PDE2010, Additional material and programme for PDE part 2, spring 2010

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1 Monday 29-03-10

Discussion of the course (by Sara) so far from a more abstract perspective. Prototype problem

$$(D) \quad -\Delta u = f \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega,$$

which consists of an inhomogeneous equation with homogeneous (Dirichlet) boundary conditions. Here $\Omega \subset \mathbb{R}^n$ is open, connected, bounded,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

$f : \Omega \to \mathbb{R}$ is given, and $u : \Omega \to \mathbb{R}$ is the unknown function.

Examples: Ω a bounded open interval in $\mathbb{R}$, Ω a bounded open rectangle in $\mathbb{R}^2$, Ω an open disk in $\mathbb{R}^2$, Ω the open unit ball

$$B = B^n = \{x_1^2 + \cdots + x_n^2 < 1\}$$

in $\mathbb{R}^n$, with boundary

$$S = S^{n-1} = \{x_1^2 + \cdots + x_n^2 = 1\}.$$

General idea. Use the eigenfunctions ϕ_n with eigenvalues λ_n of (minus) the Laplacian with zero Dirichlet boundary conditions:

$$-\Delta\phi_n = \lambda_n\phi_n \ \text{ in } \ \Omega, \quad \phi_n = 0 \ \text{ on } \ \partial\Omega.$$

Theorem: these exist, $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \ldots \uparrow \infty$, $\phi_1 > 0$ in Ω,

$$\int_\Omega \phi_n\phi_m = \delta_{mn} \quad \text{(Kronecker symbol)},$$

in other words, the eigenfunctions form an orthonormal set with respect to the inner product for functions ($f \cdot g = \int_\Omega f(x)g(x)dx$). Moreover this set is

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complete: no (square integrable) function $f : \Omega \to \mathrm{I\!R}$ can have $f \cdot \phi_n = 0$ for all n, unless $f(x) = 0$ for (almost) all $x \in \Omega$.

For Problem (D) we have

$$f = \sum_{n=1}^{\infty} a_n \phi_n \quad \Rightarrow \quad u = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} \phi_n,$$

with this (generalised) Fourier series for u having much better convergence properties than the one for f, because $\lambda_n \to \infty$.

This approach is general. The right hand side f is decomposed in its building blocks (Fourier modes) and solving Problem (D) amounts to solving for the individual blocks, and then putting the resulting solutions back together again. This is called linear superposition. The approach works for arbitrary bounded domains (domain means open connected set, no point in dropping the assumption of connectedness here) and a large class of second order elliptic operators. Chapter 6 in the book discusses the one-dimensional case.

Generically (i.e. in nonspecial cases) the eigenvalues are all different, with multiplicity one, but not in examples with symmetry, such as $\Omega = B$, $\partial\Omega = S$, which is the prototype example involving symmetry, treated by the method of separation of variables in the book.

Evolution Problems. The decomposition is also well suited for solving evolution problems involving the Laplacian. Both the wave and the heat equation with zero boundary conditions reduce to decoupled ODE's for the time dependent Fourier coefficients. These are solved with initial data at $t = 0$ derived from the Fourier decomposition of the initial data. Except for d'Alembert's method this is basically the only PDE-solving technique used in the first part of the book.

Another prototype problem in the book is

$$(H) \quad -\Delta u = 0 \ \text{ in } \ B^n, \quad u = g \ \text{ on } \ S^{n-1},$$

with $g : S^{n-1} \to \mathrm{I\!R}$ given. This is a homogeneous equation with inhomogeneous (Dirichlet) boundary conditions. It is also treated by the method of separation of variables. Here one can decompose the boundary condition in building blocks given by the eigenfunctions of the so-called Laplace-Beltrami operator on S^{n-1}.
Warning. This approach for solving Problem (H) is much less general. It does not work for arbitrary domains. A quick way to solve problems with inhomogenous boundary conditions using Fourier series is to substract from u any sufficiently smooth function U satisfying the boundary condition, which gives an inhomogeneous equation for $v = u - U$ with a homogeneous boundary condition. Only on special domains a direct approach is feasible, see Section 3.8 for Ω a rectangle and Chapters 4 and 5 for:

Problems on $\Omega = B$ the unit ball.

On a ball the building blocks can be computed in terms of special functions. This is done in Chapters 4 and 5 of the book. See also Chapter 9 (the handout) of the famous book Special Functions by Andrews, Askey and Roy for a discussion of the so-called spherical harmonics which appear here.

For $n = 2$ we have

$$x = r\cos\theta,\ y = r\sin\theta \quad \Rightarrow \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

or

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

while for $n = 3$

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \quad \Rightarrow \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} =$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}, \quad \Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Note how radial and nonradial variables separate.

A quick way to rewrite Δ in $\mathbb{R}^n$ in (new nondegenerate) coordinates $u_1, \ldots, u_n$, is as follows. Writing

$$g_{ij} = \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j},$$

the symmetric matrix G with entries g_{ij} is positive definite and invertible, and

$$\Delta = \frac{1}{\sqrt{\mathrm{detG}}}\left(\frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_n}\right)\sqrt{\mathrm{detG}}\, G^{-1} \begin{pmatrix} \frac{\partial}{\partial u_1} \\ \vdots \\ \frac{\partial}{\partial u_n} \end{pmatrix}$$

2 Thursday April Fool's Day

Discussion of the exam over Part 1 of the course. This also concerns the disk B in $\mathbb{R}^2$.

Exercise. Derive the expression for the Laplacian in (generalised) spherical coordinates in dimension $n = 2$, $n = 3$ and $n = 4$. For $n = 4$ use

$$x_1 = \cos\theta_1,\ x_2 = \sin\theta_1\cos\theta_2,\ x_3 = \sin\theta_1\sin\theta_2\cos\theta_3,\ x_4 = \sin\theta_1\sin\theta_2\sin\theta_3,$$

all θ_j between 0 and π, except the last one, θ_3 is between 0 and 2π.

Exercise. Let $-\mu \leq 0$ be an eigenvalue of

$$\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

with eigenfunction $Y(\theta, \phi)$. Show that the function $u(x, y, z) = R(r)Y(\theta, \phi)$ is harmonic if and only if

$$r^2R'' + 2rR' = \mu R,$$

and that smoothness of u in the origin is only possible if $\mu = p(p + 1)$, $p = 0, 1, 2 \ldots$. What can μ be for Δ_{S^3}, the spherical Laplacian in $\mathbb{R}^4$ you computed above? Answer: $\mu = p(p + 2)$. And for Δ_{S^4}?

Exercise. Bessel's equation of order p, Equation (1) in Section 4.7, reads

$$x^2y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0,$$

and it arises from solving Helmholtz' equation on a disk in Section 4.6 via Equation (4) and scaling by $R(r) = y(x)$, $kr^2 = x^2$. Use the previous exercise to mimic this approach for solving Helmholtz' equation on a ball in $\mathbb{R}^3$ and on a ball in $\mathbb{R}^4$. Compute the resulting analogues of the Bessel equations. These must also feature a nonnegative integer p.

The Laplace-Beltrami operator Δ_{S^1} on the circle. In dimension $n = 2$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^1}, \quad \Delta_{S^1} = \frac{\partial^2}{\partial \theta^2},$$

Exercise. Read Section 9.1 of the handout and fill in the details of the derivation of Poisson's integral formula for determining a harmonic function on the disk with boundary data given by $u(x, y) = v(r, \theta) = f(\theta)$ for $r = 1$.

Exercise. Observe how the building blocks $\cos k\theta$ and $\sin k\theta$ arise as the restriction of harmonic homogeneous polynomials to the circle. Write this out for $k = 0, 1, 2, 3$.

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The Laplace-Beltrami operator Δ_{S^2} on the 2-sphere. Same story, but more complicated, because

$$\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Note that acting on u this gives exactly the coefficient of $\frac{1}{r^2}$ in Equation (1) in Section 5.1 in the book, which first discusses the construction of solutions of $\Delta u = 0$. Later on, when we solve $\Delta u + \lambda u = 0$ with $u = 0$ on the boundary S^2, it is *only* the radial function $R(r)$ which has to be treated differently, leading to a Bessel type equation replacing Euler's equation. The nonradial part of the analysis will be exactly the same.

