemble the Laplacian.

21.3 Theorem (weak maximum principle) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let L be uniformly elliptic with bounded continuous coefficients, and $c \equiv 0$ on Ω. Suppose that for some $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$Lu \geq 0 \quad \text{in} \quad \Omega.$$

Then

$$\sup_{\Omega} u = \max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof First we assume that $Lu > 0$ in Ω and that u achieves a maximum in $x_0 \in \Omega$. Then $\nabla u(x_0) = 0$, and the Hessian of u in x_0,

$$(Hu)(x_0) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \right)_{i,j=1,\ldots,N},$$

is negative semi-definite, i.e.

$$\sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)\xi_i\xi_j \leq 0 \quad \forall \xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N.$$

We claim that consequently

$$(Lu)(x_0) = \sum_{i,j=1}^{N} a_{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \leq 0,$$

contradicting the assumption. This claim follows from a lemma from linear algebra which we state without proof.

21.4 Lemma Let $A = (a_{ij})$ and $B = (b_{ij})$ be two positive semi-definite matrices, i.e.

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq 0 \;\text{ and }\; \sum_{i,j=1}^{N} b_{ij}\xi_i\xi_j \geq 0,$$

for all $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$. Then

$$\sum_{i,j=1}^{N} a_{ij}\ b_{ij} \geq 0.$$

We continue with the proof of Theorem 21.3. Suppose $Lu \geq 0$ in Ω, and let

$$v(x) = e^{\gamma x_1}, \qquad \gamma > 0.$$

Then

$$(Lv)(x) = (a_{11}\gamma^2 + \gamma b_1)e^{\gamma x_1} \geq \gamma(\lambda\gamma - b_0)e^{\gamma x_1} > 0,$$

if $\gamma > b_0/\lambda$. Hence, by the first part of the proof, we have for all $\varepsilon > 0$ that

$$\sup_{\Omega}(u + \varepsilon v) = \max_{\partial\Omega}(u + \varepsilon v).$$

Letting $\varepsilon \downarrow 0$ completes the proof. ∎

21.5 Theorem Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let L be uniformly elliptic with bounded continuous coefficients, and $c \leq 0$ on Ω. Suppose that for some $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$Lu \geq 0 \quad \text{in} \quad \Omega.$$

Then

$$\sup_{\Omega} u = \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+,$$

where $u^+ = \max(u, 0)$ denotes the positive part of u.

Proof Let $\Omega^+ = \{x \in \Omega : u(x) > 0\}$. If $\Omega^+ = \emptyset$ there is nothing to prove. Assume $\Omega^+ \neq \emptyset$. For every component Ω_0^+ of Ω^+ we have

$$u = 0 \qquad \text{on} \qquad \partial\Omega_0^+ \setminus \partial\Omega,$$

so

$$\max_{\partial\Omega_0^+} u \leq \max_{\partial\Omega} u^+.$$

Define the operator L_0 by

$$L_0 u = Lu - cu.$$

Then $L_0 u \geq Lu$ in Ω_0^+ and by Theorem 21.3

$$\sup_{\Omega_0^+} u = \max_{\overline{\Omega_0^+}} u = \max_{\partial\Omega_0^+} u \leq \max_{\partial\Omega} u^+,$$

and this holds for every component of Ω^+. ■

21.6 Corollary Let Ω, u and L be as in Theorem 21.5. If

$$Lu = 0 \quad \text{in} \quad \Omega,$$

then

$$\sup_{\Omega} |u| = \max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Proof Exercise. ■

21.7 Definition $u \in C^2(\Omega)$ is called a *subsolution* of the equation $Lu = 0$ if $Lu \geq 0$ in Ω, and a *supersolution* if $Lu \leq 0$.

21.8 Corollary Let Ω and L be as in Theorem 21.5, and assume that $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subsolution and $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ a supersolution. Then $\underline{u} \leq \overline{u}$ on $\partial\Omega$ implies $\underline{u} \leq \overline{u}$ on Ω. (*Comparison principle*)

Proof Exercise. ■

21.9 Corollary For $f \in C(\Omega), \varphi \in C(\partial\Omega), \Omega$ and L as in Theorem 21.5, the problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases}$$

has atmost one solution in $C^2(\Omega) \cap C(\overline{\Omega})$.

Proof Exercise. ■

What we have done so far is based on the weak maximum principle. As in the case of the Laplacian, we shall also prove a strong maximum principle. We recall that Ω is said to satisfy a interior ball condition at $x_0 \in \partial\Omega$ if there exists a ball $B \subset \Omega$ such that $\overline{B} \cap \partial\Omega = \{x_0\}$.

21.10 Theorem (*Boundary Point Lemma*) Let Ω and L be as in Theorem 21.5, let $x_0 \in \partial\Omega$ be a point where the interior ball condition is satisfied by means of a ball $B = B_R(y)$, and $u \in C^2(\Omega) \cap C(\Omega \cup \{x_0\})$. Suppose that

$$Lu \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u(x) < u(x_0) \quad \forall x \in \Omega,$$

Then, if $u(x_0) \geq 0$, we have

$$\lim \inf_{\substack{x \to x_0 \\ x \in S_\delta}} \frac{u(x_0) - u(x)}{|x - x_0|} > 0 \quad \text{for all} \quad \delta > 0,$$

where
$$S_\delta = \{x \in \Omega : (y - x_0, x - x_0) \geq \delta R|x - x_0|\}.$$

For $c \equiv 0$ in Ω the same conclusion holds if $u(x_0) < 0$, and if $u(x_0) = 0$ the sign condition on c may be omitted. N.B. If the outward normal ν on $\partial\Omega$ and the normal derivative $\frac{\partial u}{\partial \nu}$ exist in x_0, then $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Proof Choose $\rho \in (0, R)$ and let $A = B_R(y) \backslash \overline{B_\rho(y)}$. For $x \in A$ we define

