1. Harmonic functions

Throughout this section, $\Omega \subset \mathbb{R}^n$ is a bounded domain.

1.1 Definition A function $u \in C^2(\Omega)$ is called *subharmonic* if $\Delta u \geq 0$ in Ω, *harmonic* if $\Delta u \equiv 0$ in Ω, and *superharmonic* if $\Delta u \leq 0$ in Ω.

1.2 Notation The measure of the unit ball in $\mathbb{R}^n$ is

$$\omega_n = |B_1| = |\{x \in \mathbb{R}^n : x_1^2 + ... + x_n^2 \leq 1\}| = \int_{B_1} dx = \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

The $(n-1)$-dimensional measure of the boundary ∂B_1 of B_1 is equal to $n\omega_n$.

1.3 Mean Value Theorem Let $u \in C^2(\Omega)$ be subharmonic, and

$$\overline{B_R(y)} = \{x \in \mathbb{R}^n : |x-y| \leq R\} \subset \Omega.$$

Then

$$u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x)dS(x),$$

where dS is the $(n-1)$-dimensional surface element on $\partial B_R(y)$. Also

$$u(y) \leq \frac{1}{\omega_n R^n} \int_{B_R(y)} u(x)dx.$$

Equalities hold if u is harmonic.

Proof We may assume $y = 0$. Let $\rho \in (0, R)$. Then

$$0 \leq \int_{B_\rho} \Delta u(x)dx = \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}(x)dS(x) = \int_{\partial B_\rho} \frac{\partial u}{\partial r}(x)dS(x) =$$

(substituting $x = \rho\omega$)

$$\int_{\partial B_1} \frac{\partial u}{\partial r}(\rho\omega)\rho^{n-1}dS(\omega) = \rho^{n-1} \int_{\partial B_1} (\frac{\partial}{\partial \rho} u(\rho\omega))dS(\omega) = \rho^{n-1} \frac{d}{d\rho} \int_{\partial B_1} u(\rho\omega)dS(\omega)$$

(substituting $\omega = x/\rho$)

$$= \rho^{n-1} \frac{d}{d\rho} \frac{1}{\rho^{n-1}} \int_{\partial B_\rho} u(x)dS(x),$$

which implies, writing

$$f(\rho) = \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho} u(x)dS(x),$$

that $f'(\rho) \geq 0$. Hence

$$u(0) = \lim_{\rho \downarrow 0} f(\rho) \leq f(R) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u(x) dS(x),$$

which proves the first inequality. The second one follows from

$$\int_{B_R} u(x)dx = \int_0^R \{\int_{\partial B_\rho} u(x)dS(x)\} d\rho \geq \int_0^R n\omega_n \rho^{n-1} u(0) d\rho = \omega_n R^n u(0).$$

This completes the proof. ■

1.4 Corollary (Strong maximum principle for subharmonic functions) Let $u \in C^2(\Omega)$ be bounded and subharmonic. If for some $y \in \Omega$, $u(y) = \sup_\Omega u$, then $u \equiv u(y)$.

Proof Exercise (hint: apply the mean value theorem to the function $\tilde{u}(x) = u(x) - u(y)$, and show that the set $\{x \in \Omega : \tilde{u}(x) = 0\}$ is open). ■

1.5 Corollary (weak maximum principle for subharmonic functions) Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is subharmonic. Then

$$\sup_\Omega u = \max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof Exercise. ■

1.6 Corollary Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be subharmonic. If $u \equiv 0$ on $\partial\Omega$, then $u < 0$ on Ω, unless $u \equiv 0$ on $\overline{\Omega}$.

Proof Exercise. ■

1.7 Corollary Let $\varphi \in C(\partial\Omega)$. Then there exists at most one function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $u = \varphi$ on $\partial\Omega$.

Proof Exercise. ■

1.8 Theorem (Harnack inequality) Let $\Omega' \subset\subset \Omega$ (i.e. $\Omega' \subset \overline{\Omega'} \subset \Omega$) be a subdomain. Then there exists a constant C which only depends on Ω' and Ω, such that for all harmonic nonnegative functions $u \in C^2(\Omega)$,

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

Proof Suppose that $\overline{B_{4R}(y)} \subset \Omega$. Then for any $x_1, x_2 \in B_R(y)$ we have

$$B_R(x_1) \subset B_{3R}(x_2) \subset B_{4R}(y) \subset \Omega,$$

so that by the mean value theorem,

$$u(x_1) = \frac{1}{\omega_n R^n} \int_{B_R(x_1)} u(x)dx \leq \frac{3^n}{\omega_n (3R)^n} \int_{B_{3R}(x_2)} u(x)dx = 3^n u(x_2).$$

Hence, x_1, x_2 being arbitrary, we conclude that

$$\sup_{B_R(y)} u \leq 3^n \inf_{B_R(y)} u.$$

Thus we have shown that for $\Omega' = B_R(y)$, with $B_{4R}(y) \subset \Omega$, the constant in the inequality can be taken to be 3^n. Since any $\Omega' \subset\subset \Omega$ can be covered with finitely many of such balls, say

$$\Omega' \subset B_{R_1}(y_1) \cup B_{R_2}(y_2) \cup \ldots \cup B_{R_N}(y_N),$$

we obtain for Ω' that $C = 3^{nN}$. ∎

Next we turn our attention to radially symmetric harmonic functions. Let $u(x)$ be a function of $r = |x|$ alone, i.e. $u(x) = U(r)$. Then u is harmonic if and only if

$$0 = \Delta u(x) = \sum_{i=1}^{n} (\frac{\partial}{\partial x_i})^2 u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(U'(r)\frac{\partial r}{\partial x_i}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\frac{x_i}{r}U'(r))$$

$$= \sum_{i=i}^{n} (\frac{1}{r}U'(r) + \frac{x_i}{r}U''(r)\frac{\partial r}{\partial x_i} - x_i U'(r)\frac{\partial}{\partial x_i}(\frac{1}{r}))$$

$$= \frac{n}{r}U'(r) + \sum_{i=1}^{n} \frac{x_i^2}{r}U''(r) - \sum_{i=1}^{n} x_i U'(r)\frac{x_i}{r^3} =$$

$$U''(r) + \frac{n-1}{r}U'(r) = \frac{1}{r^{n-1}}(r^{n-1}U'(r))',$$

implying

$$r^{n-1}U'(r) = C_1,$$

so that

$$U(r) = \begin{cases} C_1 r + C_2 & n = 1; \\ C_1 \log r + C_2 & n = 2; \\ \dfrac{C_1}{2-n}\dfrac{1}{r^{n-2}} + C_2 & n > 2. \end{cases} \tag{1.1}$$

We define the *fundamental solution* by

$$\Gamma(x) = \begin{cases} \frac{1}{2}|x| & n = 1 \\ \frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{n\omega_n(2-n)}\frac{1}{|x|^{n-2}} & n > 2, \end{cases} \tag{1.2}$$

i.e. $C_1 = 1/n\omega_n$ and $C_2 = 0$ in (1.1). Whenever convenient we write $\Gamma(x) = \Gamma(|x|) = \Gamma(r)$.

