PDE2006, exercise set 4

J. Hulshof

18th December 2006

We first review the spaces interpolating between and extrapolating from $L^2(\Omega)$ and $H_0^1(\Omega)$. Throughout $\Omega \subset I\!R^n$ is a bounded domain with smooth boundary $\partial\Omega$. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$ be the eigenvalues and $\phi_1, \phi_2, \phi_3, \ldots$ be the corresponding eigenfunctions of $-\Delta$ in $L^2(\Omega)$ with zero boundary data on $\partial\Omega$. In other words, ϕ_j is a weak solution of

$$-\Delta\phi_j = \lambda_j\phi_j \text{ in } \Omega; \quad \phi_j = 0 \text{ on } \partial\Omega.$$

These eigenvalues are the reciprocals of the eigenvalues of the solution operator A defined by $Af = u$ where u is the weak solution of

$$(P): \quad -\Delta u = f \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega$$

(see also exercise set 3). We normalise the eigenfunctions in $L^2(\Omega)$, that is

$$(\phi_i, \phi_j) = \int_{\partial\Omega} \phi_i\phi_j = \delta_{ij}.$$

Thus every $u \in L^2(\Omega)$ is uniquely written as

$$u = u_1\phi_1 + u_2\phi_2 + u_3\phi_3 + \cdots = \sum_{j=1}^{\infty} u_j\phi_j,$$

with $u_j = (u, \phi_j)$ $(j = 1, 2, \ldots)$, and $(u_1, u_2, u_3, \ldots) \in l^2$, where

$$l^2 = \{u = (u_1, u_2, u_3, \ldots) : u_j \in I\!R,\ j = 1, 2, \ldots;\ ||u||^2 = \sum_{j=1}^{\infty} u_j^2 < \infty\}$$

is the standard example of a (real) separable infinite-dimensional Hilbert space. It will be convenient to simultaneously denote by u an element in $L^2(\Omega)$ and its representing coordinate sequence in l^2.

We identify l^2 with its dual $(l^2)^*$: the continuous linear functions $F : l^2 \to I\!R$ are uniquely written as

$$F(v) =< F, v >= (u, v) = \sum_{j=1}^{\infty} u_j v_j,$$

with $u \in l^2$, with a one-to-one correspondence between $F \in (l^2)^*$ and $u \in l^2$. Likewise we identify $L^2(\Omega)$ with its dual.

Next we define a "scale" of Hilbert spaces

$$\theta^\alpha = \{u = (u_1, u_2, u_3, \ldots) : \, u_j \in I\!R, \, j = 1, 2, \ldots ; \, ||u||_\alpha^2 = \sum_{j=1}^{\infty} \lambda_j^\alpha u_j^2 < \infty\},$$

where $\alpha \in I\!R$. For $\alpha \geq 0$, the corresponding subspace of $L^2(\Omega)$ is

$$\{u = \sum_{j=1}^{\infty} u_j \phi_j \, : \, u_j \in I\!R, \, j = 1, 2, \ldots ; \, ||u||_\alpha^2 = \sum_{j=1}^{\infty} \lambda_j^\alpha u_j^2 < \infty\}.$$

Exercises 1,2,3 are optional and intended to review material discussed in the lectures.

1. Prove that the continuous linear functions $F : \theta^\alpha \to I\!R$ are uniquely written as

$$F(v) =< F, v >=< u, v >= \sum_{j=1}^{\infty} u_j v_j, \quad v \in \theta^\alpha,$$

with $u \in \theta^{-\alpha}$, with a one-to-one correspondence between $F \in (\theta^\alpha)^*$ and $u \in \theta^{-\alpha}$. Hint: first find the obvious linear isometry (norm preserving bijection) between θ^α and $\theta^0 = l^2$, and use this to show that there is a one-to-one correspondence between continuous linear functions on θ^α (i.e. elements in the dual of θ^α) and continuous linear functions on l^2 (i.e. elements in the dual of l^2).

2. Explain why $\theta^1 = H_0^1(\Omega)$ and why $H^{(-1)}(\Omega)$ may be interpreted as

$$\{u = \sum_{j=1}^{\infty} u_j \phi_j \, : \, u_j \in I\!R, \, j = 1, 2, \ldots ; \, \sum_{j=1}^{\infty} \lambda_j^{-1} u_j^2 < \infty\}$$

(these Fourier series exist in the sense that they have partial sums which, when considered as elements in $H^{(-1)}(\Omega)$, are Cauchy sequences).

3. Let $f \in L^2(\Omega)$. Let A be the solution operator corresponding to Problem (P) (see above). We already saw that $A : H^{(-1)}(\Omega) \to H_0^1(\Omega)$ is an isometry. Explain why $A : L^2(\Omega) \to \Theta^2$ is an isometry.

Remark: it can be shown that $\Theta^2 = H^2(\Omega) \cap H_0^1(\Omega)$. More generally $A : \Theta^\alpha \to \Theta^{2+\alpha}$ is an isometry.

4. For $u = u(x, t)$ we consider the wave equation with zero boundary data on $\partial\Omega$:

$$u_{tt} = \Delta u \quad \text{for } x \in \Omega, \, t \in I\!R; \quad u = 0 \quad \text{for } x \in \partial\Omega, \, t \in I\!R.$$

Writing

$$u(x, t) = u_1(t)\phi_1(x) + u_2(t)\phi_2(x) + u_3(t)\phi_3(x) + \cdots = \sum_{j=1}^{\infty} u_j \phi_j,$$

or, for short,

$$u(t) = u_1(t)\phi_1 + u_2(t)\phi_2 + u_3(t)\phi_3 + \cdots = \sum_{j=1}^{\infty} u_j \phi_j,$$

derive equations which should be satisfied by the coefficients $u_1(t), u_2(t), \ldots$ Explain why solutions with prescribed $u(0), u_t(0) \in L^2(\Omega)$ will have $u(t) \in L^2(\Omega)$ for all $t \in I\!R$.

5. Same exercise, but now for the equation

$$u_{tt} = -\Delta\Delta u \quad \text{for } x \in \Omega,\ t \in I\!R,$$

with boundary conditions

$$\Delta u = u = 0 \quad \text{for } x \in \partial\Omega,\ t \in I\!R.$$

6. How would you solve the (same) equation

$$u_{tt} = -\Delta\Delta u \quad \text{for } x \in \Omega,\ t \in I\!R$$

with boundary conditions

$$\frac{\partial u}{\partial \nu} = u = 0 \quad \text{for } x \in \partial\Omega,\ t \in I\!R,$$

when you think back of exercise 4 in series 3 (this is the problem for a clamped plate, in dimension $N = 1$ it is known as the beam equation).