Additional material for Chapter 2

1 About Chapter 1

I will use L for the Lagrangian and not F. We assume that $L = L(t, u, p)$ is as smooth as we need. Chapter 1 concerned Euler-Lagrange equations for $u = u(t) \in \mathbb{R}^n$. We saw how minimizing

$$I(u) = \int_a^b L(t, u(t), \dot{u}(t))\, dt \tag{1.1}$$

for sufficiently smooth functions $u : [a, b] \to \mathbb{R}^n$ (with $u(a)$ and $u(b)$ prescribed) leads to the Euler-Lagrange system of differential equations:

$$\frac{d}{dt}\frac{\partial L}{\partial p^i} - \frac{\partial L}{\partial u^i} = 0 \quad (i = 1, \ldots, n) \tag{1.2}$$

We also saw the Jacobi equations, obtained from (1.3.6) and the linearised Lagrangian

$$\phi = \frac{\partial^2 L}{\partial p^i \partial p^j}\pi^i \pi^j + 2\frac{\partial^2 L}{\partial p^i \partial u^j}\pi^i \eta^j + \frac{\partial^2 L}{\partial u^i \partial u^j}\eta^i \eta^j \tag{1.3}$$

The Euler-Lagrange equations of (1.3) are the Jacobi equations

$$\frac{d}{dt}\frac{\partial \phi}{\partial \pi^i} - \frac{\partial \phi}{\partial \eta^i} = 0 \quad (i = 1, \ldots, n) \tag{1.4}$$

These Jacobi equations are the linearised Euler-Lagrange equations. Verify this!

For Lagrangians independent of t we noticed a conservation law. When you multiply (1.2) by $p^i(t) = \dot{u}^i(t)$ you get

$$0 = p^i(t)\frac{d}{dt}\frac{\partial L}{\partial p^i} - \dot{u}^i(t)\frac{\partial L}{\partial u^i} = \frac{d}{dt}\left(p^i \frac{\partial L}{\partial p^i}\right) \underbrace{-\dot{p}^i(t)\frac{\partial L}{\partial p^i} - \dot{u}^i(t)\frac{\partial L}{\partial u^i}}_{-\frac{dL}{dt}}$$

$$= \frac{d}{dt}\left(p^i \frac{\partial L}{\partial p^i} - L\right)$$

2 Riemannian metrics on submanifolds of $\mathbb{R}^d$

Chapter 2 deals with the problem of finding the shortest connecting curve between two given points in an n-dimensional submanifold M of $\mathbb{R}^d$ with $d \geq n$. For this we will need knowledge of the concept of covariant differentiation on M. The nonabstract introduction with submanifolds below provides a machinery that also works in the abstract setting of general Riemannian manifolds.

Locally M is given by smooth parametrizations

$$x = f(u)$$

(coordinate charts) defined on open connected sets $U \subset \mathbb{R}^n$ with smooth[1] transitions between u and $\tilde{u}$ on $U \cap \tilde{U}$ if $f : U \to M$ and $\tilde{f} : \tilde{U} \to M$ are two different coordinate patches. A (preferably finite[2]) collection with this property that describes the whole of M is called an atlas for M.

Every such parametrization provides us with locally defined tangent vector fields

$$x_1 = \frac{\partial x}{\partial u^1}, \cdots, x_n = \frac{\partial x}{\partial u^n},$$

since for every $u \in U$ the vectors $x_i(u)$ are tangent to M in $x(u) \in M$. The inner products

$$g_{ij} = g_{ij}(u) = x_i \cdot x_j$$

are locally defined scalar fields. The coefficients define a Riemannian metric on M, the metric inherited from the standard inner product in the ambient space $\mathbb{R}^d$, as is explained next.