1.9 Theorem The fundamental solution Γ is a solution of the equation $\Delta\Gamma = \delta$ in the sense of distributions, i.e.

$$\int_{\mathbb{R}^n} \Gamma(x)\Delta\psi(x)dx = \psi(0) \qquad \forall \psi \in D(\mathbb{R}^n).$$

Proof First observe that for all $R > 0$, we have $\Gamma \in L^\infty(B_R)$ if $n = 1$, $\Gamma \in L^P(B_R)$ for all $1 \le p < \infty$ if $n = 2$, and $\Gamma \in L^P(B_R)$ for all $1 \le p < \frac{n}{n-2}$ if $n > 2$, so for all ψ in $D(\mathbb{R}^n)$, choosing R large enough, we can compute

$$\int_{\mathbb{R}^n} \Gamma(x)\Delta\psi(x)dx = \int_{B_R} \Gamma(x)\Delta\psi(x)dx = \lim_{\rho\downarrow 0}\int_{A_{R,\rho}} \Gamma(x)\Delta\psi(x)dx =$$

(here $A_{R,\rho} = \{x \in B_R : |x| > \rho\}$)

$$\lim_{\rho\downarrow 0}\{\int_{\partial A_{R,\rho}} \Gamma\frac{\partial\psi}{\partial\nu} - \int_{A_{R,\rho}} \nabla\Gamma\nabla\psi\} = \lim_{\rho\downarrow 0}\{\int_{\partial A_{R,\rho}} (\Gamma\frac{\partial\psi}{\partial\nu} - \frac{\partial\Gamma}{\partial\nu}\psi) + \int_{A_{R,\rho}} \psi\Delta\Gamma\} =$$

$$\lim_{\rho\downarrow 0}\int_{\partial B_\rho}\{\frac{-\partial\psi/\partial\nu}{n\omega_n(2-n)\rho^{n-2}} + \frac{\psi}{n\omega_n\rho^{n-1}}\} = \psi(0).$$

For $n = 1, 2$ the proof is similar. ∎

Closely related to this theorem we have

1.10 Theorem (Green's representation formula) Let $u \in C^2(\overline{\Omega})$ and suppose $\partial\Omega \in C^1$. Then, if ν is the outward normal on $\partial\Omega$, we have

$$u(y) = \int_{\partial\Omega}\{u(x)\frac{\partial}{\partial\nu}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial\nu}(x)\}dS(x) + \int_\Omega \Gamma(x-y)\Delta u(x)dx.$$

Here the derivatives are taken with respect to the x-variable.

Proof Exercise (Hint: take $y = 0$, let $\Omega_\rho = \{x \in \Omega : |x| > \rho\}$, and imitate the previous proof). ■

If we want to solve $\Delta u = f$ on Ω for a given function f, this representation formula strongly suggests to consider the convolution

$$\int_\Omega \Gamma(x - y) f(x) dx$$

as a function of y, or equivalently,

$$(\Gamma * f)(x) = \int_\Omega \Gamma(x - y) f(y) dy \tag{1.3}$$

as a function of x. This convolution is called the *Newton potential* of f.

For any harmonic function $h \in C^2(\overline{\Omega})$ we have

$$\int_\Omega h \Delta u = \int_{\partial\Omega} (h \frac{\partial u}{\partial \nu} - u \frac{\partial h}{\partial \nu}),$$

so that, combining with Green's representation formula,

$$u(y) = \int_{\partial\Omega} \{u \frac{\partial G}{\partial \nu} - G \frac{\partial u}{\partial \nu}\} + \int_\Omega G \Delta u, \tag{1.4}$$

where $G = \Gamma(x - y) + h(x)$. The trick is now to take instead of a function $h(x)$ a function $h(x, y)$ of two variables $x, y \in \overline{\Omega}$, such that h is harmonic in x, and for every $y \in \Omega$,

$$G(x, y) = \Gamma(x - y) + h(x, y) = 0 \qquad \forall x \in \partial\Omega.$$

This will then give us the solution formula

$$u(y) = \int_{\partial\Omega} u \frac{\partial G}{\partial \nu} + \int_\Omega G \Delta u.$$

In particular, if $u \in C^2(\overline{\Omega})$ is a solution of

$$(D) \begin{cases} \Delta u = f & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

then

$$u(y) = \int_{\partial\Omega} \varphi(x) \frac{\partial G(x, y)}{\partial \nu} + \int_\Omega G(x, y) f(x) dx. \tag{1.5}$$

The function $G(x, y) = \Gamma(x - y) + h(x, y)$ is called the *Green's function* for the Dirichletproblem. Of course $h(x, y)$ is by no means trivial to find. The function h is called the *regular part* of the Green's function. If we want to solve

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

(1.5) reduces to

$$u(y) = \int_{\partial\Omega} \varphi(x) \frac{\partial G(x, y)}{\partial \nu} dS(x). \tag{1.6}$$

We shall evaluate (1.6) in the case that $\Omega = B = B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$. Define the *reflection* in ∂B by

$$S(y) = \frac{y}{|y|^2} \quad \text{if} \quad y \neq 0; \quad S(0) = \infty; \quad S(\infty) = 0. \tag{1.7}$$

Here ∞ is the point that has to be added to $\mathbb{R}^n$ in order to construct the one-point compactification of $\mathbb{R}^n$. If $0 \neq y \in B$, then $\overline{y} = S(y)$ is uniquely determined by asking that 0, y and $\overline{y}$ lie (in that order) on one line l, and that the boundary ∂B of B is tangent to the cone C with top $\overline{y}$ spanned by the circle obtained from intersecting the ball B with the plane perpundicular to l going through y (you'd better draw a picture here). Indeed if $\overline{x}$ lies on this circle, then the triangles $0y\overline{x}$ and $0\overline{x}\overline{y}$ are congruent and

$$|y| = \frac{|y - 0|}{|\overline{x} - 0|} = \frac{|\overline{x} - 0|}{|\overline{y} - 0|} = \frac{1}{|\overline{y}|},$$

so that $\overline{y} = S(y)$. It is also easily checked that

$$\partial B = \{x \in \mathbb{R}^n ;\ |x - y| = |y||x - \overline{y}|\}.$$

But then the construction of $h(x, y)$ is obvious. We simply take

$$h(x, y) = -\Gamma(|y|(x - \overline{y})),$$

so that

$$G(x, y) = \Gamma(x - y) - \Gamma(|y|(x - \overline{y})). \tag{1.8}$$

Note that since $|y||\overline{y}| = 1$, and since $y \to 0$ implies $\overline{y} \to \infty$, we have, with a slight abuse of notation, that $G(x, 0) = \Gamma(x) - \Gamma(1)$. It is convenient to rewrite $G(x, y)$ as

$$\begin{aligned} G(x, y) &= \Gamma(\sqrt{|x|^2 + |y|^2 - 2xy}) - \Gamma(\sqrt{|x|^2|y|^2 + |y|^2|\overline{y}|^2 - 2|y|^2 x\overline{y}}) \\ &= \Gamma(\sqrt{|x|^2 + |y|^2 - 2xy}) - \Gamma(\sqrt{|x|^2|y|^2 + 1 - 2xy}), \end{aligned}$$

which shows that G is symmetric in x and y. In particular G is also harmonic in the y variables.

