1. Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Denote the outward normal on $\partial\Omega$ by ν. The divergence theorem says that for $v \in C^1(\overline{\Omega}, \mathbb{R}^m)$

$$\int_\Omega \nabla \cdot v = \int_{\partial\Omega} v \cdot \nu$$

Consider the problem

$$-\Delta u = f \quad \text{in } \Omega \tag{1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{2}$$

Suppose that $u \in C^2(\overline{\Omega})$ is a classical solution of Problem (1,2).

(i) Derive an integral condition (IC) that f must satisfy, and that Problem (1,2) also has a solution which satisfies the same integral condition.

(ii) Suppose that also $\phi \in C^2(\overline{\Omega})$. Evaluate

$$\int_\Omega \nabla u \cdot \nabla \phi \tag{3}$$

Writing $\nabla u = (D_1 u, \ldots, D_m u)$, let

$$H^1(\Omega) = \{u \in L^2(\Omega) : \ D_1 u, \ldots, D_m u \in L^2(\Omega)\}$$

with the (standard Sobolev) inner product norm

$$||u||_1 = \left(\int_\Omega \left(|u|^2 + |\nabla u|^2 \right) \right)^{\frac{1}{2}}$$

This space is compactly embedded in $L^2(\Omega)$, meaning that a sequence which is bounded in $H^1(\Omega)$, has a subsequence convergent in $L^2(\Omega)$.

(iii) Which integral equality for u and arbitrary $\phi \in H^1(\Omega)$ would you suggest as the defining property for a function $u \in H^1(\Omega)$ to be a weak solution of Problem (1,2)?

(iv) Show that (3) defines an inner product on

$$\tilde{H}^1(\Omega) = \{u \in H^1(\Omega) : \ u \text{ satisfies (IC)}\}$$

The inner product norm corresponding to (3) will be equivalent to the norm $||\cdot||_1$ on $\tilde{H}^1(\Omega)$, provided there exists a constant C such that for all $u \in \tilde{H}^1(\Omega)$ the following inequality holds:

$$\int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2$$

(v) Show, arguing by contradiction and using the compactness of the embedding $\tilde{H}^1(\Omega) \to L^2(\Omega)$, that there is no sequence $u_n \in \tilde{H}^1(\Omega)$ which has $\int_\Omega |u_n|^2 = 1$ and $\int_\Omega |\nabla u_n|^2 \to 0$. Deduce that indeed both norms are equivalent on $\tilde{H}^1(\Omega)$.

(vi) Let $f \in L^2(\Omega)$ satisfy (IC). Show, applying the Riesz representation theorem in $\tilde{H}^1(\Omega)$, that Problem (1,2) has a weak solution in $H^1(\Omega)$ which is unique up to an additive constant.

(vii) Evaluate the consequences of dropping the assumption that f satisfies (IC). Is it really the proof in $\tilde{H}^1(\Omega)$ that fails?

2. With Ω as above consider the problem

$$\Delta\Delta u = f \quad \text{in } \Omega \tag{4}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{5}$$

Let

$$H^2(\Omega) = \{u \in L^2(\Omega) : \, D_i u, D_{ij} u \in L^2(\Omega), \, i,j = 1 \ldots m\}$$

with the (standard Sobolev) inner product norm

$$||u||_2 = \left(\int_\Omega \left(|u|^2 + \sum_{i=1}^{m} |D_i u|^2 + \sum_{i,j=1}^{m} |D_{ij} u|^2 \right) \right)^{\frac{1}{2}}$$

The space $H_0^2(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^2(\Omega)$. Discuss how you would formulate (and establish the unique) existence of weak solutions of Problem (4,5) in $H_0^2(\Omega)$. Hint: show for functions $u, v \in C_c^\infty(\Omega)$ that $\int_\Omega \sum_{i,j=1}^m D_{ij} u D_{ij} v = \int_\Omega \Delta u \Delta v$, that on $C_c^\infty(\Omega)$ the corresponding norm is equivalent to the $|| \cdot ||_2$-norm, and apply the Riesz theorem to the appropriate weak formulation in $H_0^2(\Omega)$.