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}, \quad r = |x - y|,$$

where $\alpha > 0$ is to be specified later on. Then

$$\frac{\partial v}{\partial x_i}(x) = -2\alpha e^{-\alpha r^2}(x_i - y_i),$$

$$\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = 4\alpha^2 e^{-\alpha r^2}(x_i - y_i)(x_j - y_j) - 2\alpha e^{-\alpha r^2}\delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$, so

$$(Lv)(x) = e^{-\alpha r^2}\Big\{\sum_{i,j=1}^{N} 4\alpha^2(x_i - y_i)(x_j - y_j)a_{ij}(x)$$

$$-\sum_{i=1}^{N} 2\alpha(a_{ii}(x) + b_i(x)(x_i - y_i)) + c(x)\Big\} - c(x)e^{-\alpha R^2}.$$

Hence, if c is nonpositive, by definition 21.2 and (21.3),

$$(Lv)(x) \geq e^{-\alpha r^2}\Big\{4a^2\lambda r^2 - 2\alpha(N\Lambda + Nb_0 r) + c\Big\} \geq 0 \quad \text{in} \quad A,$$

provided α is chosen sufficiently large.

Now let
$$w_\varepsilon(x) = u(x_0) - \varepsilon v(x) \quad x \in \overline{A}, \quad \varepsilon > 0.$$

Then

$$(Lw_\varepsilon)(x) = -\varepsilon(Lv)(x) + c(x)u(x_0) \leq -\varepsilon(Lv)(x) \leq 0 \leq Lu(x) \quad \forall x \in A,$$

if $c(x)u(x_0) \leq 0$ for all $x \in \Omega$. Because $u(x) < u(x_0) \quad \forall x \in \partial B_\rho(y)$ we can choose $\varepsilon > 0$ such that

$$w_\varepsilon(x) = u(x_0) - \varepsilon v(x) \geq u(x) \quad \forall x \in \partial B_\rho(y),$$

while for $x \in \partial B_R(y)$

$$w_\varepsilon(x) = u(x_0) - \varepsilon v(x) = u(x_0) \geq u(x).$$

Hence

$$\begin{cases} Lw_\varepsilon \leq Lu & \text{in} \quad A \\ w_\varepsilon \geq u & \text{on} \quad \partial A, \end{cases}$$

so that by the comparison principle (Corollary 21.8) $u \leq w_\varepsilon$ in A, whence

$$u(x_0) - u(x) \geq \varepsilon v(x) \quad \forall x \in A.$$

Since

$$\nabla v(x_0) = 2\alpha e^{-\alpha r^2}(y - x_0),$$

this completes the proof for the case that $c \leq 0$ and $u(x_0) \geq 0$. Clearly the case $c \equiv 0$ and $u(x_0)$ arbitrary is also covered by this proof. Finally, if c is allowed to change sign, and $u(x_0) = 0$, we replace L by $\hat{L}u = Lu - c_+ u$. ∎

21.11 Theorem (Strong Maximum Principle, Hopf) Let Ω and L be as in Theorem 21.5 and let $u \in C^2(\Omega)$ satisfy

$$Lu \geq 0 \quad \text{in} \quad \Omega.$$

(i) If $c \equiv 0$ in Ω then u cannot have a global maximum in Ω, unless u is constant.

(ii) If $c \leq 0$ in Ω then u cannot have a global nonnegative maximum in Ω, unless u is constant.

(iii) If u has a global maximum zero in Ω, then u is identically equal to zero in Ω.

Proof Suppose $u(y_0) = M$ and $u(x) \leq M$ for all $x \in \Omega$. Let

$$\Omega_- = \{x \in \Omega; u(x) < M\},$$

and assume that $\Omega_- \neq \emptyset$. Then $\partial\Omega_- \cap \Omega \neq \emptyset$, so there exists $y \in \Omega_-$ with $d(y, \partial\Omega_-) < d(y, \partial\Omega)$. Hence we may choose a maximal $R > 0$ such that $B_R(y) \subset \Omega_-$, and on $\partial B_R(y) \subset \Omega$ there must be a point x_0 where $u(x_0) = M$. In view of the boundary point lemma we have $\nabla u(x_0) \neq 0$, contradicting the assumption that M is a global maximum. ∎

21.12 Corollary Let Ω and L be as in Theorem 21.5 and let $u \in C^2(\Omega)$ satisfy

$$Lu = 0 \quad \text{in} \quad \Omega.$$

(i) If $c \equiv 0$ in Ω then u cannot have a global extremum in Ω, unless u is constant.

(ii) If $c \leq 0$ in Ω then $|u|$ cannot have a global maximum in Ω, unless u is constant.

21.13 Theorem (a priori estimate) Let L and Ω be as in Theorem 21.5, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $f \in C(\Omega)$. If

$$Lu \geq f \quad \text{in} \quad \Omega,$$

then

$$\sup_{\Omega} u = \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C \sup_{\Omega} |f_-|,$$

where C is a constant only depending on λ, b_0, the diameter of Ω.

Proof We may assume that

$$\Omega \subset \{x \in \mathbb{R}^N; 0 < x_1 < d\}.$$

Let L_0 be defined by $L_0 u = Lu - cu$. For $x \in \Omega$ and α a positive parameter we have

$$L_0 e^{\alpha x_1} = (\alpha^2 a_{11} + \alpha b_1) e^{\alpha x_1} \geq (\lambda \alpha^2 - b_0 \alpha) e^{\alpha x_1} = \lambda \alpha (\alpha - \frac{b_0}{\lambda}) e^{\alpha x_1} \geq \lambda,$$

if $\alpha = \frac{b_0}{\lambda} + 1$. Now define $v(x)$ by

$$v = \max_{\partial\Omega} u_+ + \frac{1}{\lambda}(e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} |f_-|.$$

Then

$$Lv = L_0 v + cv \leq -\sup_{\Omega} |f_-|,$$

so

$$L(v - u) \leq -(\sup_{\Omega} |f_-| + f) \leq 0 \quad \text{in} \quad \Omega.$$

Clearly $v - u \geq 0$ on $\partial\Omega$, whence, by the weak maximum principle, $v - u \geq 0$, and

$$u \leq \max_{\partial\Omega} u^+ + \frac{1}{\lambda} e^{\alpha d} \sup_{\Omega} |f^-| \quad \text{in} \ \ \Omega.$$