Next we compute $\partial G/\partial\nu$ on ∂B. We write

$$r = |x-y|; \quad \overline{r} = |x-\overline{y}|; \quad \frac{\partial}{\partial\nu} = \nu\cdot\nabla = \sum_{i=1}^{n}\nu_i\frac{\partial}{\partial x_i}; \quad \frac{\partial r}{\partial x_i} = \frac{x_i-y_i}{r}; \quad \frac{\partial\overline{r}}{\partial x_i} = \frac{x_i-\overline{y}_i}{\overline{r}},$$

so that since $G = \Gamma(r) - \Gamma(|y|\overline{r})$,

$$\frac{\partial\Gamma(r)}{\partial\nu} = \Gamma'(r)\frac{\partial r}{\partial\nu} = \frac{1}{n\omega_n r^{n-1}}\sum_{i=1}^{n}x_i\frac{x_i-y_i}{r} = \frac{1-xy}{n\omega_n r^n},$$

and

$$\frac{\partial\Gamma(|y|\overline{r}|)}{\partial\nu} = \Gamma'(|y|\overline{r})|y|\frac{\partial\overline{r}}{\partial\nu} = \frac{1}{n\omega_n|y|^{n-2}\overline{r}^{n-1}}\frac{\partial\overline{r}}{\partial\nu} = \frac{1}{n\omega_n|y|^{n-2}\overline{r}^{n-1}}\sum_{i=1}^{n}x_i\frac{x_i-\overline{y}_i}{\overline{r}}$$

$$= \frac{1}{n\omega_n}|y|^{2-n}\overline{r}^{-n}\sum_{i=1}^{n}x_i(x_i-\overline{y}_i) = \quad \text{(substituting } r=|y|\overline{r}\text{)}$$

$$\frac{1}{n\omega_n}|y|^{2-n}(\frac{|y|}{r})^n\sum_{i=1}^{n}(x_i^2-x_i\overline{y}_i) = \frac{1}{n\omega_n r^n}\{|y|^2-xy\},$$

whence

$$\frac{\partial G(x,y)}{\partial\nu(x)} = \frac{1}{n\omega_n r^n}(1-|y|^2) = \frac{1-|y|^2}{n\omega_n|x-y|^n}. \tag{1.9}$$

1.11 Theorem (Poisson integration formula) Let $\varphi\in C(\partial B)$. Define $u(y)$ for $y\in B$ by

$$u(y) = \frac{1-|y|^2}{n\omega_n}\int_{\partial B}\frac{\varphi(x)}{|x-y|^n}dS(x),$$

and for $y\in\partial B$, by $u(y)=\varphi(y)$. Then $u\in C^2(B)\cup C(\overline{B})$, and $\Delta u = 0$ in B.

Proof First we show that $u\in C^\infty(B)$ and that $\Delta u=0$ in B. We have

$$u(y) = \frac{1-|y|^2}{n\omega_n}\int_{\partial B}\frac{\varphi(x)}{|x-y|^n}dS(x) = \int_{\partial B}K(x,y)\varphi(x)dS(x),$$

where the integrand is smooth in $y\in B$, and $K(x,y)$ is positive, and can be written as

$$K(x,y) = \frac{\partial G(x,y)}{\partial\nu(x)} = \sum_{i=1}^{n}x_i\frac{\partial G(x,y)}{\partial x_i}.$$

Thus $u \in C^\infty(B)$ and

$$\Delta u(y) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial y_j} = \sum_{j=1}^{n} \Big(\frac{\partial}{\partial y_j}\Big)^2 \int_{\partial B} \sum_{i=1}^{n} x_i \frac{\partial G(x,y)}{\partial x_i} \varphi(x) dS(x)$$

$$= \int_{\partial B} \{ \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \frac{\partial^2 G(x,y)}{\partial y_j^2} \} \varphi(x) dS(x) = 0.$$

Next we show that $u \in C(\overline{B})$. Observe that

$$\int_{\partial B} K(x,y) dS(x) = 1,$$

because $\tilde{u} \equiv 1$ is the unique harmonic function with $\tilde{u} \equiv 1$ on the boundary. We have to show that for all $x_0 \in \delta B$

$$\lim_{\substack{y \to x_0 \\ y \in B}} u(y) = \varphi(x_0) = u(x_0),$$

so we look at

$$u(y) - u(x_0) = \int_{\partial B} K(x,y)(\varphi(x) - \varphi(x_0)) dS(x).$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(x_0)| < \varepsilon \ \text{ for all } x \in \delta B \text{ with } \ |x - x_0| < \delta.$$

Thus we have, with $M = \max_{\partial B} |\varphi|$, that

$$|u(y) - u(x_0)| \le \int_{x \in \partial B,\ |x - x_0| < \delta} K(x,y) |\varphi(x) - \varphi(x_0)| dS(x)$$

$$+ \int_{x \in \partial B,\ |x - x_0| \ge \delta} K(x,y) |\varphi(x) - \varphi(x_0)| dS(x) \le$$

$$\int_{\partial B} K(x,y) \varepsilon dS(x) + \int_{x \in \partial B,\ |x - x_0| \ge \delta} K(x,y) 2M dS(x) =$$

$$\varepsilon + 2M \int_{x \in \partial B,\ |x - x_0| \ge \delta} K(x,y) dS(x) \le \quad (\text{choosing } \ y \in B \text{ with } \ |y - x_0| < \frac{\delta}{2}\ )$$

$$\varepsilon + 2M \int_{x \in \partial B,\ |x - y| \ge \frac{\delta}{2}} \frac{1 - |y|^2}{n\omega_n |x - y|^n} dS(x) \le \varepsilon + 2M \frac{1 - |y|^2}{n\omega_n} \int_{\partial B} \Big(\frac{2}{\delta}\Big)^n dS(x) =$$

$$\varepsilon + 2M \Big(\frac{2}{\delta}\Big)^n (1 - |y|^2) \to \varepsilon \ \text{ as } \ y \to x_0.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. ■

1.12 Remark On the ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ the Poisson formula reads

$$u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{\varphi(x)}{|x-y|^n} dS(x).$$

1.13 Corollary A function $u \in C(\Omega)$ is harmonic if and only if

$$u(y) = \frac{1}{\omega_n R^n} \int_{B_R(y)} u(x)dx$$

for all $B_R(y) \subset\subset \Omega$.