In terms of local coordinates $u^1, \ldots, u^n$ tangent vector fields V on M are described by

$$V = V^i x_i = V^i(u) x_i(u) = V^1(u) x_1(u) + \cdots + V^n(u) x_n(u), \tag{2.1}$$

in which we use a summation convention for repeated lower and upper indices. Two such vectors fields have inner product

$$V \cdot W = V^i x_i \cdot W^j x_j = V^i W^j x_i \cdot x_j = V^i W^j g_{ij},$$

called the *first fundamental form*. Don't forget the u-dependence which is usually dropped from the notation and pay attention to the double use of subscripts: as indices in g_{ij} and as derivatives in x_i. The inner product of two tangent vector fields on M defines a scalar field[3] on M. The map

$$(V, W) \to V \cdot W \tag{2.2}$$

is well defined, independent of the choice of coordinates, and bilinear over the scalar fields, which makes the map a (symmetric) tensor. The scalars are real valued (smooth) functions $\phi, \psi : M \to \mathbb{R}$ and we have

$$(\phi V) \cdot (\psi W) = \phi\psi \, (V \cdot W)$$

The formula's hide the fact that this linearity differs from the usual linearity over $\mathbb{R}$ because the dependence on $x \in M$ is suppressed in the notation. The map (2.2) is a *Riemannian metric*, with metric coefficients $g_i j$ in local coordinates.

3 Covariant differentiation

If we differentiate a vector field V as given by (2.1) we get contributions from u-dependence in $V^i(u)$ and from u-dependence in $x_i(u)$. The tangential part of the resulting derivative is what is by definition the covariant derivative. The partial derivative of (2.1) with respect to u^j can be written as

$$\frac{\partial V}{\partial u^j} = \frac{\partial V^i}{\partial u^j} x_i + V^i x_{ij}, \quad x_{ij} = \frac{\partial x_i}{\partial u^j} = \frac{\partial^2 x}{\partial u^j \partial u^i} = \frac{\partial^2 x}{\partial u^i \partial u^j} = x_{ji} \tag{3.1}$$

[1]See Section 1.4 in the book
[2]This is related to the concept of compactness
[3]A real valued function

In the case that $M = \mathbb{R}^n = \mathbb{R}^d$ with $x^i = u^i$, the tangent vectors x_i are the unit base vectors e_i so that $x_{ij} = 0$ and the covariant partial derivatives of V are just the partial derivatives V. The same holds if $x(u)$ in linear in u. In all other cases we decompose x_{ij} as

$$x_{ij} = \Gamma^l_{ij} x_l + \textit{normal parts}$$

The coefficients Γ^l_{ij} are called the *Christoffel symbols*. Taking the inner product with x_k we get

$$\Gamma_{ijk} := x_{ij} \cdot x_k = \Gamma^l_{ij} x_l \cdot x_k = \Gamma^l_{ij} g_{lk}$$

Thus Γ_{ijk} is obtained from Γ^l_{ij} using g_{lk}. Introducing $g^{kl} = g^{lk}$ by

$$g_{lk} g^{km} = \delta^m_l,$$

we also obtain Γ^m_{ij} from Γ_{ijk}:

$$g^{mk} \Gamma_{ijk} = \Gamma^l_{ij} g_{lk} g^{km} = \Gamma^l_{ij} \delta^m_l = \Gamma^m_{ij}$$

The relation between both Γ-symbols is given by

$$\Gamma_{ijk} = \Gamma^l_{ij} g_{lk}, \quad \Gamma^m_{ij} = g^{mk} \Gamma_{ijk}$$

The metric coefficients are used to raise and lower the exponents[4].

Next we determine Γ_{ijk}. Differentiating g_{ij} with respect to u^k we get

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial u^k} = \frac{\partial}{\partial u^k}(x_i \cdot x_j) = x_{ki} \cdot x_j + x_{jk} \cdot x_i = \Gamma_{kij} + \Gamma_{jki}$$

Note the two cyclic permutations kij and jki of ijk on the right. Using cyclic permutation, we have the following three equivalent forms of the resulting statement:

$$\begin{aligned} g_{ij,k} &= \Gamma_{kij} + \Gamma_{jki} \\ g_{jk,i} &= \Gamma_{ijk} + \Gamma_{kij} \\ g_{ki,j} &= \Gamma_{jki} + \Gamma_{ijk} \end{aligned}$$

Multiplying by $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$ and adding up we get

$$\Gamma_{ijk} = \frac{1}{2}\left(g_{jk,i} + g_{ki,j} - g_{ij,k}\right)$$

Using the symmetry $g_{ij} = g_{ji}$ it follows that

$$\Gamma_{ijk} = \frac{1}{2}\left(g_{jk,i} + g_{ik,j} - g_{ij,k}\right), \quad \Gamma^m_{ij} = \frac{1}{2} g^{mk}\left(g_{jm,i} + g_{im,j} - g_{ij,m}\right) \tag{3.2}$$