This is why it makes good sense to first discuss solutions of $\Delta u = 0$, for which the radial part is so easy, and the nonradial part so interesting. Separation of the radial variable,

$$u(x, y, z) = R(r)Y(\theta, \phi),$$

shows that the nonradial part $Y(\theta, \phi)$ must be an eigenfunction of Δ_{S^2}, and leads to Euler's easy equation (3) with power solutions involving the parameter μ. Recall (one of the exercises above) that, even before looking at the problem for Y, smoothness of the solution $u(x, y, z)$ only allows a discrete set of μ's, namely $\mu = n(n + 1)$ where n is (not the dimension but) an index which runs over the nonnegative integers.

How to find the eigenfunctions Y of Δ_{S^2}? Two approaches. The first is separation of variables. Section 9.2 in the handout gives a concise treatment which parallels Chapter 5 in the book. It derives the associated Legendre equation with parameters n and m, separating

$$Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$$

to get (5) and (6) for $\Phi(\phi)$ and $\Theta(\theta)$. Note that not all the eigenfunctions will be of this form, but the solutions thus obtained do provide us with a complete basis of eigenfunctions, as is explained in the handout. The first equation (5) is easy, the usual $\frac{1}{2}$, $\cos m\phi$ and $\sin m\phi$ for $\Phi(\phi)$, but (6) is more involved. Via $s=\cos\theta$, s between -1 and $+1$, it results in (12) in Section 5.1 of the book, with $\mu=n(n+1)$, or the y-equation just before (3.9.5) in the handdout. This is the Associated Legendre Equation. It can be solved using powerseries, all of which are too bad in $s=\pm1$ to provide smooth solutions on S^2, except for the polynomial ones, which are given, scattered through Chapter 5 in the book, by (for $m=0$) Rodrigues' formula in Section 5.6, (3) in Section 5.3 ($m=0,\dots,n$), and just below (3), confusingly, for $m=-1,\dots,-n$. Baptized as

$$P_n^m(s),$$

each of these provides us with two linearly independent spherical harmonics

$$Y(\theta,\phi)=P_n^m(\cos\theta)\cos m\phi \quad\text{and}\quad Y(\theta,\phi)=P_n^m(\cos\theta)\sin m\phi$$

(except for $m=1$), so that we obtain (with $m=0,\dots,n$) $2n+1$ linearly independent harmonic functions:

$$r^nP_n^m(\cos\theta)\cos m\phi,\quad r^nP_n^m(\cos\theta)\sin m\phi,$$

or, in complex notation, with suitable normalizing factors, the Spherical Harmonics in (4) of Section 5.3.

In the handout (ALE),

$$(ALE)\quad (1-s^2)y''(s)-2sy'(s)+(n(n+1)-\frac{m^2}{1-s^2})y(s)$$

is rewritten, setting

$$y(s)=(1-s^2)^{\frac{m}{2}}v(s),$$

ignore (3.9.5), as

$$(EC_{n,m})\quad (1-s^2)v''-2(1+m)sv'+(n-m)(n+m+1)v=0$$

for $v(s)$. Here $m=0,\dots,n$. See also the explanation below (5) in Section 5.7.

Exercise. Derive Equation $(EC_{n,m})$.

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A generating function. The polynomial solutions are, upto a constant, the ultraspherical polynomials

$$C_{n-m}^{m+\frac{1}{2}}(s).$$

These are defined (compare Exercise 31 in Section 5.5) by

$$\frac{1}{(1-2s\rho+\rho^2)^\alpha}=\sum_{k=0}^\infty C_k^\alpha(s)\rho^k,$$

where, through

$$\alpha = \frac{N-2}{2},$$

α is related to the space dimension N in which

$$u(r,\theta;\rho) = \frac{1}{d^{N-2}} = \frac{1}{(r^2 - 2\rho r\cos\theta + \rho^2)^{\frac{N-2}{2}}}$$

is a solution of $\Delta u = 0$, symmetric around the polar axis (say, as in the handout, the x_1-axis, but with $N = 3$ usually the z-axis) and singular in the point with $x_1 = \rho$ on the positive polar axis. The polar angle θ is the angle between the positive polar axis and the vector corresponding to a point x with length r, and d is the distance from x to the singularity. If $\rho = 1$ we simply have the radial singular harmonic function

$$u(r) = \frac{1}{r^{N-2}}.$$

Shifting this single and very special singular solution along the polar axis we generate all the harmonic polynomials we encountered in the separtion of variables approach by Taylor expansion, see below. It is now wonder that the role of this single solution is fundamental. With the appropriate normalizing factor, which makes Δu a Dirac point mass, it is called the fundamental solution of $\Delta u = 0$, and will appear in the construction of Green's functions in Chapter 12, which only treats the exceptional case $N = 2$, when the formula's above break down.

Writing the shifted solution as a power series in the parameter ρ, assuming $\rho < r$, we obtain

$$u(r,\theta;\rho) = \frac{1}{(r^2 - 2\rho r\cos\theta + \rho^2)^{\frac{N-2}{2}}} = \sum_{k=0}^{\infty} C_k^{\frac{N-2}{2}}(\cos\theta)\frac{\rho^k}{r^{N-2+k}},$$

where each of the terms in the sum defines a singular harmonic function. For $\rho > r$ (symmetry between ρ and r) we may write this solution as a power series in r,

$$u(r,\theta;\rho) = \frac{1}{(r^2 - 2\rho r\cos\theta + \rho^2)^{\frac{N-2}{2}}} = \sum_{k=0}^{\infty} C_k^{\frac{N-2}{2}}(\cos\theta)\frac{r^k}{\rho^{N-2+k}},$$

and each term is a harmonic function, which is in fact a harmonic homogeneous polynomial of degree k. Bonnet's recurrence relations simply follow from manipulations with the power series (use the one with $r = 1$).

Exercise. Do this for N=3, when you will recover the Legendre functions.

Tricks for solving the ODE. A direct way to solve (EC_m^n) is explained below.

Exercise. Differentiate Equation $(EC_{n,m})$ and show that v' satisfies $(EC_{n,m+1})$.

Exercise. Solve Equation $(EC_{n,m})$ for $v'(s)$ when $m = -n-1$. This gives a simple polynomial for $v'(s)$. Differentiating this polynomial $m+n$ times we obtain a solution of $(EC_{n,m})$. Finally a multiplication by $(1-s^2)^{\frac{m}{2}}$ gives, upto a huge factor, the solution P_n^m.

Exercise. Compare the results to Section 5.6 Rodrigues formula (1) and Section 5.7 Rodrigues formula (1). Prove formula (3) in Section 5.6, Bonnet's recurrence relation.

Exercise. Prove the orthogonality relations of P_n (Section 5.6 Thm 1).

Exercise. Prove the orthogonality relations of P_n^m (Section 5.6 Exercise 16).

Second approach for the eigenfunctions Y of Δ_{S^2}. One can also work directly with harmonic polynomials, see Section 9.3 and 9.4 in de handout. This works for all dimensions. In dimension N the number of independent harmonic polynomials of degree k is given by the number

$$c_{k,N} = (2k + N - 2)! \frac{(k + N - 3))!}{k!(N - 2)!},$$

which, for $N = 3$ gives $c_k = 2k+1$, consistent with the numbers we found above! The remarkable statement proved in the handout is that the homogeneous harmonic polynomial of degree k in dimension N which is symmetric around a (given) polar axis exists and is a multiple of the zonal harmonic function

$$C_k^{\frac{N-2}{2}}(\cos\theta)r^k,$$

where θ is the polar angle. With $c_{k,N}$ rotated copies one can make a vector basis of the homogeneous harmonic polynomial of degree k in $\mathbb{R}^N$. You may wonder, restrictiing attention to $N = 3$, how such a basis relates to the basis constructed using the associated Legendre functions. The handout continues on that theme.

Eigenfunctions of the Laplacian on ball.