Proof Exercise (hint: use Poisson's formula in combination with the weak maximum principle which was proved using the mean value (in-)equalities). ▪

1.14 Corollary Uniform limits of harmonic functions are harmonic.

Proof Exercise. ▪

1.15 Corollary (Harnack convergence theorem) For a nondecreasing sequence of harmonic functions $u_n : \Omega \to \mathbb{R}$ to converge to a harmonic limit function u, uniformly on compact subsets, it is sufficient that the sequence $(u_n(y))_{n=1}^{\infty}$ is bounded for just one point $y \in \Omega$.

Proof Exercise (hint: use Harnack's inequality to establish convergence). ▪

1.16 Corollary If $u : \Omega \to \mathbb{R}$ is harmonic, and $\Omega' \subset\subset \Omega$, $d =$ distance $(\Omega', \partial\Omega)$, then

$$\sup_{\Omega'} |\nabla u| \leq \frac{n}{d} \sup_{\Omega} |u|.$$

For higher order derivatives the factor $\frac{n}{d}$ has to be replaced by $(\frac{ns}{d})^s$, where s is the order of the derivative.

Proof Since $\Delta\nabla u = \nabla\Delta u = 0$, we have by the mean value theorem for $y \in \Omega'$

$$\big|\nabla u(y)\big| = \big|\frac{1}{\omega_n d^n} \int_{B_d(y)} \nabla u(x)dx\big| =$$

(by the vector valued version of Gauss' Theorem)

$$\big|\frac{1}{\omega_n d^n} \int_{\partial B_d(y)} u(x)\nu(x)dS(x)\big| \leq \frac{1}{\omega_n d^n} n\omega_n d^{n-1} \sup_{B_d(y)} |u(x)||\nu(x)| = \frac{n}{d} \sup_{B_d(y)} |u(x)|,$$

since ν is the unit normal. ∎

1.17 Corollary (Liouville) If $u : \mathbb{R}^n \to \mathbb{R}^+$ is harmonic, then $u \equiv$ constant.

Proof We have

$$\big|\nabla u(y)\big| = \Big|\frac{1}{w_n R^n}\int_{\partial B_R(y)} u(x)\nu(x)dS(x)\Big| \leq \frac{1}{w_n R^n}\int_{\partial B_R(y)} |u(x)\nu(x)|dS(x)$$

$$= \frac{n}{R}\frac{1}{n\omega_n R^{n-1}}\int_{\partial B_R(y)} u(x)dS(x) = \frac{n}{R}u(y)$$

for all $R > 0$. Thus $\nabla u(y) = 0$ for all $y \in \mathbb{R}^n$, so that $u \equiv$ constant. ∎

We have generalized a number of properties of harmonic functions on domains in $\mathbb{R}^2$, which follow from the following theorem for harmonic functions of two real variables.

1.18 Theorem Let $\Omega \subset \mathbb{R}^2$ be simply connnected. Suppose $u \in C(\Omega)$ is harmonic. Then there exists $v : \Omega \to \mathbb{R}$ such that

$$F(x + iy) = u(x, y) + iv(x, y)$$

is an analytic function on Ω. In particular $u, v \in C^\infty(\Omega)$ and $\Delta u = \Delta v = 0$ in Ω.

1.19 Exercise Let $u : \Omega \to \mathbb{R}$ be harmonic. Show that the function $v = |\nabla u|^2$ is subharmonic in Ω.

1.20 Exercise Adapt the proof of the mean value theorem to show that under the same assumptions, for dimension $n > 2$,

$$u(y) = \frac{1}{\omega_n R^n}\int_{B_R(y)} u(x)dx - a_n \int_{B_R(y)} G(|x-y|; R)\Delta u(x)dx,$$

where

$$G(r, R) = \frac{1}{r^{n-2}} - \frac{1}{R^{n-2}} + \frac{n-2}{2}\frac{r^2 - R^2}{R^n},$$

and $a_n > 0$ can be computed explicitly. Derive a similar formula for $n = 2$.

1.21 Exercise Show directly that the function

$$K(x, y) = \frac{R^2 - |y|^2}{n\omega_n R|x-y|^n}$$

is positive and harmonic in $y \in B = B_R$, and, without to much computation, that $\int_B K(x,y)dx = 1$. (These are the three essential ingredients in the proof of Theorem 1.11 and Remark 1.13).

1.22 Exercise Use Corollary 1.16 to prove that, if u is harmonic in Ω and $B_{2R}(x_0) \subset\subset \Omega$, then

$$|u(x) - u(y)| \leq |x-y|^\alpha (2R)^{1-\alpha} \frac{n}{R} \sup_{B_{2R}(x_0)} |u| \quad \forall x, y \in B_R(x_0).$$

1.23 Exercise (Schwarz reflection principle) Let Ω be a domain which is symmetric with respect to $x_n = 0$, and let $\Omega^+ = \Omega \cap \{x_n > 0\}$. Show that a function u which is harmonic in Ω^+, and continuous at $\Omega \cap \{x_n = 0\}$, has a unique harmonic extension to Ω, provided $u = 0$ on $\Omega \cap \{x_n = 0\}$.

2. Perron's method

2.1 Theorem Let Ω be bounded and suppose that the exterior ball condition is satisfied at every point of $\partial\Omega$, i.e. for every point $x_0 \in \partial\Omega$ there exists a ball B such that $\overline{B} \cap \overline{\Omega} = \{x_0\}$. Then there exists for every $\varphi \in C(\partial\Omega)$ exactly one harmonic function $u \in C(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$.

For the proof of Theorem 2.1 we need to extend the definition of sub- and superharmonic to continuous functions.

2.2 Definition A function $u \in C(\Omega)$ is called *subharmonic* if $u \leq h$ on B for every ball $B \subset\subset \Omega$ and every $h \in C(\overline{B})$ harmonic with $u \leq h$ on ∂B. The definition of superharmonic is likewise.

Clearly this is an extension of Definition 1.1, that is, every $u \in C^2(\Omega)$ with $\Delta u \geq 0$ is subharmonic in the sense of Definition 2.2. See also the exercises at the end of this section.

2.3 Theorem Suppose $u \in C(\overline{\Omega})$ is subharmonic, and $v \in C(\overline{\Omega})$ is superharmonic. If $u \leq v$ on $\partial\Omega$, then $u < v$ on Ω, unless $u \equiv v$.