∎

21.14 Corollary In the same situation, if $Lu = f$, then

$$\sup_{\Omega} |u| = \max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u| + C \sup_{\Omega} |f|.$$

Another important consequence of the strong maximum principle is what is widely known as Serrin's sweeping principle, which is very useful in the study of semilinear elliptic equations of the form

$$Lu + f(x, u) = 0.$$

21.15 Theorem (Serrin's sweeping principle) Let L and Ω be as in Theorem 21.5, and suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous, and has a continuous partial derivative f_u, which is locally bounded in $u \in \mathbb{R}$, uniformly in $x \in \Omega$. Suppose that there exists a family (of supersolutions) $\{\overline{u}_\lambda,\ \lambda \in [0,1]\} \subset C^2(\Omega) \cap C(\overline{\Omega})$, with $\overline{u}_\lambda$ varying continuously with λ in the sense that the map $\lambda \in [0,1] \to \overline{u}_\lambda \in C(\overline{\Omega})$ is continuous. Suppose also that, for all $\lambda \in [0,1]$,

$$L\overline{u}_\lambda + f(x, \overline{u}_\lambda) \le 0 \ \text{ in } \Omega, \text{and } \ u_\lambda > 0 \text{ on } \partial\Omega,$$

and that there exists a (subsolution) $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$, with

$$L\underline{u} + f(x, \underline{u}) \ge 0 \ \text{ in } \Omega, \text{and } \ \underline{u} \le 0 \text{ on } \partial\Omega.$$

Then, if $\underline{u} \le \overline{u}_\lambda$ in Ω for some $\lambda = \lambda_0 \in [0,1]$, it follows that $\underline{u} < \overline{u}_\lambda$ in Ω for all $\lambda \in [0,1]$.

Proof We first prove the statement for $\lambda = \lambda_0$. Assume it is false. In view of the assumptions this means that the function $w = \overline{u}_\lambda - \underline{u}$ has a global maximum zero in some interior point of Ω. But w is easily seen to satisfy the equation

$$Lw + c_\lambda(x) w \le 0 \ \text{ in } \Omega,$$

with

$$c_\lambda(x) = \int_0^1 f_u(x, s\overline{u}_\lambda(x) + (1-s)\underline{u}(x))ds,$$

which is a bounded continuous function in Ω. By Theorem 21.11(iii), it follows that w is identically equal to zero, a contradiction. Thus the statement is proved for $\lambda = \lambda_0$.

Next vary λ starting at $\lambda = \lambda_0$. As long as $\underline{u} < \overline{u}_\lambda$ there is nothing to prove, the only thing that can go wrong is, that for some $\lambda = \lambda_1$, with $|\lambda_1 - \lambda_0|$ chosen minimal, $\overline{u}_\lambda$ touches $\underline{u}$ again from below. But this is ruled out by the same argument as above. ∎

22. Maximum principles for parabolic equations

We consider solutions $u(x,t)$ of the equation

$$u_t = Lu, \tag{22.1}$$

where

$$Lu = \sum_{i,j=1}^N a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i \frac{\partial u}{\partial x_i} + cu. \tag{22.2}$$

Throughout this section we assume that L has bounded continuous coefficients $a_{ij}(x,t) = a_{ji}(x,t)$, $b_i(x,t)$, $c(x,t)$ defined for (x,t) in a set of the form $Q_T = \Omega \times (0,T]$, with $T > 0$, and Ω a domain in $\mathbb{R}^N$. The set $\Gamma_T = \overline{Q_T} \backslash Q_T$ is called the parabolic boundary of Q_T. If the elliptic part Lu is uniformly elliptic in Q_T, that is if there exist numbers $0 < \lambda \leq \Lambda < \infty$, independent of $(x,t) \in Q_T$, such that

$$\lambda |\xi|^2 \leq (A(x,t)\xi, \xi) \leq \Lambda |\xi|^2 \quad \forall (x,t) \in Q_T \quad \forall \xi \in \mathbb{R}^N, \tag{22.3}$$

where

$$A(x,t) = \big(a_{ij}(x,t)\big)_{i,j=1,\ldots,N} = \begin{pmatrix} a_{11}(x,t) & \cdots & a_{1N}(x,t) \\ \vdots & & \vdots \\ a_{N1}(x,t) & \cdots & a_{NN}(x,t) \end{pmatrix},$$

then equation (22.1) is called uniformly parabolic in Q_T.

22.2 Notation

$$C^{2,1}(Q_T) = \{u : Q_T \to \mathbb{R};\ u, u_t, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(Q_T)\}.$$

Our goal is again to exclude the existence of maxima and minima of (sub- and super-) solutions u in Q_T.

22.3 Theorem Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and $c \equiv 0$, and suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies the inequality $u_t \leq Lu$ in Q_T. Then

$$\sup_{Q_T} u = \max_{\overline{Q_T}} u = \max_{\Gamma_T} u.$$

Proof First we assume that $u_t > Lu$ in Q_T and that u achieves a maximum in $P = (x_0, t_0) \in Q_T$. Then the first order x-derivatives of u vanish in P, and

$$(Hu)(P) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(P)\right)_{i,j=1,\ldots,N}$$

is negative semi-definite, i.e.

$$\sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j}(P) \xi_i \xi_j \leq 0 \quad \forall \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N.$$

Consequently

$$(Lu)(P) = \sum_{i,j=1}^{N} a_{ij}(P) \frac{\partial^2 u}{\partial x_i \partial x_j}(P) \leq 0,$$

implying $u_t(P) < 0$, so u cannot have a maximum in P.