A generating function for the Bessel functions $J_n(r)$. We first recall Theorem 2 in Section 4.9, written as, using polar coordinates in $\mathbb{R}^2$,

$$\cos y + i\sin y = e^{iy} = e^{ir\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(r)e^{in\theta} = \sum_{n=-\infty}^{\infty} J_n(r)(\cos n\theta + i\sin n\theta).$$

Note that consequently $J_{-n}(r) = (-1)^n J_n(r)$, see also Exercise 4.7.16. Each term in the sum satisfies

$$\Delta u + u = 0.$$

Bessel functions for $N = 3$. Recall that separation of variables for Helmholtz equation in polar coordinates (OOPS, the book uses k where the whole world uses λ, and then writes $k = \lambda^2$)

$$\Delta u + \lambda^2 u = 0, \quad u(x, y) = R(r)\Theta(\theta),$$

gave (see Section 4.6. $u \leftrightarrow \phi$, $\lambda^2 \leftrightarrow k$)

$$\Theta(\theta) = \cos m\theta \quad \text{and} \quad \sin m\theta,$$

and

$$R''(r) + \frac{1}{r}R'(r) + (\lambda^2 - \frac{\mu}{r^2})R(r) = 0, \quad \mu = m^2.$$

In the exercise above on Bessel's analogue for $N = 3$ you found that separation of variables

$$u(x, y, z) = R(r)Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$$

for

$$\Delta u + \lambda^2 u = 0$$

in $\mathbb{R}^3$ leads to

$$R''(r) + \frac{2}{r}R'(r) + (\lambda^2 - \frac{\mu}{r^2})R(r) = 0, \quad \mu = m(m+1),$$

so that via

$$R(r) = y(x), \quad \lambda r = x,$$

(we already know that only $k = \lambda^2 > 0$ allows solutions u which are zero on the boundary of the ball), we arrive at

$$x^2 y''(x) + 2xy'(x) + (x^2 - m(m+1))y(x) = 0,$$

which, after putting

$$y(x) = \frac{w(x)}{\sqrt{x}}$$

gives (see Section 4.8, Example 2)

$$x^2 w''(x) + xw'(x) + (x^2 - (m + \frac{1}{2})^2)w(x) = 0,$$

Bessel's equation with index $m + \frac{1}{2}$, and regular solution

$$y(x) = (\frac{\pi}{2x})^{\frac{1}{2}} J_{m+\frac{1}{2}}(x) = j_m(x).$$

As before λ has to be fitted with the zero's of j_n, giving rise to Theorem 1 in Section 4.5. The angular part is exactly the same as for the case that $\lambda = 0$.

A generating function for $N = 3$? Can you think of nice analogue of the generating function above?

Exercises from Chapter 5. Section 5.1: 1,3,4. Section 5.2 Example 2, 1,2,7,10,12. Section 5.3: 1,7. Section 5.4: 5, 9.

Finally, check the grey box properties of the Legendre function in Section 5.6, and verify that they are consistent with

$$\frac{1}{(1 - 2s\rho + \rho^2)^{\frac{1}{2}}} = \sum_{k=0}^{\infty} P_k(s)\rho^k,$$

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I explained the role of symmetry in relation to the real eigenvalues we encounter throughout the book when we do separation of variables. This is part of Section 5.1.7 below. What follows below is part of what every math student should be aware of. May be skipped if you already know what Hilbert spaces are, and what the Riesz Representation Theorem says. The exercises in the sections before Section 5.1.7 are not part of the programme for this course.

5.1 Some functional analysis and measure theory

Functional analysis is a toolkit for solving equations in which the unknowns are functions rather than numbers. For instance, we may want to find a function $f = f(x)$ such that, for every $x \in [0, 1]$,

$$f(x) - \int_0^x \sin(x-t)f(t)dt = \cos(x).$$

This is an example of a linear integral equation. The left hand side defines a function of a function, which we refer to as a (linear) functional acting on the variable function f. In the nonlinear integral equation

$$f(x) - \int_0^x \sin(x-t)f(t)^2 dt = \cos(x),$$

the left hand side defines a nonlinear functional.

Most of the equations we solved in analysis and linear algebra required finding a solution as a number or a finite set of numbers, which, substituted in some given function, would make it zero, or would maximise or minimise it. Consequently we learned in linear algebra and analysis all sorts of things about linear and nonlinear functions defined on subsets of $\mathbb{R}^m$ and $\mathbb{C}^m$, finite dimensional vector spaces over the real or complex numbers, equipped with a natural (inner product) norm. Our lifes were made easy by the fact that bounded closed sets in $\mathbb{R}^m$ and $\mathbb{C}^m$ are compact so that bounded sequences have convergent subsequences. Another fact taking completely for granted was the continuity of linear functions.

In the infinite-dimensional setting needed to solve problems such as the integral equations above, we first need good normed vector spaces in which our solutions are to be found. There are many different possibilities to assign a norm to a function, leading to different spaces. There are many candidates for $\mathbb{R}^\infty$ so to speak. It should be emphasized that in applications it is usually the norm which appears first, and then leads to the introduction of vector space on which the norm is well defined. This space will not be of much use unless Cauchy sequences in this space are convergent (with, by definition, the limit belonging to the same space). The theory of even only linear functionals is a subtle issue in which linear algebra and analysis (epsilons and delta's) merge.

5.1.1 Banach spaces

We begin with the concept of a (real) Banach space and related concepts.

Definition 1 *Let X be a normed vector space. A sequence x_n in X, indexed by $n \in \mathbb{N}$, is called*

convergent in X if $\quad \exists\, \bar{x} \in X \quad \forall\, \epsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n \geq N \quad ||x_n - x|| \leq \epsilon$;

Cauchy if $\quad \forall\, \epsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m, n \geq N \quad ||x_n - x_m|| \leq \epsilon$.

If all Cauchy sequences in X are convergent, then X is called a complete normed space or a Banach space. A set $\mathcal{O} \subset X$ is called open if

$$\forall\, \bar{x} \in \mathcal{O} \quad \exists\, \epsilon > 0 \quad B(\bar{x}, \epsilon) = \{x \in X : ||x - \bar{x}|| < \epsilon\} \subset \mathcal{O}$$

A set $\mathcal{G}$ is called closed if $\mathcal{G}^c = \{x \in X : x \notin \mathcal{G}\}$ is open.

The open sets thus defined form a topology: the empty set $\emptyset$ is open, X is open, arbitrary unions of open sets are open, and finite intersections of open sets are open. Equivalent norms on X should give the same convergent and Cauchy sequences, and the same open sets.

Exercise 1 So how would you define two norms on X to be equivalent?

Exercise 2 Let Y be a Banach space and let $X \subset Y$ be a linear subspace. Prove that X is closed if and only if X is Banach.

If a normed space X is not complete, it can be made complete by "adding" to X all limits of Cauchy sequences which are not convergent in X. If X is a subspace of some larger Banach space Y then X is complete if and only if X is closed in Y:

Exercise 3 Let Y be a Banach space and let $X \subset Y$ be a linear subspace. Prove that X is closed if and only if X is Banach.

5.1.2 Finite-dimensional spaces

The first example of a Banach space is $\mathbb{R}^m$, which has finite dimension m. Note that the dimension of a vector space X is the supremum (possibly $+\infty$) of all n for which there exist n linearly independent $x_1, \ldots, x_n \in X$, i.e.

$$\lambda_1 x_1 + \cdots \lambda_n x_n = 0 \quad \Rightarrow \quad \lambda_1 = \cdots = \lambda_n = 0$$

There are many norms on $\mathbb{R}^m$, but they are all equivalent. This follows from the fact that in $\mathbb{R}^m$ bounded closed sets are compact (equivalent: bounded sequences have convergent subsequences):

Exercise 4 (i) Show that in every normed space the function $x \to ||x||$ is continuous. (ii) Assume that $x_1, \ldots, x_n \in X$ are linearly independent. Define the map $L : \mathbb{R}^n \to X$ by $L(\xi) = L(\xi_1, \ldots, \xi_n) = ||\xi_1 x_1 + \cdots \xi_n x_n||$. Show that L is continuous and that there exists $0 < m \leq M < \infty$ such that $m \leq L(\xi) \leq M$ for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ with $|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2 = 1$. (iii) Show by scaling that $m|\xi| \leq L(\xi) \leq M|\xi|$ for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. (iv) Show that on a finite-dimensional vector space all norms are equivalent. (v) Show that every finite-dimensional normed space is complete.

Exercise 5 For a sequence x_n in a Banach space X let $s_n = x_1 + \cdots + x_n$. Show that the sequence s_n is convergent if

$$\sum_{N=1}^{\infty} ||x_n|| < \infty.$$

In view of Exercise 3 above every finite-dimensional normed space X has the Heine-Borel property: closed bounded subsets are compact. The Heine-Borel property characterises finite-dimensional spaces: in any infinite-dimensional normed space it is possible, given a $0 < \delta < 1$, by using what is known as Riesz' lemma, to find a sequence x_n with $||x_n|| = 1$ and $||x_n - x_m|| \geq \delta$ if $m \neq n$. Such a sequence is bounded (it lies on the unit sphere), but cannot have a convergent subsequence because all mutual distances are larger than $\delta > 0$.