Proof First we prove that $u \leq v$ in Ω. If not, then the function $u - v$ must have a maximum $M > 0$ achieved in some interior point x_0 in Ω. Since $u \leq v$ on $\partial\Omega$ and $M > 0$, we can choose a ball $B \subset\subset \Omega$ centered in x_0, such that $u - v$ is not identical to M on ∂B. Because of the Poisson Integral Formula, there exist harmonic functions $\overline{u}, \overline{v} \in C(\overline{B})$ with $\overline{u} = u$ and $\overline{v} = v$ on ∂B. By definition, $\overline{u} \geq u$ and $\overline{v} \leq v$. Hence $\overline{u}(x_0) - \overline{v}(x_0) \geq M$, while on ∂B we have $\overline{u} - \overline{v} = u - v \leq M$. Because $\overline{u}$ and $\overline{v}$ are harmonic it follows that $\overline{u} - \overline{v} \equiv M$ on B, and therefore the same holds for $u - v$ on ∂B, a contradiction.

Next we show that also $u < v$ on Ω, unless $u \equiv v$. If not, then the function $u - v$ must have a zero maximum achieved in some interior point x_0 in Ω, and, unless $u \equiv v$, we can choose x_0 and B exactly as above, reading zero for M. Again this gives a contradiction. ∎

Using again the Poisson Integral Formula we now introduce

2.4 Definition Let $u \in C(\Omega)$ be subharmonic, and let $B \subset\subset \Omega$ be a ball. The unique function $U \in C(\Omega)$ defined by

(i) $U = u$ for $\Omega \backslash B$;

(ii) U is harmonic on B,

is called the *harmonic lifting* of u in B.

2.5 Proposition The harmonic lifting U on B in Definition 2.4 is also subharmonic in Ω.

Proof Let $B' \subset\subset \Omega$ be an arbitrary closed ball, and suppose that $h \in C(\overline{B}')$ is harmonic in B', and $U \leq h$ on $\partial B'$. We have to show that also $U \leq h$ on B'. First observe that since u is subharmonic $U \geq u$ so that certainly $u \leq h$ on $\partial B'$, and hence $u \leq h$ on B'. Thus $U \leq h$ on $B' \backslash B$, and also on the boundary $\partial\Omega'$ of $\Omega' = B' \cap B$. But both U and h are harmonic in $\Omega' = B' \cap B$, so by the maximum principle for harmonic functions, $U \leq h$ on $\Omega' = B' \cap B$, and hence on the whole of B'. ∎

2.6 Proposition If $u_1, u_2 \in C(\Omega)$ are subharmonic, then $u = \max(u_1, u_2) \in C(\Omega)$ is also subharmonic.

Proof Exercise. ∎

2.7 Definition A function $u \in C(\overline{\Omega})$ is called a *subsolution* for $\varphi : \partial\Omega \to \mathbb{R}$ if u is subharmonic in Ω and $u \leq \varphi$ in $\partial\Omega$. The definition of a supersolution is likewise.

2.8 Theorem For $\varphi : \partial\Omega \to \mathbb{R}$ bounded let S_φ be the collection of all subsolutions, and let

$$u(x) = \sup_{v \in S_\varphi} v(x), \quad x \in \Omega.$$

Then $u \in C(\Omega)$ is harmonic in Ω.

Proof Every subsolution is smaller then or equal to every supersolution. Since $\sup_{\partial\Omega} \varphi$ is a supersolution, it follows that u is well defined. Now fix $y \in \Omega$ and choose a sequence of functions $v_1, v_2, v_3, \ldots \in S_\varphi$ such that $v_n(y) \to u(y)$ as $n \to \infty$. Because of Proposition 2.6

we may take this sequence to be nondecreasing in $C(\Omega)$, and larger then or equal to $\inf_{\partial\Omega}\varphi$. Let $B \subset\subset \Omega$ be a ball with center y, and let V_n be the harmonic lifting of v_n on B. Then $v_n \le V_n \le u$ in Ω, and V_n is also nondecreasing in $C(\Omega)$. By the Harnack Convergence Theorem, the sequence V_n converges on every ball $B' \subset\subset B$ uniformly to a harmonic function $v \in C(B)$. Clearly $v(y) = u(y)$ and $v \le u$ in B. The proof will be complete if we show that $v \equiv u$ on B for then it follows that u is harmonic in a neighbourhood of every point y in Ω. So suppose $v \not\equiv u$ on B. Then there exists $z \in B$ such that $u(z) > v(z)$, and hence we can find $\overline{u} \in S_\varphi$ such that $v(z) < \overline{u}(z) \le u(z)$. Define $w_n = \max(v_n, \overline{u})$ and let W_n be the harmonic lifting of w_n on B. Again it follows that the sequence W_n converges on every ball $B' \subset\subset B$ uniformly to a harmonic function $w \in C(B)$, and clearly $v \le w \le u$ in B, so $v(y) = w(y) = u(y)$. But v and w are both harmonic, so by the strong maximum principle for harmonic functions they have to coincide. However, the construction above implies that $v(z) < \overline{u}(z) \le w(z)$, a contradiction. ∎

Next we look at the behaviour of the harmonic function u in Theorem 2.8 near the boundary.

2.9 Definition Let $x_0 \in \partial\Omega$. A function $w \in C(\overline{\Omega})$ with $w(x_0) = 0$ is called a barrier function in x_0 if w is superharmonic in Ω and $w > 0$ in $\overline{\Omega}\backslash\{x_0\}$.

2.10 Proposition Let u be as in Theorem 2.8, and let $x_0 \in \partial\Omega$, and suppose there exists a barrier function w in x_0. If φ is continuous in x_0, then $u(x) \to \varphi(x_0)$ if $x \to x_0$.

Proof The idea is to find a sub- and a supersolution of the form $u^\pm = \varphi(x_0) \pm \epsilon \pm kw(x)$. Fix $\epsilon > 0$ and let $M = \sup_{\partial\Omega}|\varphi|$. We first choose $\delta > 0$ such that $|\varphi(x) - \varphi(x_0)| < \epsilon$ for all $x \in \partial\Omega$ with $|x - x_0| < \delta$, and then $k > 0$ such that $kw > 2M$ on $\overline{\Omega}\backslash B_\delta(x_0)$. Clearly then u^- is a sub- and u^+ is a supersolution, so that $\varphi(x_0) - \epsilon - kw(x) \le u(x) \le \varphi(x_0) + \epsilon + kw(x)$ for all $x \in \Omega$. Since $\epsilon > 0$ was arbitrary, this completes the proof. ∎

2.11 Exercise Finish the proof of Theorem 2.1, and prove that the map

$$\varphi \in C(\partial\Omega) \to u \in C(\overline{\Omega})$$

is continuous with respect to the supremum norms in $C(\partial\Omega)$ and $C(\overline{\Omega})$.

2.12 Exercise Show that for a function $u \in C(\Omega)$ the following three statements are equivalent:

(i) u is subharmonic in the sense of Definition 2.2;

(ii) for every nonnegative compactly supported function $\phi \in C^2(\Omega)$ the inequality

$$\int_\Omega u\Delta\phi \ge 0$$

holds;

(iii) u satisfies the conclusion of the Mean Value Theorem, i.e. for every $B_R(y) \subset\subset \Omega$ the inequality

$$u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u(x) dS(x)$$

holds.