Now suppose that we only know that $u_t \leq Lu$ in Q_T. Let

$$v(x) = e^{\gamma x_1}, \qquad \gamma > 0.$$

Then

$$(Lv)(x) = (a_{11}\gamma^2 + \gamma b_1)e^{\gamma x_1} \geq \gamma(\lambda\gamma - b_0)e^{\gamma x_1} > 0,$$

if $\gamma > b_0/\lambda$, where $b_0 = \sup_{Q_T} |b|$. Hence, by the first part of the proof, we have for all $\varepsilon > 0$ that

$$\sup_{Q_T}(u + \varepsilon v) = \max_{\Gamma_T}(u + \varepsilon v).$$

Letting $\varepsilon \downarrow 0$ completes the proof. ∎

22.4 Theorem Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and let $c \leq 0$, and suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies the inequality $u_t \leq Lu$ in Q_T. Then

$$\sup_{Q_T} u = \max_{\overline{Q_T}} u \leq \max_{\Gamma_T} u^+.$$

Proof Exercise. ∎

22.5 Corollary Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies the inequality $u_t \leq Lu$ in Q_T. If $u \leq 0$ on Γ_T. Then also $u \leq 0$ on Q_T.

Proof Exercise. Hint: consider the function

$$v(x,t) = e^{-kt}u(x,t),$$

with $k > 0$ sufficiently large. ∎

22.6 Remark Everything we have done so far remains valid if Q_T is replaced by $D \cap \{t \leq T\}$, where D is a bounded domain in $\mathbb{R}^{N+1}$.

As in the elliptic case we also have a strong maximum principle, , and $\rho(x) \leq x - x_0$. By the boundary point lemma this last inequality must be strict, for otherwise the first order derivatives of u cannot vanish simultaneously in P, contradicting $u \leq M$ in Dbut this requires a little more work.

22.7 Theorem (*Boundary Point Lemma*) Let L be a uniformly elliptic operator with bounded coefficients in a domain $D \subset \mathbb{R}^{N+1}$ and $c \leq 0$ in D. Suppose the interior ball condition in a point $P = (x_0, t_0) \in \partial D$ is satisfied by means of a ball $B = B_R((x_1, t_1))$ with $x_1 \neq x_0$. Let $u \in C^{2,1}(D) \cap C(D \cup \{P\})$ satisfy

$$Lu - u_t \geq 0 \quad \text{in} \quad D \ \text{ and } \ u(x,t) < u(P) \quad \forall (x,t) \in D.$$

Then, if $u(P) \geq 0$, we have

$$\lim \inf_{\substack{(x,t)\to P \\ (x,t)\in S_\theta}} \frac{u(P) - u(x,t)}{|(x,t) - P|} > 0,$$

where S_θ is a cone with top P and opening angle $\theta < \pi$, intersecting D and radially symmetric around the line through P and (x_1, t_1). For $c \equiv 0$ in D the same conclusion holds if $u(P) < 0$, and if $u(P) = 0$ the sign condition on c may be omitted. N.B. If the outward normal ν on ∂D and the normal derivative $\frac{\partial u}{\partial \nu}$ exist in P, then $\frac{\partial u}{\partial \nu}(P) > 0$.

Proof Without loss of generality we assume that the center of the disk B is in the origin. Let B_0 be a small disk centered at P such that $\overline{B_0}$ does not have any point in common with the t-axis. This is possible because P is not at the top or the bottom of B. Consequently there exists $\rho > 0$ such that $|x| \geq \rho$ for all $(x,t) \in K = B \cap B_0$. Consider

$$v(x,t) = e^{-\alpha(x^2+t^2)} - e^{-\alpha R^2}, \tag{22.4}$$

which is zero on ∂B. Then

$$Lv - v_t = e^{-\alpha(x^2+t^2)} \Big\{ \sum_{i,j=1}^{N} 4\alpha^2 x_i x_j a_{ij} - \sum_{i=1}^{N} 2\alpha(a_{ii} + b_i x_i) + c + 2\alpha t \Big\} - ce^{-\alpha R^2}.$$

Hence, if c is nonpositive, we have on K that

$$Lv - v_t \geq e^{-\alpha R^2} \Big\{ 4\alpha^2 \lambda \rho^2 - 2\alpha(N\Lambda + N b_0 R) + c - 2\alpha R \Big\} > 0, \tag{22.5}$$

choosing $\alpha > 0$ sufficiently large, because c is bounded. For the function

$$w_\varepsilon(x,t) = u(x,t) + \varepsilon v(x,t),$$

it then also follows that $Lw_\varepsilon - w_t > 0$ in K. Since $v = 0$ on ∂B, and $u < u(P)$ on B, we can choose $\varepsilon > 0$ so small that $w_\varepsilon \leq u(P)$ on ∂K. Applying Theorem 22.3 (22.4) and keeping in mind Remark 22.6, it follows that $w_\varepsilon \leq u(P)$ on K, so that

$$u(x,t) \leq u(P) - \varepsilon v(x,t) \quad \forall (x,t) \in K. \tag{22.6}$$

This completes the proof. ■

22.8 Theorem Let L be uniformly elliptic with bounded coefficients in a domain $D \subset \mathbb{R}^{N+1}$ and $c \leq 0$ in D. Suppose $u \in C^{2,1}(D)$ satisfies

$$Lu - u_t \geq 0 \quad \text{and} \quad u(x,t) < M \quad \forall (x,t) \in D.$$

If $u(P) = M \geq 0$ for some $P = (x_0, t_0) \in D$, then $u \equiv M$ on the component of $D \cap \{t = t_0\}$ containing P. For $c \equiv 0$ in D the condition $M \geq 0$ can be omitted.