5.1.3 Spaces of continuous functions

Functional analysis is (linear and nonlinear) analysis in infinite-dimensional complete normed spaces (Banach spaces). Complete because we can hardly prove anything if Cauchy sequences are not convergent. Functions on such spaces are often called functionals, so as to distinguish them from the familiar functions defined on subsets of $\mathbb{R}^m$. Many Banach spaces consist of such ordinary functions, for instance the space $C([0,T])$ consisting of all real valued continuous functions on a closed bounded interval $[0,T]$, also denoted as $C^0([0,T])$.

Exercise 6 Prove that $C([0,T])$ equipped with the maximum norm is a Banach space. Construct a bounded sequence which does not have a convergent subsequence. Show also that

$$||x||_2 = \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}}$$

defines a norm on $C([0,T])$, but that with this norm the space is not complete.

$C([0,T])$ is of course infinite-dimensional, but not too infinite-dimensional. It is a separable normed space: there exists a sequence x_n in $X = C([0,T])$ such that every element in X is the limit of some subsequence of x_n (equivalent: there exists a countable dense subset). Non-separable spaces are to large for man to handle and should be avoided.

Exercise 7 Let $x \in C([0,T])$. Use the uniform continuity of $x(t)$ on $[0,T]$ to show that the (continuous) piecewise linear functions are dense in $C([0,T])$ and prove that $C([0,T])$ is separable.

Existence and uniqueness results for differential equations

$$\frac{dx}{dt} = f(x),$$

with f Lipschitz continuous, given some initial value $x(0) = x_0$, are proved using Banach's fixed point theorem in $C([0,T])$, after rewriting de ODE as an integral equation.

5.1.4 Lebesgue spaces

The next important class of function spaces are the Lebesque spaces. Roughly speaking $L^p(\Omega)$, where $1 \le p < \infty$, is the space of all (Lebesque measurable) functions $u : \Omega \to \mathbb{R}$ for which the p-norm

$$||u||_p = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}$$

is well-defined. In applications one restricts to Ω open and connected, with varying smoothness assumptions on the boundary of Ω.

If $||u - v||_p = 0$ then $u(x) = v(x)$ for almost every $x \in \Omega$. By general agreement u and v are then considered equivalent: we do not bother about modifications of u on sets of measure zero. Needless to say that whenever possible we choose the nicest u among all functions equivalent to a particluar measurable function.

The Lebesgue spaces need the concept of Lebesgue measure, which extends the definition of

$$\mu(I) = (b_1 - a_1)\cdots(b_n - a_n) \quad \text{for blocks} \quad I = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

to a collection Λ_m of so-called Lebesgue measurable subsets E of $\mathbb{R}^m$. We briefly recall the essential step. The starting point is that for measurable subsets $F \subset \mathbb{R}^m$ it should certainly be true that

$$F \subset \cup_{n=1}^{\infty} I_n \quad \Rightarrow \quad \mu(F) \leq \sum_{n=1}^{\infty} \mu(I_n),$$

so that an obvious definition of $\mu(F)$ would be

$$\mu(F) = \inf_{F \subset \cup_{n=1}^{\infty} I_n} \sum_{n=1}^{\infty} \mu(I_n) \in [0, \infty],$$

being the best we can do using our blocks. This definition makes sense for every subset $F \subset \mathbb{R}^m$, and obviously one has

$$\forall\, E, F \subset \mathbb{R}^m \quad \mu(F) \leq \mu(F \cap E) + \mu(F \cap E^c),$$

but it is impossible to prove that this statement remains true when $\leq$ is replaced by $=$, as one would expect or hope.

The Lebesgue measurable subsets $E \subset \mathbb{R}^m$ are precisely those subsets E with

$$\forall\, F \subset \mathbb{R}^m \quad \mu(F) = \mu(F \cap E) + \mu(F \cap E^c)$$

(cutting a loaf of bread F with E does not miraculously increase the amount of bread....). In fact it is sufficient to check that equality holds for all blocks F. The collection Λ_m and the map $\mu : \Lambda_m \to [0, \infty]$ allow all countable operations one could reasonably expect to be allowed.

Remark. For the construction of nonmeasurable subsets one needs Zorn's lemma ($\Leftrightarrow$ Axiom of choice, uncountable version), which goes beyond the ability of mortal man. Thus for all practical purposes, all subsets of $\mathbb{R}^m$ we encounter in daily life of applicable math are measurable, at least as long as we resist the temptation of Zorn.

Integrals are defined by approximation with measurable stepfunctions (linear combinations of characteristic function of measurable sets). Each $f \in L^1(\mathbb{R}^N)$ may be approximated in L^1-norm with a sequence of compactly supported smooth functions. In other words, the compactly supported smooth functions are dense in $L^1(\mathbb{R}^N)$. The same statement holds in $L^p(\Omega)$, where $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$ is open.

5.1.5 Hilbert space theory

Banach spaces in which the norm comes from an inner product are called Hilbert spaces. A fundamental theorem for Hilbert spaces is:

Theorem 1 *Let H be a Hilbert space and $K \subset H$ a closed convex subset. For every point $x \in H$ there exists a unique point $u \in K$ which is closer to x than any other point of K. The point $u \in K$ is called the projection of x on K, denoted by $u = P_K x$. An important special case is that of K being a linear subspace of H.*

The proof takes a minimizing sequence u_n and uses the parallellogram law to show that this sequence is Cauchy. In fact:

Exercise 8 Prove that $||P_K x_1 - P_K x_2|| \leq ||x_1 - x_2||$ for all $x_1, x_2 \in H$. Note that in general P_K is not a linear map.

Finite-dimensional Hilbert spaces can, as far as their Hilbert space structure is concerned, be identified with $\mathbb{R}^m$, whereas every separable infinite-dimensional Hilbert space can be identified with the standard Hilbert space

$$l^2 = \{x = (x_1, x_2, x_3, \dots) : ||x||_2^2 = \sum_{i=1}^{\infty} |x_i|^2 < \infty\}.$$

It is easy to see that the unit ball in l^2 is not compact, because the unit basis vectors form a sequence which is bounded, while all mutual distances equal $\sqrt{2}$.

There is another remarkable difference between l^2 and $\mathbb{R}^m$, as the following nontrivial exercise shows.

Exercise 9 Show there exists a continuous map from the closed unit ball in l^2 to the closed unit sphere which leaves the sphere pointwise invariant (goodbye Brouwer).

5.1.6 Compact linear maps

We need the following theorem and the notion of compactness for an operator.

Theorem 2 *Let X and Y be normed spaces. For a linear map $A : X \to Y$ continuity in any point is equivalent to*

$$||A||_{op} = \sup\{||Ax||_Y : ||x||_X \leq 1\} < \infty.$$

This property of a linear map A is often called boundedness of A: bounded on the unit ball (on all balls in fact), but not on X of course. Note that then also $||Ax_1 - Ax_2||_Y \leq ||A||_{op} \, ||x_1 - x_2||_X$ for all $x_1, x_2 \in X$.

A special case is $Y = \mathbb{R}$. The vector space X^* consisting of all continuous linear $F : X \to \mathbb{R}$ is denoted by X^* and

$$||F|| = \sup\{|Fx| : ||x||_X \leq 1\}$$

defines a norm on X^*. This dual space X^* can be quite different from X. However: the **Riesz representation theorem** states that the continous linear real valued functions F on a Hilbert space are precisely the functions $x \to (x, y)$, and that the norm of F equals the norm of the corresponding y. So, as normed vector spaces, H and H^* may be identified.

Also as a consequence of Riesz the closed hyperplanes in H are precisely the sets of the form

$$\{x \in H : (x, y) = c\} \qquad y \in H, \quad c \in \mathbb{R},$$

just as in $\mathbb{R}^n$. The proof of Riesz' representation theorem relies on the projection theorem above.

We say that $A : X \to Y$ is called compact if the image sequence of any bounded sequence in X always has a convergent subsequence in Y.

Exercise 10 Show that A compact and linear implies A continuous. To avoid confusion assume that Y is Banach.