Hint: In order to deal with (ii) show that it is equivalent to the existence of a sequence $(\Omega_n)_{n=1}^{\infty}$ of strictly increasing domains, and a corresponding sequence of subharmonic functions $(u_n)_{n=1}^{\infty} \in C^{\infty}(\Omega_n)$, with the property that for every compact $K \subset \Omega$ there exists an integer N such that $K \subset \Omega_N$, and moreover, the sequence $(u_n)_{n=N}^{\infty}$ converges uniformly to u on K.

Finally we formulate an optimal version of Theorem 2.1.

2.13 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Then there exists for every $\varphi \in C(\partial\Omega)$ exactly one harmonic function $u \in C(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$. The map $\varphi \in C(\partial\Omega) \to u \in C(\overline{\Omega})$ is continuous with respect to the supremum norm.

Proof Exercise.

3. Potential theory

We recall that the fundamental solution of Laplace's equation is given by

$$\Gamma(x) = \Gamma(|x|) = \begin{cases} \dfrac{1}{2\pi} \log(|x|) & \text{if } n = 2; \\ \dfrac{1}{n(2-n)\omega_n} |x|^{2-n} & \text{if } n > 2, \end{cases}$$

and that the Newton potential of a bounded function $f : \Omega \to \mathbb{R}$ is defined by

$$w(x) = \int_{\Omega} \Gamma(x-y) f(y) dy.$$

Note that we have interchanged the role x and y in the previous section.

When $n = 3$, one can view $w(x)$ as the gravitational potential of a body Ω with density function f, that is, the gravitational field is proportional to $-\nabla w(x)$. This gradient is well defined because of the following theorem.

3.1 Theorem Let $f \in L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open and bounded, and let $w(x)$ be the Newton potential of f. Then $w \in C^1(\mathbb{R}^n)$ and

$$\frac{\partial w(x)}{\partial x_i} = \int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i} f(y)dy.$$

Proof First observe that

$$\frac{\partial \Gamma(x-y)}{\partial x_i} = \frac{x_i - y_i}{n\omega_n |x-y|^n} \quad \text{so that} \quad |\frac{\partial \Gamma(x-y)}{\partial x_i}| \leq \frac{1}{n\omega_n |x-y|^{n-1}}.$$

Hence

$$\int_{B_R(y)} |\frac{\partial \Gamma(x-y)}{\partial x_i}| dx \leq \int_{B_R(y)} \frac{dx}{n\omega_n |x-y|^{n-1}} = \int_{B_R(0)} \frac{dx}{n\omega_n |x|^{n-1}}$$

$$= \int_0^R \frac{1}{r^{n-1}} r^{n-1} dr = R < \infty,$$

and

$$\frac{\partial \Gamma(x-y)}{\partial x_i} \in L^1(B_R(y)) \quad \text{for all} \quad R > 0.$$

Thus the function

$$v_i(x) = \int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i} f(y)dy$$

is well defined for all $x \in \mathbb{R}^n$.

Now let $\eta \in C^\infty([0,\infty))$ satisfy

$$\begin{cases} \eta(s) = 0 \quad \text{for} \quad 0 \leq s \leq 1; \\ 0 \leq \eta'(s) \leq 2 \quad \text{for} \quad 1 \leq s \leq 2; \\ \eta(s) = 1 \quad \text{for} \quad s \geq 2, \end{cases}$$

and define

$$w_\varepsilon(x) = \int_\Omega \Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) f(y)dy.$$

Then the integrand is smooth in x, and its partial derivates of any order with respect to x are also in $L^\infty(\Omega)$. Thus $w_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$\frac{\partial w_\varepsilon(x)}{\partial x_i} = \int_\Omega \frac{\partial}{\partial x_i}\big(\Gamma(x-y)\eta(\frac{|x-y|}{\varepsilon}) f(y)\big)dy =$$

$$\int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i}\eta(\frac{|x-y|}{\varepsilon}) f(y)dy + \int_\Omega \Gamma(x-y)\eta'(\frac{|x-y|}{\varepsilon})\frac{|x_i-y_i|}{\varepsilon|x-y|} f(y)dy.$$

We have for $n > 2$, and for all $x \in \mathbb{R}^n$, that

$$\Big|\frac{\partial w_\varepsilon(x)}{\partial x_i} - v_i(x)\Big| = \Big|\int_\Omega \frac{\partial\Gamma(x-y)}{\partial x_i}\big(\eta(\frac{|x-y|}{\varepsilon}) - 1\big)f(y)dy$$

$$+\int_\Omega \Gamma(x-y)\eta'(\frac{|x-y|}{\varepsilon})\frac{x_i - y_i}{\varepsilon|x_i - y_i|}f(y)dy\Big| \le$$

$$\|f\|_\infty\Big\{\int_{|x-y|\le 2\varepsilon}\frac{1}{n\omega_n|x-y|^{n-1}}dy + \int_{|x-y|\le 2\varepsilon}\frac{1}{n(n-2)\omega_n|x-y|^{n-2}}\frac{2}{\varepsilon}dy\Big\}$$

$$= \|f\|_\infty\Big\{\int_0^{2\varepsilon}\frac{1}{r^{n-1}}r^{n-1}dr + \int_0^{2\varepsilon}\frac{1}{(n-2)r^{n-2}}\frac{2}{\varepsilon}r^{n-1}dr\Big\}$$

$$= \|f\|_\infty\big\{2\varepsilon + \frac{1}{n-2}\frac{1}{\varepsilon}(2\varepsilon)^2\big\} = \|f\|_\infty\big(2 + \frac{4}{n-2}\big)\varepsilon,$$

so that

$$\frac{\partial w_\varepsilon}{\partial x_i} \to v_i \quad \text{uniformly in } \mathbb{R}^n \quad \text{as} \quad \varepsilon \downarrow 0.$$

Similarly one has

$$|w_\varepsilon(x) - w(x)| = \Big|\int_\Omega \Gamma(x-y)\big(\eta(\frac{|x-y|}{\epsilon}) - 1\big)f(y)dy\Big|$$

$$\le \|f\|_\infty \int_0^{2\varepsilon}\frac{1}{(n-2)r^{n-2}}r^{n-1}dr = \|f\|_\infty\frac{\varepsilon^2}{2(n-2)},$$

so that also $w_\varepsilon \to w$ uniformly on $\mathbb{R}^n$ as $\varepsilon \downarrow 0$. This proves that $w, v_i \in C(\mathbb{R}^n)$, and that $v_i = \partial w/\partial x_i$. The proof for $n = 2$ is left as an exercise. ∎

The next step would be to show that for $f \in C(\Omega)$, $w \in C^2(\Omega)$ and $\Delta w = f$. Unfortunately this is not quite true in general. For a counterexample see Exercise 4.9 in [GT]. To establish $w \in C^2(\Omega)$ we introduce the concept of *Dini continuity*.