Proof Suppose the result is false. Then there exist two points $P = (x_0, t_0)$ and $P_1 = (x_1, t_0)$ in D such that $u(P) = M$, $u(P_1) < M$, and $u < M$ on the line segment l joining P and P_1 We can choose P and P_1 in such a way that the distance of l to the boundary of D is larger then the length of l. For notational convenience we argue in the remainder of the proof as if $N = 1$ and x_1 is to the right of x_0. Then for every $x \in (x_0, x_1)$ let $\rho = \rho(x)$ be the largest radius such that $u < M$ on the ball $B_\rho(x)$. Clearly $\rho(x)$ is well defined. By definition, $u = M$ in some point P_x on the boundary of $B_{\rho(x)}(x)$, and, applying the boundary point lemma, it follows that P_x is either the top or the bottom of $B_\rho(x)$, so $P_x = (x, t_0 \pm \rho(x))$. Now let $\delta > 0$ be small, and consider $x + \delta$. Then, again by the boundary point lemma, P_x cannot be in the closure of $B_{\rho(x+\delta)}(x + \delta)$. Hence

$$\rho(x + \delta)^2 < \rho(x)^2 + \delta^2. \tag{22.7}$$

Substituting δ/m for δ, and $x + j\delta/m$ for $j = 0, \ldots, m - 1$ in (22.7), we obtain, summing over j,

$$\rho(x + \delta)^2 < \rho(x)^2 + \frac{\delta^2}{m}, \tag{22.8}$$

for all m so that $\rho(x)$ is nonincreasing in x. Letting $x \downarrow x_0$ it follows that $u(x_0, t_0) < M$, a contradiction. ∎

22.9 Theorem Let L be uniformly elliptic with bounded coefficients in a domain $D \subset \mathbb{R}^{N+1}$ and $c \leq 0$ in D. Suppose $u \in C^{2,1}(D)$ satisfies $u_t \leq Lu$ in D and, for $M \geq 0$ and $t_0 < t_1$, that $u < M$ in $D \cap \{t_0 < t < t_1\}$. Then also $u < M$ on $D \cap \{t = t_1\}$. For $c \equiv 0$ in D the condition $M \geq 0$ can be omitted, and the result is also true when D is replaced by Q_T.

Proof Suppose there exists a point $P = (x_1, t_1)$ in $D \cap \{t = t_1\}$ with $u(P) = M$. For notational convenience we assume that P is the origin, so $x_1 = 0$ and $t_1 = 0$. Consider the function

$$v(x,t) = e^{-(|x|^2 + \alpha t)} - 1, \tag{22.9}$$

which is zero on, and positive below the parabola $\alpha t = -|x|^2$. Then

$$Lv - v_t = e^{-(|x|^2+\alpha t)} \Big\{ \sum_{i,j=1}^{N} 4x_i x_j a_{ij} - \sum_{i=1}^{N} 2(a_{ii} + b_i x_i) + c + \alpha \Big\} - c.$$

Hence, if c is nonpositive,

$$Lv - v_t \geq e^{-(|x|^2+\alpha t)}\Big\{4\lambda|x|^2 - 2(N\Lambda + Nb_0|x|) + c + \alpha\Big\}. \tag{22.10}$$

Now let B be a small ball with center in the origin. Choosing $\alpha > 0$ sufficiently large, we can make the right hand side of (22.10) positive in B. Consider on $K = B \cap \{\alpha t < -|x|^2\}$ the function

$$w_\varepsilon(x,t) = u(x,t) + \varepsilon v(x,t),$$

then by similar reasoning as before we can choose $\varepsilon > 0$ so small that $w_\varepsilon \leq M$ on ∂K. By Theorem 22.3/22.4 it follows again that $w_\varepsilon \leq M$ thoughout K, so

$$u(x,t) \leq M - \varepsilon v(x,t) \quad \forall (x,t) \in K, \tag{22.11}$$

implying $u_t > 0$ in the origin, a contradiction. This completes the proof. Note that we did not need u to be defined for $t > 0$ in the proof, which corresponds to $t > t_1$ in the Theorem. Thus the proof applies equally well to Q_T with $t_1 = T$. ■

22.10 Theorem (*Strong Maximum Principle*) Let L be uniformly elliptic with bounded coefficients in a domain $D \subset \mathbb{R}^{N+1}$ and $c \leq 0$ in D. Suppose $u \in C^{2,1}(D)$ satisfies $u_t \leq Lu$ in D and, for some $M \geq 0$ that $u \leq M$ in D. If $P \in D$ and $u(P) = M$, then $u = M$ in every point P' in D which can be joined to P by a continuous curve in D along which, running from P' to P, t is nondecreasing. For $c \equiv 0$ in D the condition $M \geq 0$ can be omitted.

Proof By continuity the curve can always be chosen in such a way that it is piecewise linear with either x or t constant along every piece. By the previous two theorems $u = M$ along such a curve. ■

22.11 Remark For Q_T the statement in Theorem 22.10 is simply that either $u < M$ in Q_T, or $u \equiv M$ in Q_T.

Next we give some applications of the results above to semilinear parabolic equations. Instead of (22.1) we now consider

$$u_t = Lu + f(x,t,u), \tag{22.12}$$

where f is a given continous function of the variables x and t, as well as the unknown u.

22.12 Proposition (*Comparison Principle*) Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and $c \leq 0$, and suppose that $f(x,t,u)$ is nonincreasing in u. If $u, v \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfy $u_t \leq Lu + f(x,t,u)$, $v_t \geq Lv + f(x,t,v)$ in Q_T, and $u \leq v$ in Γ_T, then $u \leq v$ throughout Q_T.

Proof Exercise. Hint: apply Theorem 22.3 to the function $z = u - v$. ■

The condition that f is nonincreasing in u is rather restrictive. However, if f satisfies a one-sided uniform Lipschitz condition,

$$f(x,t,v) - f(x,t,u) \leq K(v-u), \quad \forall x,t,u,v, \quad v > u, \tag{22.13}$$

we can substitute

$$u(x,t) = e^{Kt}w(x,t)$$

in (22.12). This yields

$$w_t = Lw + e^{-Kt}f(x,t,e^{Kt}w) - Kw = Lw + g(x,t,w), \tag{22.14}$$

and clearly $g(x,t,w)$ is nonincreasing in w.

22.13 Theorem (*Strong Comparison Principle*) Let L be uniformly elliptic in Q_T with bounded coefficients, Ω be bounded, and suppose that f satisfies (22.13). If $u, v \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfy $u_t \leq Lu + f(x,t,u)$, $v_t \geq Lv + f(x,t,v)$ in Q_T, and $u \leq v$ in Γ_T, then $u < v$ throughout Q_T, unless $u \equiv v$ in Q_T.