The strikingly easily proved **Hilbert-Schmidt theorem** (look it up, or see Section 5.3 below) states that in a Hilbert space H **every compact symmetric** (i.e. $(Ax, y) = (x, Ay)$ for all $x, y \in H$) **linear operator** $A : H \to H$ comes with a **basis of eigenvectors corresponding to a sequence of real eigenvalues which converge to zero** . This statement generalises the corresponding theorem for symmetric matrices.

Linear differential operators in ODE and PDE applications are never continuous from a (subspace of a) normed space X into itself. Such operators are called unbounded. In avoiding these unbounded operators and still solve PDE's the next section is crucial.

5.1.7 General framework for eigenvalue problems

The framework below applies to many examples in the book.

Let H and V be Hilbert spaces such that $V \subset H$. The inner product on H is denoted by single brackets, the inner product on V by double brackets. We shall write

$$(u, u) = |u|^2 \quad \text{for} \quad u \in H \qquad \text{and} \qquad ((u, u)) = ||u||^2 \quad \text{for} \quad u \in V$$

Throughout this subsection we assume that V is dense in H, and that V is compactly embedded in H, meaning that the inclusion map $i : V \to H$ defined by $i(x) = x$ is compact (and thus also bounded).

(i) Let $f \in H$. Then $v \to (f, v)$ defines a continous linear $\mathbb{R}$-valued map not only on H but also on V, by composing it with $i : V \to H$:

$$v \in V \to v \in H \to (f, v) \in \mathbb{R}.$$

Thus (for free!) there exists a unique $u \in V$ such that $((u, v)) = (f, v)$ for all $v \in V$. Denote $u = Af$. It is easy to show (do it!) that

(ii) $A : H \to V$ is injective.

(iii) $A : H \to H$ is linear, symmetric and compact.

(iv) $A : V \to V$ is linear, symmetric and compact.

(v) $A : H \to H$ is positive, i.e. $(Af, f) > 0$ if $f \neq 0$.

(vi) $A : V \to V$ is positive, i.e. $((Af, f)) > 0$ if $f \neq 0$.

By the Hilbert-Schmidt theorem:

(vii) H has an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ of eigenvectors of A corresponding to positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$, with $\lambda_n \to 0$ as $n \to \infty$, where

$$\lambda_1 = \max_{f \in H} \frac{(Af, f)}{(f, f)}$$

and, more generally, for $n > 1$,

$$\lambda_n = \max_{f \in H, (f, \phi_1) = \cdots = (f, \phi_{n-1}) = 0} \frac{(Af, f)}{(f, f)}$$

(viii) V also has an orthonormal basis $\{\psi_1, \psi_2, \ldots\}$ of eigenvectors of A, which are multiples of $\{\phi_1, \phi_2, \ldots\}$.

Exercise 11 What are these multiples? What are the corresponding eigenvalue formula's for $A : V \to V$? Evaluate these formula's in terms of norms only, i.e. without A appearing in the formula's. Hint: use the definition of A.

Let

$$H = L^2(0, 1) = \{f : (0, 1) \to \mathbb{R} \mid f \text{ is measurable}, \int_0^1 f^2 < \infty\},$$

equipped with the inner product $(f, g) = \int_0^1 fg$. We say that $g \in L^1_{loc}(0, 1)$ (locally integrable functions) is a weak derivative of f if

$$\int_0^1 gv = -\int_0^1 fv'$$

for all $v \in C^1([0, 1])$ with $v(0) = v(1) = 0$. One can show that g is unique if it exists, and that $f(x) - f(y) = \int_y^x g$ for all $0 < y < x < 1$. We write $g = f'$. On

$$V = \{f \in C([0, 1]) \mid f(0) = f(1) = 0, \ f' \text{ exists}, \ f' \in L^2(0, 1)\}$$

we take the inner product $((f, g)) = \int_0^1 f'g'$.

Exercise 12 Now let $f \in C([0, 1])$ and suppose that we look for $u \in C^2([0, 1])$ with $-u'' = f$ and $u(0) = u(1) = 0$. This boundary value problem can be solved by means of direct integration and the appropriate choice of integration constants. Show that $u = Af$, with A as above. In other words, with the operator A we solve this boundary value problem. What are the eigenfunctions and eigenvalues of A?

Exercise 13 The λ's thus obtained correspond to the reciprocals of the λ's in Chapter 6. In particular, Equation (1) on page 333 should be considered with y replaced by u, λy replaced by f, q replaced by $-q$ so as to get

$$-(p(x)u'(x))' + q(x)u(x) = r(x)f(x).$$

Take boundary conditions $u(a) = u(b) = 0$. Multiply by a smooth $v(x)$ with $v(a) = v(b) = 0$. Show that for p and r continuous and positive, and q continuous and nonnegative, the approach above works: what do you you use for $(\cdot, \cdot)$ and $((\cdot, \cdot))$? Relate the eigenvalues λ in Equation (1) on page 333 to the eigenvalues above. You should get something which involves maximizing or minimizing the quotient of

$$\int ru^2 \quad \text{and} \quad \int pu'^2 + qu^2.$$

5.2 Homework set 3

Hand in May 10 during before the course

1. Section 5.5, Exercise 25 (Reduction of order, this exercise show that only the polynomial solutions appear in the spherical harmonics).

2. Section 5.5, Exercise 31, the first one, also: 31(c) Derive Bonnet's recurrence formula (3) on Page 309 directly from 31(b) by differentiating the power series.

3. For every continuous $f : [0,1] \to \mathbb{R}$ the unique solution $u : [0,1] \to \mathbb{R}$ of $-u''(x) = f(x)$ with $u(0) = u(1) = 0$ can be computed directly. Show that
$$u(x) = \int_0^1 G(x,y)f(y)dy,$$
with $G(x,y) = x(1-y)$ for $0 \leq x \leq y$ and $G(x,y) = y(1-x)$ for $y \leq x \leq 1$ or so.

4. For every continuous compactly supported $f : \mathbb{R} \to \mathbb{R}$ (i.e. $f(x) = 0$ outside some bounded interval) the unique bounded solution $u : \mathbb{R} \to \mathbb{R}$ of $-u''(x) + u(x) = f(x)$ is given by a formula of the form
$$u(x) = \int_{-\infty}^{\infty} G(x,y)f(y)dy.$$
Determine $G(x,y)$. Hint use varation of variables
$$u(x) = A(x)e^x + B(x)e^{-x},$$
etc.

5.3 Extra: symmetric linear maps and quadratic forms

Let H be a Hilbert space and $A : H \to H$ linear. Then continuity of A is equivalent to
$$||A|| = \sup\{||Ax|| : ||x|| \leq 1\} < \infty.$$
The quadratic form
$$Q(x) = (Ax, x)$$
has the property that
$$M = \sup\{|Q(x)| : ||x|| \leq 1\} \leq ||A||,$$
since $|Q(x)| = |(Ax,x)| \leq ||Ax||\,||x|| \leq ||A||\,||x||\,||x||$. In fact:

Theorem 3 *If A is linear, continuous and symmetric (i.e. $(Ax,y) = (x,Ay)$) then $M = ||A||$.*

We remark that if H is a Hilbert space and $A : H \to H$ is linear and symmetric (i.e. $(Ax,y) = (x,Ay)$), then the Hellinger-Toeplitz theorem asserts that A is continuous. This relies on the concept of weak topologies and will not be treated here.

Exercise 14 Proof this theorem by showing that $||A|| \leq M$. Hint: use

$$4(Ax, y) = (A(x + y), x + y) - (A(x - y), x - y),$$

$$|(A(x \pm y), x \pm y)| \leq ||A|| \, ||x \pm y||^2,$$

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2,$$

and put $y = Ax/||Ax||$ with $||x|| \leq 1$ to conclude.

The key point is that any $x \in H$ with $||x|| = 1$ wich realizes M is an eigenvector with eigenvalue $\pm M$. This can be proved by copying the proof of this statement for the case that $H = \mathbb{R}^2$ and A is a symmetric 2x2 matrix. To prove that such x always exists we need extra information.

Theorem 4 *If A is linear, compact and symmetric then M is achieved (and thus A has an eigenvector).*

To prove this theorem choose a sequence x_n with $||x_n|| = 1$ and $Q(x_n) \to \pm M$. Without loss of generality we assume that $Q(x_n) \to +M$. The definition of A being compact states that if x_n is bounded then the sequence Ax_n has a convergent subsequence. Thus we may choose our sequence in such a way that Ax_n converges. We claim that consequently also x_n converges (to a limit which realizes M). This follows from

$$Ax_n - ||A||x_n \to 0,$$

which we infer from

$$0 \leq ||(Ax_n - ||A||x_n||)||^2 = (Ax_n - ||A||x_n, Ax_n - ||A||x_n) =$$

$$||Ax_n||^2 - 2||A||(Ax_n, x_n) + ||A||^2||x_n||^2,$$

in which the first term is less then $||A||^2 = M^2$, the third term equals $||A||^2 = M^2$, and the middle term converges to $-2||A||M = -M^2$.