3.2 Definition $f : \Omega \to \mathbb{R}$ is called (locally) Dini continuous in Ω, if for every $\Omega' \subset\subset \Omega$ there exists a measurable function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\int_0^R \frac{\varphi(r)}{r}dr < \infty \qquad \text{for all } R > 0,$$

such that

$$|f(x) - f(y)| \le \varphi(|x-y|)$$

for all x, y in Ω'. If the function φ can be chosen independent of Ω', then f is called uniformly Dini continuous in Ω.

3.3 Definition $f : \Omega \to \mathbb{R}$ is called (uniformly) Hölder continuous with exponent $\alpha \in (0, 1]$ if f is (uniformly) Dini continuous with $\varphi(r) = r^\alpha$.

3.4 Theorem Let Ω be open and bounded, and let $f \in L^\infty(\Omega)$ be Dini continuous. Then $w \in C^2(\Omega)$, $\Delta w = f$ in Ω, and for every bounded open set $\Omega_0 \supset \Omega$ with smooth boundary $\partial\Omega_0$,

$$\frac{\partial^2 w(x)}{\partial x_i \partial x_j} = \int_{\Omega_0} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}(f(y) - f(x))dy - f(x) \int_{\partial\Omega_0} \frac{\partial \Gamma(x-y)}{\partial x_i} \nu_j(y) dS(y),$$

where $\nu = (\nu_1, ..., \nu_n)$ is the outward normal on $\partial\Omega_0$, and f is assumed to be zero on the complement of Ω.

Proof We give the proof for $n \geq 3$. Note that

$$\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j} = \frac{1}{n\omega_n} \frac{|x-y|^2 \partial_{ij} - n(x_i - y_i)(x_j - y_j)}{|x-y|^{n+2}},$$

so that

$$\left|\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}\right| \leq \frac{1}{\omega_n} \frac{1}{|x-y|^n},$$

which is insufficient to establish integrability near the singularity $y = x$. Let

$$u_{ij}(x) = \int_{\Omega_0} \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}(f(y) - f(x))dy - f(x) \int_{\partial\Omega_0} \frac{\partial \Gamma(x-y)}{\partial x_i} \nu_j(y) dS(y).$$

Since f is Dini continuous, it is easy to see that $u_{ij}(x)$ is well defined for every $x \in \Omega$, because the first integrand is dominated by

$$\frac{1}{\omega_n} \frac{\varphi(r)}{r^n}$$

and the second integrand is smooth. Now let

$$v_{i\varepsilon}(x) = \int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i} \eta\left(\frac{|x-y|}{\varepsilon}\right) f(y) dy.$$

Then

$$\left|v_{i\varepsilon}(x) - \frac{\partial w(x)}{\partial x_i}\right| = \left|\int_\Omega \frac{\partial \Gamma(x-y)}{\partial x_i} \left\{\eta\left(\frac{|x-y|}{\varepsilon}\right) - 1\right\} f(y) dy\right|$$

$$\leq \|f\|_\infty n\omega_n \int_0^{2\varepsilon} \frac{1}{n\omega_n r^{n-1}} r^{n-1} dr = 2\|f\|_\infty \varepsilon,$$

so that $v_{i\varepsilon} \to \partial w / \partial x_i$ uniformly in $\mathbb{R}^n$ as $\varepsilon \downarrow 0$. Extending f to Ω_0 by $f \equiv 0$ in Ω^c, we find for $x \in \Omega$, using the smoothness of $(\partial\Gamma/\partial x_i)\eta f$, that

$$\frac{\partial v_{i\varepsilon}(x)}{\partial x_j} = \int_{\Omega_0} \frac{\partial}{\partial x_j} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta\left(\frac{|x-y|}{\varepsilon}\right) f(y) dy =$$

$$\int_{\Omega_0} \{f(y)-f(x)\} \frac{\partial}{\partial x_j} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta(\frac{|x-y|}{\varepsilon}) dy +$$

$$f(x) \int_{\Omega_0} \frac{\partial}{\partial x_j} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta(\frac{|x-y|}{\varepsilon}) dy =$$

$$\int_{\Omega_0} \{f(y)-f(x)\} \frac{\partial}{\partial x_j} \frac{\partial \Gamma(x-y)}{\partial x_i} \eta(\frac{|x-y|}{\varepsilon}) dx - f(x) \int_{\partial\Omega_0} \frac{\partial \Gamma(x-y)}{\partial x_i} \nu_j(y) dS(y),$$

provided $2\varepsilon < d(x, \partial\Omega)$, so that

$$\big|u_{ij}(x) - \frac{\partial v_{i\varepsilon}(x)}{\partial x_j}\big| = \big| \int_{\Omega_0} \{f(y)-f(x)\} \frac{\partial}{\partial x_j} \big(1 - \eta(\frac{|x-y|}{\varepsilon})\big) \frac{\partial \Gamma(x-y)}{\partial x_i} dy\big| =$$

$$\big| \int_{\Omega_0} \{ \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j} \big(1 - \eta(\frac{|x-y|}{\varepsilon})\big) - \eta'\big(\frac{|x-y|}{\varepsilon}\big) \frac{x_j - y_j}{\varepsilon |x-y|} \frac{\partial \Gamma(x-y)}{\partial x_i} \} \times$$

$$\{f(y)-f(x)\} dy\big| \le \int_{|x-y|\le 2\varepsilon} \{ \frac{1}{\omega_n |x-y|^n} + \frac{2}{\varepsilon n \omega_n |x-y|^{n-1}} \} \varphi(|x-y|) dy \le$$

$$\int_0^{2\varepsilon} (\frac{n}{r^n} + \frac{2}{\varepsilon r^{n-1}}) \varphi(r) r^{n-1} dr \le$$

$$n \int_0^{2\varepsilon} \frac{\varphi(r)}{r} dr + 2 \int_0^{2\varepsilon} \frac{r}{\varepsilon} \frac{\varphi(r)}{r} dr \le (n+2) \int_0^{2\varepsilon} \frac{\varphi(r)}{r} dr,$$

implying

$$\frac{\partial v_{i\varepsilon}}{\partial x_j} \to u_{ij} \quad \text{as} \quad \varepsilon \downarrow 0,$$

uniformly on compact subsets of Ω. This gives $v_i \in C^1(\Omega)$ and

$$u_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} = \frac{\partial^2 w(x)}{\partial x_i \partial x_j}.$$