Proof Exercise, use (22.14) and the lemma below. ■

22.14 Lemma Let L be uniformly elliptic with bounded coefficients in a domain $D \subset \mathbb{R}^{N+1}$, and let f satisfy (22.13). Suppose $u, v \in C^{2,1}(D)$ satisfies $u_t \leq Lu + f(x,t,u)$, $v_t \geq Lv + f(x,t,v)$, and $u \leq v$ in D. If $P \in D$ and $u(P) = v(P)$, then $u = v$ in every point P' in D which can be joined to P by a continuous curve in D along which, running from P' to P, t is nondecreasing.

Proof Exercise. ■

Next we shall give a monotonicity property for sub- and supersolutions of semilinear *autonomous* equations, that is, the coefficients a, b, c, and nonlinearity f are independent of t.

22.15 Theorem (*Monotonicity*) Let L be uniformly elliptic in Q_T with bounded coefficients independent of t, Ω be bounded, and suppose that $f = f(x,u)$ satisfies (22.13). Suppose $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $0 \leq L\underline{u} + f(x,\underline{u})$ in Ω, and $\underline{u} = 0$ on $\partial\Omega$, and suppose $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies $u_t = Lu + f(x,u)$ in Q_T, $u = 0$ on $\partial\Omega \times (0,T]$, and $u(x,0) = \underline{u}(x,0)$ for all $x \in \Omega$, then $u_t \geq 0$ in Q_T.

Proof By Theorem 22.13 we have $\underline{u} \leq u$ in Q_T. Define $v(x,t) = u(x,t+h)$, where $0 < h < T$. Again applying Theorem 22.13 we have $u \leq v$ in Q_{T-h}, i.e. $u(x,t+h) \geq u(x,t)$. ■

Remark The assumption that $\underline{u} = 0$ on $\partial\Omega$ can be relaxed to $\underline{u} \leq 0$, but then we can no longer talk about a solution $u \in C(\overline{Q_T})$, because obviously u will be discontinuous in the set of cornerpoints $\partial\Omega \times \{0\}$. The result however remains true for solutions $u \in C^{2,1}(Q_T)$ which are continuous up to both the lateral boundary $\partial\Omega \times (0, T]$, and $\Omega \times \{0\}$, and in addition have the property that for every cornerpoint $y_0 \in \partial\Omega \times \{0\}$,

$$0 \geq \limsup_{y \in Q_T, y \to y_0} u(y) \geq \liminf_{y \in Q_T, y \to y_0} u(y) \geq \underline{u}(y_0).$$

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1. Consider the problem

$$(D)\begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0; \\ u(0,t) = u(1,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & 0 \le x \le 1. \end{cases}$$

Ignoring the initial condition for the moment, find all solutions with separate variables, that is $u(x,t) = f(x)g(t)$. Use these solutions to write down a Fourier series expansion for the solution of (D) in terms of the Fourier series expansion of u_0.

2. Consider the problem

$$\begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0; \\ u_x(0,t) = u(1,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & 0 \le x \le 1. \end{cases}$$

Carry out the same programme as in 1.

3. Consider the problem

$$(D)\begin{cases} u_{tt} = u_{xx} & 0 < x < 1,\ t > 0; \\ u(0,t) = u(1,t) = 0 & t > 0; \\ u(x,0) = \alpha(x) & 0 \le x \le 1; \\ u_t(x,0) = \beta(x) & 0 \le x \le 1. \end{cases}$$

Ignoring the initial conditions, find all solutions with separate variables, that is $u(x,t) = f(x)g(t)$. Use these solutions to write down a Fourier series expansion for the solution of (D) in terms of the Fourier series expansions of α and β.

4. Write the Laplacian in $\mathbf{R}^2$ in polar coordinates ($x = r\cos\theta,\ x = r\sin\theta$). Consider the problem

$$\begin{cases} u_t = u_{xx} + u_{yy} & x^2 + y^2 < 1,\ t > 0; \\ u(x,y,t) = 0 & x^2 + y^2 = 1,\ t > 0; \\ u(x,y,0) = u_0(x,y) & x^2 + y^2 \le 1. \end{cases}$$

Ignoring the initial conditions, find solutions with separate variables, that is $u(x,t) = f(r)h(\theta)g(t)$.

N.B. Let $n = 0, 1, 2, 3, \ldots$. The Bessel function of order n is given by $J_n(x) =$

$$\frac{x^n}{2^n n!}\left(1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \frac{x^6}{2\cdot 4\cdot 6\cdot(2n+2)(2n+4)(2n+6)} + \ldots,\right.$$

and solves the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0. \qquad (B_n)$$

Any solution of (B_n) which is continuous in $x=0$, is a multiple of $J_n(x)$. The series expansion of $J_n(x)$ is valid for all x. For $x>0$, $J_n(x)$ has only simple zeros, which form a sequence

$$\alpha_1^{(n)} < \alpha_2^{(n)} < \alpha_3^{(n)} < \alpha_4^{(n)} < \ldots \to \infty.$$

Finally we mention the reduction formulas

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x) \text{ and } \frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x).$$

5. Evaluate the problem

$$\begin{cases} u_{tt} = u_{xx} & x>0,\ t>0; \\ u(0,t) = \sin t & t>0; \\ u(x,0) = 0 & x>0; \\ u_t(x,0) = 0 & x>0. \end{cases}$$

6. Evaluate the problem

$$\begin{cases} u_{tt} = u_{xx} & 0<x<3\pi,\ t>0; \\ u(0,t) = \sin t & 0<t<\pi; \\ u(0,t) = 0 & t\geq\pi; \\ u(3\pi,t) = 0 & t>0; \\ u(x,0) = 0 & 0<x<3\pi; \\ u_t(x,0) = 0 & 0<x<3\pi. \end{cases}$$