In particular, if $A : H \to H$ is linear, compact, symmetric and positive (meaning $(Ax, x) > 0$ for nonzero x) then

$$\lambda_1 = \sup_{||x|| \leq 1} (Ax, x) = \max_{||x|| \leq 1} (Ax, x)$$

is a postive eigenvalue. The argument may then be repeated for $A : H_1 \to H_1$, where H_1 is the subspace of vectors orthogornal to the corresponing eigenvector. This produces a nonincreasing sequence of positive eigenvalues λ_n with mutually orthogornal (unit) eigenvectors. The compactness of A implies that $\lambda_n \to 0$ (why?). The eigenvectors thus obtained form a Hilbert basis of H.

5.4 Green's function for the Laplacian on a ball

The Gauss divergence theorem says that for $\Omega \subset \mathbb{R}^N$ bounded and open with continuously differentiable boundary $\partial\Omega$ and $V : \bar{\Omega} \to \mathbb{R}^N$ continuously differentiable,

$$\int_\Omega \nabla \cdot V = \int_{\partial\Omega} V \cdot n,$$

where n is the outward unit normal on $\partial\Omega$. Note that $n = n(x)$. In the integral on the left we dropped the usual $dx = dx_1 \dots dx_N$ from the notation, and in the integral on the right we did not write the usual dS or $dS(x)$. So take care what the integration variable is, because below we'll have x and y. I will try to be consistent and stick to x as integration variable(s).

With $V = u\nabla v$ this becomes

$$\int_\Omega (\nabla u \cdot \nabla v + u\Delta v) = \int_{\partial\Omega} u\frac{\partial v}{\partial n},$$

from which, interchanging u and v and subtracting, we get

$$\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}),$$

the two Green's formulas. These formula's do not directly apply to the radially symmetric function defined by (with some abuse of notation)

$$\Gamma(x) = \Gamma(|x|) = \Gamma(r), \quad \Gamma'(r) = -\frac{1}{|\partial B_r|},$$

where $|\partial B_r|$ is the measure of the N-dimensional sphere with radius r, and $\Gamma(r)$ is the primitive of $\Gamma(r)$ (take the integration constant such that $\Gamma(\infty) = 0$ if $N \geq 3$, $\Gamma(r) = -\frac{1}{2\pi}\log r$ for $N = 2$, $\Gamma(r) = -\frac{r}{2}$ for $N = 1$), but applying Green's formula with $u = \Gamma$ and $v = 1$ on $\Omega = B_R(0)$ gives

$$\int_{B_R(0)} \Delta\Gamma = \int_{\partial B_R(0)} \Gamma'(R) = -\int_{\partial B_R(0)} \frac{1}{|\partial B_R(0)|} = -1,$$

while $\Delta\Gamma = 0$ outside the origin. Apparently

$$-\Delta\Gamma = \delta,$$

where δ is a unit point mass Dirac "function" located at the origin. Whatever that means, we have in fact that, provided $0 \in \Omega$, Ω nice as above, for C^2-functions $\psi : \bar{\Omega} \to \mathbb{R}$, that

$$\psi(0) = \int_{\partial\Omega} (\Gamma\frac{\partial\psi}{\partial n} - \psi\frac{\partial\Gamma}{\partial n}) - \int_\Omega \Gamma\Delta\psi,$$

the latter integral being "improper" in the origin.

Exercise 15 Apply Green's formula to Ω with a small ball $B_\epsilon(0)$ cut out and take the limit $\epsilon \to 0$ to prove this formula.

Next, write, with more abuse of notation,

$$\Gamma(x, y) = \Gamma(x - y).$$

Then, provided $u \in \Omega$, and replacing ψ by u,

$$u(y) = \int_{\partial\Omega} (\Gamma(x, y)\frac{\partial u(x)}{\partial n} - u(x)\frac{\partial\Gamma(x, y)}{\partial n})dS(x) - \int_\Omega \Gamma(x, y)\Delta u(x),$$

which is called Green's representation formula.

Since, for any C^2-function $h(x, y)$ which is harmonic in x,

$$0 = \int_{\partial\Omega} (h(x, y)\frac{\partial u(x)}{\partial n} - u(x)\frac{\partial h(x, y)}{\partial n})dS(x) - \int_{\Omega} h(x, y)\Delta u(x),$$

we have,

$$u(y) = -\int_{\partial\Omega} (u(x)\frac{\partial G(x, y)}{\partial n})dS(x) - \int_{\Omega} G(x, y)\Delta u(x),$$

provided also

$$G(x, y) = \Gamma(x, y) + h(x, y) = 0 \quad \text{for} \quad x \in \partial\Omega.$$

The art is to find such a function $h(x, y)$. Once this is done, we can express u inside Ω in terms of its values on the boundary and its Laplacian inside the domain.

Finding $h(x, y)$ in closed form is easy for the case that $\Omega = B_R(0)$. For $y \in B_R(0)$ define

$$\bar{y} = \frac{R^2}{|y|^2}y,$$

the mirror image of y under reflection in the sphere $\partial B_R(0)$. The, irrespective of y, we have

$$\partial B_R(0) = \{x \in \mathrm{IR}^N : |x - y| = \frac{|y|}{R}|x - \bar{y}|\},$$

so that

$$h(x, y) = -\Gamma(\frac{|y|}{R}|x - \bar{y}|)$$

does the job, i.e.

$$G(x, y) = \Gamma(|x - y|) - \Gamma(\frac{|y|}{R}|x - \bar{y}|).$$

Exercise 16 Show that for $G(x, y)$ and $B_R(0)$ as above,

$$\frac{\partial G(x, y)}{\partial n} = -\frac{R^2 - |y|^2}{R|\partial B_1(0)|\,|x - y|^N},$$

and relate this to Theorem 2 on Page 642. It will be convenient to write $r = |x - y|$, $\bar{r} = |x - \bar{y}|$ and use (check) that

$$\frac{\partial r}{\partial x_i} = \frac{x_i - y_i}{r} \quad \Rightarrow \quad \frac{\partial r}{\partial n} = \sum_{i=1}^{N} \frac{x_i}{R}\frac{x_i - y_i}{r},$$

and likewise for $\bar{r}$.

5.5 Mean value properties for harmonic functions

Let $\Omega \subset \mathrm{IR}^N$ be open and $u : \Omega \to \mathrm{IR}$ twice continuously differentiable. Starting with the Gauss divergence theorem applied to a ball

$$\bar{B}_r(x_0) = \{x \in \mathrm{IR}^N : |x - x_0| \le r\} \subset \Omega,$$

we have

$$\int_{|x-x_0|\le r} \Delta u \, dx = \int_{|x-x_0|=r} \frac{\partial u}{\partial n} \, dS(x) =$$

(write $x = x_0 + r\xi$, express the normal derivative as r-derivative and change integration variable to ξ)

$$r^{N-1} \int_{|\xi|=1} \frac{\partial}{\partial r} u(x_0 + r\xi) \, dS(\xi) = r^{N-1} \frac{d}{dr} \int_{|\xi|=1} u(x_0 + r\xi) \, dS(\xi),$$

by interchanging integration and r-differentiation. Thus

$$\frac{d}{dr} \int_{|\xi|=1} u(x_0 + r\xi) \, dS(\xi) = \frac{1}{r^{N-1}} \int_{|x-x_0|\le r} \Delta u \, dx.$$

This says that **the average value of $u(x)$ on a sphere $|x - x_0| = r$ changes with r according to the integral of Δu. If u is harmonic then this average value does not change with r!** Clearly the average goes to $u(x_0)$ if $r \to 0$. Conclusion: **harmonic functions have the mean value property that in every point they coincide with their averages over spheres (and balls) centered in this point.** Of course this only holds for radii smaller than the radius of the largest such open ball contained in the domain.