It remains to show that $\Delta w = f$. Fix $x \in \Omega$ and let $\Omega_0 = B_R(x) \supset \Omega$. Then

$$\Delta w(x) = \sum_{i=1}^n u_{ii}(x) = \sum_{i=1}^n \frac{\partial^2 w(x)}{\partial x_i^2} = -f(x) \sum_{i=1}^n \int_{\partial B_R(x)} \frac{\partial \Gamma(x-y)}{\partial x_i} \nu_i(y) dS(y) =$$

$$f(x) \int_{\partial B_R(x)} \sum_{i=1}^n \frac{\partial \Gamma(x-y)}{\partial y_i} \nu_i(y) dS(y) = f(x) \int_{\partial B_R(0)} \frac{\partial \Gamma}{\partial \nu} dS =$$

$$f(x) n \omega_n R^{n-1} \frac{1}{n \omega_n R^{n-1}} = f(x),$$

and this completes the proof. ∎

3.5 Definition Let f be locally integrable on Ω. A function $u \in C(\Omega)$ is called a weak C_0-solution of $\Delta u = f$ in Ω if, for every compactly supported $\psi \in C^2(\Omega)$, the equality

$$\int_\Omega u\Delta\psi = \int_\Omega \psi f$$

holds.

3.6 Exercise Let $f \in C(\overline{\Omega})$, and let $w \in C^1(\mathbb{R}^n)$ be the Newton potential of f. Show that w is a weak C_0-solution of $\Delta u = f$ in Ω, and that the map

$$f \in C(\overline{\Omega}) \to w \in C(\overline{\Omega})$$

is compact with respect to the supremum norm in $C(\overline{\Omega})$.

4. Existence results; the method of sub- and supersolutions

We begin with some existence results which follow from the previous results. The first one combines the results of Perron's method (Theorem 2.1 and Exercise 2.13) with the continuity of the second derivatives of the Newton potential of a Dini continuous function (Theorem 3.4).

4.1 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Then the problem

$$\begin{cases} \Delta u = f & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

has a unique classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for every bounded Dini continuous $f \in C(\overline{\Omega})$ and for every $\varphi \in C(\partial\Omega)$.

Proof Exercise (hint: write $u = \tilde{u} + w$, where w is the Newton potential of f). ∎

The previous theorem gives a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. We recall that for f locally integrable on Ω, the function $u \in C(\overline{\Omega})$ is called a weak C_0-solution of $\Delta u = f$ in Ω if, for every compactly supported $\psi \in C^2(\Omega)$, the equality $\int u\Delta\psi = \int \psi f$ holds. The next theorem combines Perron's method with Exercise 3.5.

4.2 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Then the problem

$$\begin{cases} \Delta u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak C_0-solution for every $f \in C(\overline{\Omega})$. The map

$$f \in C(\overline{\Omega}) \to u \in C(\overline{\Omega})$$

is compact with respect to the supremum norm in $C(\overline{\Omega})$.

Proof Exercise.

4.3 Exercise Compute the solution of

$$\begin{cases} \Delta u = -1 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the case that

$$\Omega = \Omega_{\epsilon,R} = \{x \in \mathbb{R}^n : \ \epsilon < |x| < R\}.$$

4.4 Exercise Let Ω be bounded and suppose that for some $\epsilon > 0$ the exterior ball condition is satisfied at every point of $\partial\Omega$ by means of a ball with radius $r \geq \epsilon$. For $f \in C(\overline{\Omega})$ let u be the unique weak C_0-solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that

$$|u(x)| \leq C||f||_\infty \text{dist}(x, \partial\Omega),$$

where C is a constant which depends only on ϵ and the diameter of Ω.

The concept of weak solutions allows one to obtain existence results for semilinear problems without going into the details of linear regularity theory, which we shall discuss later on in this course. We consider the problem

$$(D) \quad \begin{cases} \Delta u = f(x, u) & \text{in } \Omega; \\ u = 0 & \text{on } \quad \partial\Omega, \end{cases}$$

where $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is continuous.

4.5 Definition A function $\underline{u} \in C(\overline{\Omega})$ is called a weak C_0-subsolution of (D), if $\underline{u} \leq 0$ on $\partial\Omega$, and if for every compactly supported nonnegative $\psi \in C^2(\Omega)$, the equality

$$\int_\Omega \underline{u}\Delta\psi \geq \int_\Omega \psi f(x, \underline{u}(x))dx$$

holds. A C_0-supersolution $\overline{u}$ is defined likewise, but with reversed inequalities. A function u which is both a C_0-subsolution and a C_0-supersolution, is called a C_0-solution of (D).

4.6 Theorem Let Ω be bounded and suppose that there exists a barrier function in every point of $\partial\Omega$. Let $f : \mathbb{R} \times \Omega \to \mathbb{R}$ be continuous. Suppose that Problem (D) admits a C_0-subsolution $\underline{u}$ and a C_0-supersolution $\overline{u}$, satisfying $\underline{u} \leq \overline{u}$ in Ω. Then Problem (D) has at least one C_0-solution u with the property that $\underline{u} \leq u \leq \overline{u}$.

Sketch of the proof The proof is due to Clement and Sweers and relies on an application of Schauder's fixed point theorem. Let

$$[\underline{u}, \overline{u}] = \{u \in C(\overline{\Omega}) : \ \underline{u} \leq u \leq \overline{u}\}.$$

In order to define the map T we first replace f by f^* defined by $f^*(x,s) = f(x,s)$ for $\underline{u}(x) \leq s \leq \overline{u}(x)$, $f^*(x,s) = f(x, \overline{u}(x))$ for $s \geq \overline{u}(x)$, and $f^*(x,s) = f(x, \underline{u}(x))$ for $s \leq \underline{u}(x)$. It then follows from the maximum principle that every solution for the problem with f^* must belong to $[\underline{u}, \overline{u}]$. Writing f for f^* again, the map T is now defined by $T(v) = u$, where u is the weak C_0-solution of the problem

$$\begin{cases} \Delta u = f(x, v(x)) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $e \in C^2(\overline{\Omega})$ be the (positive) solution of

$$\begin{cases} \Delta e = -1 & \text{in } \Omega; \\ e = 0 & \text{on } \partial\Omega. \end{cases}$$

Then e is bounded in Ω by some constant M. We introduce the set

$$A_k = \{u \in C(\overline{\Omega}) : \ |u(x)| \leq k e(x) \ \forall x \in \Omega\}.$$

From the maximum principle it follows again that $T : A_k \to A_k$ is well defined, provided k is larger then the supremum of $f = f^*$. The compactness of T follows from Theorem 4.2. Hence there exists a fixed point, which is the solution we seek. ∎

4.7 Exercise Fill in the details of the proof.

4.8 Exercise Prove the existence of a positive weak solution in the case that $f(u) = -u^\beta(1-u)$ with $0 < \beta < 1$.

4.9 Exercise Prove the existence of a positive weak solution in the case that $f(u) = -u^\beta$ with $0 < \beta < 1$.

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