7. Evaluate the problem

$$(D)\begin{cases} u_{tt} = u_{xx} & 0<x<3\pi,\ t>0; \\ u(0,t) = u(3\pi,t) = 0 & t>0; \\ u(x,0) = 0 & 0\leq x\leq\pi; \\ u(x,0) = \sin x & \pi\leq x\leq 2\pi; \\ u(x,0) = 0 & 2\pi\leq x\leq 3\pi; \\ u_t(x,0) = 0 & 0\leq x\leq 3\pi. \end{cases}$$

8. Evaluate the problem

$$(D)\begin{cases} u_{tt} = u_{xx} & 0<x<4\pi,\ t>0; \\ u(0,t) = u(4\pi,t) = 0 & t>0; \\ u(x,0) = 0 & 0\leq x\leq\pi; \\ u(x,0) = \sin x & \pi\leq x\leq 2\pi; \\ u(x,0) = 0 & 2\pi\leq x\leq 4\pi; \\ u_t(x,0) = 0 & 0\leq x\leq 4\pi. \end{cases}$$

9. Let a, b, c be fixed real numbers, and consider the equation

$$u_{tt} + au_x + bu_t + cu = u_{xx}, \qquad u = u(x,t).$$

Show that this equation can be reduced to an equation without first order terms by substituting

$$u(x,t) = e^{Ax+Bt} w(x,t).$$

10. Consider the equation

$$u_{tt} = u_{xx} - u, \qquad u = u(x,t).$$

Let ω and k be two parameters. Substitute

$$u(x,t) = U(kx - \omega t),$$

and evaluate. In particular, classify all bounded solutions of this form. What are their initial values for u and u_t?

11. Let c be a fixed real number.
(i) Show that the equation

$$u_{tt} + cu = u_{xx}, \qquad u = u(x,t),$$

can be reduced to the equation

$$w_{\xi\eta} + \lambda w = 0, \qquad w = w(\xi,\eta),$$

where $\lambda = c/4$.
(ii) Let $\lambda > 0$. Substitute $w(\xi,\eta) = f(2\sqrt{\lambda\xi\eta})$, and derive an equation for f. What is the solution of this equation satisfying $f(0) = 1$?
(iii) Find a solution of

$$\begin{cases} u_{tt} = u_{xx} - cu & -t < x < t, \ t > 0; \\ u(-t,t) = u(t,t) = 1 & t > 0. \end{cases}$$

(iv) Let $U(x,t)$ be the solution of (iii), and let

$$E(x,t) = \begin{cases} \frac{1}{2}U(x,t) & -t \le x \le t, \quad t \ge 0; \\ 0 & \text{otherwise.} \end{cases}$$

Compute $E_{tt} - E_{xx}$ in the sense of distributions. (Hint: think of the case $\lambda = 0$.)
(v) Find a solution formula for the Cauchy problem

$$\begin{cases} u_{tt} = u_{xx} - u & x,t \in \mathbb{R}; \\ u(x,0) = \alpha(x) & x \in \mathbb{R}; \\ u_t(x,0) = \beta(x) & x \in \mathbb{R}. \end{cases}$$

12. In the context of the previous exercise discuss the case $\lambda < 0$, and find a solution formula for the Cauchy problem

$$\begin{cases} u_{tt} = u_{xx} + u & x, t \in \mathbb{R}; \\ u(x,0) = \alpha(x) & x \in \mathbb{R}; \\ u_t(x,0) = \beta(x) & x \in \mathbb{R}. \end{cases}$$

N.B. Let $n = 0, 1, 2, 3, \ldots$. The modified Bessel function of order n is given by $I_n(x) =$

$$\frac{x^n}{2^n n!}\left(1 + \frac{x^2}{2 \cdot (2n+2)} + \frac{x^4}{2 \cdot 4 \cdot (2n+2)(2n+4)} + \frac{x^6}{2 \cdot 4 \cdot 6 \cdot (2n+2)(2n+4)(2n+6)} + \ldots,\right.$$

and solves the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - (1 + \frac{n^2}{x^2})y = 0.$$

13. Let $\varepsilon > 0$. Consider the equation

$$u_t = \varepsilon u_{xx} + u u_x, \qquad u = u(x,t).$$

Substitute

$$u(x,t) = U(x - ct), \quad c \in \mathbb{R},$$

and evaluate. In particular, find all bounded solutions of this form.

14. Let $\lambda > 0$. Consider the equation

$$u_t = u_{xx} + \lambda u(1-u), \qquad u = u(x,t).$$

Substitute

$$u(x,t) = U(x - ct), \quad c \in \mathbb{R},$$

and evaluate. In particular, find all bounded solutions of this form.

15. Let $\lambda > 0$, $a \in (0,1)$. Consider the equation

$$u_t = u_{xx} + \lambda u(1-u)(u-a), \qquad u = u(x,t).$$

Substitute

$$u(x,t) = U(x - ct), \quad c \in \mathbb{R},$$

and evaluate. In particular, (try to) find all bounded solutions of this form.

16. Consider the nonlinear diffusion equation

$$u_t = (u^m)_{xx}, \quad u = u(x,t) \geq 0, \quad m > 1.$$

Subsitute

$$u(x,t) = t^{-\alpha} U(\eta), \quad \eta = \sqrt{\beta} x t^{-\beta}.$$

(i) What is the condition on α and β you need in order to derive an ordinary differential equation for $U(\eta)$.
(ii) What is the condition on α and β you need in order to have that the integral of $u(x,t)$ with respect to x (if it exists) is constant in time?
(iii) Solve the differential equation for $U(\eta)$ for α and β satisfying both the conditions obtained in (i) and (ii), when $U(0) > 0$ and $U'(0) = 0$.
(iv) What would be the corresponding solution of the parabolic equation? Evaluate its properties, e.g. in relation to the fundamental solution of $u_t = u_{xx}$.