Exercise 17 A twice continuously differentiable function $u : \Omega \to \mathbb{R}$ is called *subharmonic* if $\Delta u \ge 0$ in Ω. Show that in every point $x_0 \in \Omega$ the value $u(x_0)$ is less or equal than its local averages over closed balls $\bar{B}_r(x_0) \subset \Omega$. If a continuous function $u : \Omega \to \mathbb{R}$ with this propery has a point x_0 for which

$$u(x_0) \ge u(x) \quad \forall x \in \Omega$$

then, provided $\Omega \subset \mathbb{R}^N$ is also connected, $u(x)$ is constant in Ω. Prove this by showing that in such a case $\{x \in \Omega : \ u(x) = u(x_0)\}$ is both open and closed.

Exercise 18 A continuous function $u : \Omega \to \mathbb{R}$ with the mean value property is harmonic: it is twice continuously differentiable and satisfies $\Delta u = 0$ in Ω. Prove this by taking a fixed but arbitrary closed ball in Ω, apply Poisson's integral formula to construct a function U which is continuous on the closed ball, harmonic inside, and coincides with U on the boundary of the ball, and then apply the previous exercise to show that u and U coincide on the ball. Remark: since the Poisson integral formula allows differentiation under the integral the first order derivatives of u are also continuous and harmonic. Hence all derivatives of u are continuous and harmonic.

Exercise 19 Show that orthogornal transformations leave harmonic functions harmonic.

Exercise 20 Show the divergence theorem implies that for $\Omega \subset \mathbb{R}^N$ bounded and open with continuously differentiable boundary $\partial\Omega$ and $u : \bar{\Omega} \to \mathbb{R}$ continuously differentiable,

$$\int_\Omega \nabla u = \int_{\partial\Omega} un,$$

where n is the outward unit normal on $\partial\Omega$. Both integrals are vectorvalued integrals. Hint: apply Gauss to $V = ue_i$ where e_i is the i-th standard basis vector.

Exercise 21 Use the previous exercise to show

$$|\nabla u(x_0)| \leq \frac{N}{\rho}|u(x_0)|$$

if $u : \Omega \to \mathbb{R}$ is nonnegative and harmonic and $\bar{B}_\rho(x_0) \subset \Omega$ where $\rho > 0$.

Exercise 22 Show that a bounded harmonic function $u : \mathbb{R}^N \to \mathbb{R}$ is constant.

Exercise 23 Show that a bounded sequence of harmonic functions $u_n : \Omega \to \mathbb{R}$ is equicontinuous on

$$\Omega_\rho = \{x \in \Omega : \bar{B}_\rho(x) \subset \Omega.$$

Thus we can take a subsequence which converges uniformly on every compact subset $K \subset \Omega$. Show this and that the limit function is also harmonic.

Exercise 24 Let $u : \Omega \to [0, \infty)$ be harmonic, $\rho > 0$, Ω_ρ as above and $x \in \Omega_\rho, y \in \Omega_{2\rho}$ with $|x - y| \leq \rho$. Use the mean value property of u on $\bar{B}_\rho(x)$ and $\bar{B}_{2\rho}(y)$ to show that $u(x) \leq 2^N u(y)$.

Exercise 25 Use the previous exercise to show that for all bounded open $\Omega \subset \mathbb{R}^N$ there exists a constant C which depends only on Ω and ρ such that

$$\max_\Omega u \leq C \min_\Omega u$$

for all harmonic $u : \Omega \to [0, \infty)$.

6 Delta-functions and Green's functions

The basic idea to present here is that (partial) differential equations (P)DE's of the form $Lu = f$, $f : \Omega \to \mathbb{R}$ given, $u : \Omega \to \mathbb{R}$ to be found, satisfying the (P)DE as well as appropriate boundary conditions (BC's), can besolved by first solving the (P)DE with f replaced by a point mass in arbitrary point $y \in \Omega$. Just as we may sometimes write a function f as $f(\cdot)$, meaning that we can put something on the position of the dot, the informal notation for a point mass in y is

$$\delta(\cdot - y).$$

In $\mathbb{R}^3$ you should think of

$$x \to \delta(x - y)$$

as the mass distribution or density (which depends on the variable x) of a point mass located in y. The Dirac delta-function is not a actually a mathematical function. The theory of distributions developes the approriate formalism in which the physically "obvious" formula

$$f(x) = \int_\Omega \delta(x - y) f(y) dy \tag{6.1}$$

is also mathematically correct.

Denoting by $x \to G(x, y)$ the solution of $Lu = \delta(\cdot - y)$ which satisfies the BC's, the solution formula

$$u(x) = \int_\Omega G(x, y) f(y) dy$$

does make sense without this formalism, as we shall discuss a bit below. Note that this presentation has the x as variable in u, and the y as integration variable in the decomposition of both u and f. In the Green's function approach leading to Poisson's integral formula the roles of x and y were the other way around. This is only because we chose to start with integrals in which x was the integration variable. Note that we may interchange the role of x and y if we like, but have to be careful not to get confused.

6.1 Dirac delta-functions in one variable

Let $\delta_n : \mathbb{R} \to \mathbb{R}$ be a sequence of smooth nonnegative even functions with

$$\delta_n(x) = 0 \text{ for } |x| \geq \frac{1}{n}, \quad \int_{-\infty}^{\infty} \delta_n(x)dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} \delta_n(x)dx = 1. \tag{6.2}$$

It should be clear that we want to think of the limit

$$\delta = \lim_{n\to\infty} \delta_n$$

as an object which has a meaning, but that "functions" are not the appropriate concept to be used here. Note that for any $\epsilon > 0$ the function δ_n has the property that

$$\delta_n(x) = 0 \text{ for } |x| \geq \epsilon \ \text{ and } \int_{-\epsilon}^{\epsilon} \delta_n(x)dx = 1 \ \text{ if } \ n \geq \frac{1}{\epsilon},$$

so there is a real motivation to want to say that the limit object should satisfy

$$\delta(x) = 0 \text{ for } |x| \neq 0 \ \text{ and } \int_{0^-}^{0^+} \delta(x)dx = 1. \tag{6.3}$$

Before continuing we observe that (6.1) makes sense for any continuous function $f : \mathbb{R} \to \mathbb{R}$ as a limit case.

Exercise 26 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\delta_n : \mathbb{R} \to \mathbb{R}$ be a sequence of smooth nonnegative functions satisfying (6.2).

1. Define
$$H_n(x) = \int_{-\infty}^{x} \delta_n(y)dy,$$
and show that $H_n(0) = \frac{1}{2}$, $H_n(x) = 0$ for $x < -\frac{1}{n}$, $H_n(x) = 1$ for $x > \frac{1}{n}$, whence $H = \lim_{n\to\infty} H_n$ can be defined as function. It is called the Heavide function. Reflect a moment on the relevance of $H(0) = \frac{1}{2}$ and note that it makes sense to think of H as a primitive function of the delta-"function".

2. Accept for a fact that thus $H(x-y)$ is a primitive function of $\delta(x-y)$.

3. Define
$$f_n(x) = \int_{-\infty}^{\infty} \delta_n(x-y)f(y)dy$$
Prove that $f_n(x) \to f(x)$ as $n \to \infty$. Why is the convergence uniform on bounded x-intervals?

Exercise 27 Now assume that $f : \mathbb{R} \to \mathbb{R}$ is piecewise continuous, with only finitely many points of discontinuity on every bounded interval, and that in each such point the left and right limits exist. Prove that $f_n(x) \to \frac{1}{2}(f(x^-) + f(x^+))$ in every $x \in \mathbb{R}$.

6.2 Green's functions in one variable

We consider differential operators L of the form

$$L = \frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x),$$

meaning that for n times differentiable functions $u = u(x)$:

$$(Lu)(x) = u^{(n)}(x) + a_{n-1}(x)u^{(n-1)}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x),$$

or, for short,

$$Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1u' + a_0u,$$

Note and accept the bad notation with x, a consequence of us not liking to write D instead of $\frac{d}{dx}$. As already indicated, we want to solve

$$(Lu)(x) = u^{(n)}(x) + a_{n-1}(x)u^{(n-1)}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) = \delta(x-y),$$

in order to solve $Lu = f$.