17. Let $p > 1$. Consider the following problem.

$$(P) \begin{cases} \Delta u + u^p = 0 & \text{in} \quad B, \\ u > 0 & \text{in} \quad B, \\ u = 0 & \text{in} \quad \partial B, \end{cases}$$

where $B = \{x \in \mathbb{R}^n : x_1^2 + \ldots x_n^2 < 1\}$. We restrict our attention to radially symmetric solutions $u = U(r)$, $r = (x_1^2 + \ldots x_n^2)^{1/2}$.
(i) Show that the transformation

$$X = \frac{rU'(r)}{U(r)}; \quad Y = r^2 U(r)^{p-1}; \quad t = \log r$$

leads to the quadratic system

$$(Q) \begin{cases} X' = \dfrac{dX}{dt} = X(2 - n - X) - Y, \\ Y' = \dfrac{dY}{dt} = Y(2 + (p-1)X). \end{cases}$$

(ii) Show that radial solutions correspond to orbits of (Q) in the upper half plane coming out of $(0,0)$.
(iii) Show that there is only one such orbit γ.
(iv) Show that for $p = \frac{n+2}{n-2}$ all orbits of (Q) are symmetric in the line $X = \frac{2-n}{2}$.
(v) Show that γ escapes to infinity in finite time through the second quadrant along the negative X-axis, if and only if $p < \frac{n+2}{n-2}$ ($p < \infty$ if $n = 2$).
(vi) Show that (P) has a radial solution if and only if $p < \frac{n+2}{n-2}$ ($p < \infty$ if $n = 2$).
(vii) Show that this solution is unique.
(viii) Show that (P) has a radial solution for every annulus if $n > 2$ and $p = \frac{n+2}{n-2}$.

18. Show that for a function $u \in C(\Omega)$ the following three statements are equivalent:

(i) $u \leq h$ on B for every ball $B \subset\subset \Omega$ and every harmonic $h \in C(\overline{B})$ with $u \leq h$ on ∂B;

(ii) for every nonnegative compactly supported function $\phi \in C^2(\Omega)$ the inequality

$$\int_\Omega u\Delta\phi \geq 0$$

holds;

(iii) u satisfies the conclusion of the Mean Value Theorem, i.e. for every $B_R(y) \subset\subset \Omega$ the inequality

$$u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x)dS(x)$$

holds.

Hint: In order to deal with (ii) show that it is equivalent to the existence of a sequence $(\Omega_n)_{n=1}^\infty$ of strictly increasing domains, and a corresponding sequence of subharmonic functions $(u_n)_{n=1}^\infty \in C^\infty(\Omega_n)$, with the property that for every compact $K \subset \Omega$ there exists an integer N such that $K \subset \Omega_N$, and moreover, the sequence $(u_n)_{n=N}^\infty$ converges uniformly to u on K.

19. Find a flow in $\Omega = \{(x, y) : \ y > 0; \ y > 1 \text{ if } x = 0\}$, which at infinity has a velocity equal to one parallel to the x-axis. Discuss the behaviour of the velocity near the points $(0, 0)$ and $(0, 1)$.

20. Find a flow in $\Omega = \{(x, y) : \ y > 0; \ x^2 + y^2 > 1\}$, which at infinity has a velocity equal to one parallel to the x-axis. Hint: consider the complex function $z \to z + \frac{1}{z}$.

21. Find a flow in $\Omega = \{(x, y) : \ x^2 + y^2 > 1\}$, which at infinity has a velocity equal to one parallel to the x-axis.

22. Find nontrivial flows in $\Omega = \{(x, y) : \ x > 0; \ y > 0\}$. Discuss the behaviour of the velocity near the corner point $(0, 0)$.

23. Find nontrivial flows in $\Omega = \{(x, y) : \ y > 0\} \cup \{(x, y) : \ x < 0\}$. Discuss the behaviour of the velocity near the corner point $(0, 0)$.

24. Compute the solution of

$$\begin{cases} \Delta u + 1 = 0 & \text{in} \quad B, \\ u = 0 & \text{in} \quad \partial B, \end{cases}$$

where $B = \{x \in \mathbb{R}^n : \ x_1^2 + \ldots x_n^2 < 1\}$.

25. Let $0 < \epsilon < 1$. Compute the solution of

$$\begin{cases} \Delta u + 1 = 0 & \text{in} \quad \Omega_\epsilon, \\ u = 0 & \text{in} \quad \partial\Omega_\epsilon, \end{cases}$$

where $\Omega_\epsilon = \{x \in \mathbb{R}^n : \epsilon < x_1^2 + \dots x_n^2 < 1\}$. Examine the behaviour of the solution as $\epsilon \to 0$.

26. Show that

$$\begin{cases} \Delta u + 1 + x^2 = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 < 1\}$, has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and that $0 < u(x, y) < \frac{1}{4}$ for all (x, y) in Ω.

27. Show that a harmonic function $u \in C(\mathbb{R}^n)$ for which

$$\inf_{|x|<R} u(x) \geq -M(R),$$

with

$$\lim_{R\to\infty} \frac{M(R)}{R} = 0,$$

has to be constant. Hint: take a fixed y, let $R > |y|$ and consider the function $u(x) + M(R)$ on $B_d(y)$, where $d = R - |y|$.

28. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : y^2 < 1\}$. Show that the only bounded harmonic function $u \in C(\overline{\Omega})$ with $u \equiv 0$ on $\partial\Omega$, is $u \equiv 0$. Hint: consider the functions $w_\epsilon(x) = \epsilon \cosh x \cos y$.

29. Show that

$$\begin{cases} \Delta u + 1 = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : y^2 < 1\}$, has a unique bounded solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and compute u.

30. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < \pi\}$. Find a bounded harmonic function $u \in C(\overline{\Omega} - \{(0, 0)\})$ satisfying $u(x, \pi) = 0$, and $u(x, 0) = H(x)$. Hint: consider the complex function $z \to e^z - 1$.

31. Let Ω be a bounded domain, let $y \in \Omega$, and let $\Omega' = \{x \in \Omega : x \neq y\}$. Show that the only bounded harmonic function $u \in C(\overline{\Omega} - \{y\})$ with $u \equiv 0$ on $\partial\Omega$, is $u \equiv 0$. Hint: consider ($n \geq 3$) functions of the form $M(\frac{\epsilon}{|x-y|})^{n-2}$ on $\Omega - B_\epsilon(y)$.