6.2.1 $2nd$ order problems on a bounded interval

Exercise 28 Take $Lu = u''$, $y = 0$. Accepting (6.3) show that $Lu = u'' = \delta$ must imply

$$u' = C_- \text{ for } x < 0, \quad u' = C_+ \text{ for } x > 0, \quad C_+ - C_- = 1,$$

and solve, for fixed but arbitrary $0 < y < 1$,

$$u''(x) = \delta(x-y), \quad u(0) = u(1) = 0.$$

Denote the solution by $u(x) = G(x, y)$. Observe how this solution consists of 2 parts: a function defined for $x < y$ which solves $Lu = 0$ on the left and satisfies the left boundary condition $u(0) = 0$, and a function defined for $x < y$ which solves $Lu = 0$ on the right and satisfies the right boundary condition $u(1) = 0$. In other words, **a part which has the properties demanded to the left of y, and a part which has the properties demanded to the right of y, coinciding in y with a jump equal to 1 in the first derivative**.

Exercise 29 Look at the previous exercise and at exercise 3 of the third home work set in Section 5.2 above and observe that the $G(x, y)$ just computed agrees with the $G(x, y)$ there, except for a minus sign. You have now verified that $u'' = f$ with $u(0) = u(1) = 0$ is solved by

$$u(x) = \int_0^1 (G(x, y)f(y)dy.$$

The function $G(x, y)$ is called the Green's function for $u'' = f$ with boundary conditions $u(0) = u(1) = 0$.

Exercise 30 Using the explanation in Exercise 28 find the Green's function for $u'' = f$ with boundary conditions $u(0) = u'(1) = 0$.

Exercise 31 Can you find a Green's function for $u'' = f$ with boundary conditions $u'(0) = u'(1) = 0$?

Next we consider a first example where we cannot solve $Lu = f$ by just taking two primitives, and solve $Lu(x) = \delta(x-y)$ with the given BC's to obtain $G(x, y)$.

Exercise 32 To find the Greens' function for $u'' - u' = f$ with boundary conditions $u(0) = u'(1) = 0$, take $0 < y < 1$ fixed.

- solve $u''(x) - u'(x) = 0$ with $u(0) = 0$ for $x < y$.
- solve $u''(x) - u'(x) = 0$ with $u'(1) = 0$ for $x > y$.
- what should the jumps in u and u' be?
- use the two free constants to get the jumps right.

The resulting two functions together form the Greens' function $G(x, y)$. It is the (unique) solution of $u''(x) - u'(x) = \delta(x - y)$ with $u(0) = u'(1) = 0$.

We have not yet defined what this really means, nor will we do so. Observe though that, denoting the solution by $u(x)$ and not by $G(x, y)$, the lower order term $u'(x)$ is piecewise continuous. It has no contribution if we integrate it from $y-\epsilon$ to $y+\epsilon$ and take the limit $\epsilon \to 0$, whereas both $u''(x)$ and $\delta(x-y)$ (must/will) do. If we accept that the primitive of $u''(x)$ is $u'(x)$ and the primitive of $\delta(x-y)$ is $H(x-y)$, both $u''(x)$ and $\delta(x-y)$ have in fact exactly the same contribution if we integrate from $y - \epsilon$ to $y + \epsilon$ and take the limit $\epsilon \to 0$. The upshot of this reasoning is that the lower order term u' does/should not matter for the procedure to determine the Green's function. For any second order problem of the form $u''(x) + a(x)u'(x) + b(x)u(x) = f(x)$ with a zero-boundary condition on the left and a zero boundary condition on the right *may or may not* admit a Green's function obtained as in Exercise 32. In Exercise 31 the method does not work.

Exercise 33 See what goes wrong if you try to find a Green's function for $u'' + u = f$ with boundary conditions $u(0) = u(\pi) = 0$ following the method outlined in Exercise 32. Observe that the solutions on the left and the right have a nontrivial joint solution. In other words, $u'' + u = 0$ with boundary conditions $u(0) = u(\pi) = 0$ has a non-trivial solution.

Exercise 34 Convince yourself of the fact that the problem $u''(x)+a(x)u'(x)+b(x)u(x) = f(x)$ with $u(0) = u(1) = 0$ has a Green's function if and only if $u''(x) + a(x)u'(x) + b(x)u(x) = 0$ with $u(0) = u(1) = 0$ has no non-trivial solution.

Exercise 35 Compute a Green's function for $u'' + u = f$ with boundary condition $u(0) = u(L) = 0$ for $0 < L < \pi$.

6.2.2 Green's functions on $\mathbb{R}$

If we solve equations of the form $Lu = f$ on the whole of $\mathbb{R}$ there are no boundaries and no boundary conditions. How should we identify a unique solution? Informally speaking, we can expect to have to demand that $u(x)$ (and possibly derivatives) is (are) well-behaved as $x \to \pm\infty$. In many physical problems this can only mean: not too large.

Surely the answer will also depend on the assumptions we put on f. In addition to f being continous we will have to put some assumptions on $f(x)$ concerning its behaviour as $x \to \pm\infty$. However, if we go for a Green's function approach, we start by solving $Lu = \delta(\cdot - y)$, and the behaviour of this particular right hand side as $x \to \pm\infty$ is of no consideration at all. Thus we simply ignore f and proceed with trying to identify a solution of $Lu = \delta(\cdot - y)$ which is unique in having a property not shared by the other solutions of $Lu = \delta(\cdot - y)$. *Note that there may be different properties which do the trick.*

Exercise 4 of the third home work set in Section 5.2 above is illuminating. We will examine the difference between solving $u'' = f$ on $\mathbb{R}$ and $u'' - u = f$ on $\mathbb{R}$. In both cases there are two free constants in the general solution to get rid of. By the reasoning above we should first look at the difference between solving

$$u''(x) - u(x) = \delta(x - y)$$

and

$$u''(x) = \delta(x - y).$$

Note how, in both cases, we first end up with considering solutions of the homogeneous equation!

Exercise 36 Which solutions of $u'' - u = 0$ have exceptional behaviour as $x \to -\infty$ and which solutions have exceptional behaviour as $x \to \infty$? Use these solutions to construct a (unique) solution $u(x) = G(x, y)$ of $u''(x) - u(x) = \delta(x - y)$ which has the exceptional behaviour (that it goes to zero) as $x \to -\infty$ *and* as $x \to \infty$. Hint: first take $y = 0$ and determine $G(x, 0)$. Afterwards show that $G(x, y) = G(x-y)$ (and observe that this nice reduction is possible because there are no x-dependent coefficients in the equation.

Exercise 37 Mimic the approach above for $u''(x) = \delta(x - y)$ and observe it fails.

Exercise 38 If you want to solve $u''(x)+au'(x)+bu(x) = f(x)$ using a Green's function which is bounded, what is the condition on a and b that you need?

6.2.3 Green's functions for higher order problems

Solving

$$(Lu)(x) = u^{(n)}(x) + a_{n-1}(x)u^{(n-1)}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) = \delta(x - y)$$

requires solving

$$Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1 u' + a_0 u = 0$$

for $x < y$ and for $x > y$. We first examine the case that $y = 0$. Denote a solution defined for $x \le 0$ by $u_-(x)$ and a solution defined for $x \ge 0$ by $u_+(x)$.

The matching conditions in 0 which allow to combine $u_-(x)$ and $u_+(x)$ into a solution of

$$Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1 u' + a_0 u = \delta$$

are the jump condition

$$u_+^{(n-1)}(0) = u_-^{(n-1)}(0) + 1,$$

and the (zero) jump conditions

$$u_+(0) = u_-(0), \quad u'_+(0) = u'_-(0), \quad u''_+(0) = u''_-(0), \ldots, u_+^{(n-2)}(0) = u_-^{(n-2)}(0).$$

Exercise 39 Find a Green's function for $u'''(x) = f(x)$ with boundary condition $u(0) = u'(0) = u(1) = 0$.

Exercise 40 Find a Green's function for $u'''(x) = f(x)$ with boundary condition $u(0) = u'(0) = u'(1) = 0$.

Exercise 41 Is there a Green's function for $u'''(x) = f(x)$ with boundary condition $u(0) = u'(0) = u''(1) = 0$?

Exercise 42 If possible find a Green's function for $u'''(x) = f(x)$ with $u(0) = u''(0) = u'(1) = 0$.

Exercise 43 Find a Green's function for $u''''(x) = f(x)$ with boundary condition $u(0) = u'(0) = u(1) = u'(1) = 0$.

Exercise 44 If possible find a Green's function for $u''''(x) = f(x)$ with $u(0) = u'''(0) = u'(1) = u''(1) = 0$.

Exercise 45 If possible find a bounded Green's function for $u'''' + u(x) = f(x)$ on $\mathbb{R}$.