These formula's express the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ji}$ in terms of the metric coefficients g_{ij} and their first order derivatives, and can be used to write (3.1) as

$$\frac{\partial V}{\partial u^j} = \frac{\partial V^i}{\partial u^j} x_i + V^i \Gamma^l_{ij} x_l + \textit{normal parts}$$

[4]Just as with tensor coefficients, though the Γ's are not tensor coefficients

The tangential part is thus

$$D_{u^j}V := (\frac{\partial V}{\partial u^j})_T = \left(\frac{\partial V^l}{\partial u^j} + V^i\Gamma^l_{ij}\right)x_l, \quad V = V^i x_i \tag{3.3}$$

This is called the covariant derivative of V with respect to u^j. Both V and $D_{u^j}V$ are tangent vector fields, with components

$$V^i \quad \text{and} \quad (D_{u^j}V)^l = \frac{\partial V^l}{\partial u^j} + V^i\Gamma^l_{ij}$$

4 Tangent vectors as derivatives

Next we introduce the view point on tangent vectors as directional derivatives. Since every tangent vector defines a directional derivative, it has become customary to identify such first order differential operators with their direction vectors. In short, we think of

$$x_i = \frac{\partial x}{\partial u^i} \quad \text{and} \quad \frac{\partial}{\partial u^i}$$

as essentially the same objects. To see how this works in a point $x_0 \in M$ we use integral curves starting at x_0, that is, solutions of

$$\dot\gamma(t) = X(\gamma(t)), \quad \gamma(0) = x_0 \in M, \tag{4.1}$$

where X is a tangent vector field defined near x_0. The differential equation in (4.1) is called the *flow equation* for X. Using coordinates u, with $u = u_0$ corresponding to x_0, the expressions in (4.1) evaluate as

$$\gamma(t) = x(u(t)), \quad \dot\gamma(t) = \frac{\partial x}{\partial u^i}(u(t))\dot u^i(t) = \dot u^i(t)x_i, \quad X(\gamma(t)) = X^i(u(t))x_i,$$

so the system to be solved for $u = u(t)$ to obtain the integral curves is

$$\dot u^i = X^i(u), \quad u(0) = u_0. \tag{4.2}$$

The solution $u = u(t)$ exists locally and is unique. We have $\dot u^i(0) = X^i(u_0)$ and $X_0 := X(x_0) = \dot\gamma(0) = \dot u^i(0)x_i = X^i(u_0)x_i$. On scalar fields (functions) $\phi : M \to \mathbb{R}$, given in local coordinates as

$$\phi = \phi(u^1, \ldots, u^n),$$

the vector field X now acts through

$$\frac{d}{dt}|_{t=0}\phi(u(t)) = \frac{\partial \phi}{\partial u^i}(u_0)\dot u^i(0) = X^i_0\frac{\partial \phi}{\partial u^i}(u_0)$$

at ϕ in $u = u_0$, i.e. as the directional derivative

$$X^i_0\frac{\partial}{\partial u^i} \quad \text{corresponding to the direction vector} \quad X^i_0 x_i$$

in $u = u_0$. The derivative only depends on the value of the vector field in x_0. Since the point $x_0 = x(u_0)$ was arbitrary we have

$$X = X^i \frac{\partial}{\partial u^i} \quad \text{corresponding to the tangent field} \quad X = X^i x_i = X^i \frac{\partial x}{\partial u^i}.$$

The two expressions above are merely different representations of the tangent vector field X (both in local coordinates):

If ϕ is extended to a neighbourhood of M in $\mathbb{R}^d$, the directional derivative

$$\frac{\partial \phi}{\partial X} = X^i \frac{\partial \phi(x(u))}{\partial u^i}$$

is computed by multiplying the components

$$X^i \frac{\partial x^k}{\partial u^i}$$

of the *tangent field* X with the partial derivatives

$$\frac{\partial \phi}{\partial x^k}$$

As *differential operator*

$$X = X^i \frac{\partial}{\partial u^i}$$

X acts on scalar fields like $\phi = \phi(u)$ and produces a scalar field $X\phi$, the derivative of ϕ in the direction of X. This directional derivative is denoted by

$$\nabla_X \phi = X\phi, \quad \text{replacing the notation} \quad \frac{\partial \phi}{\partial X}$$

We already use the notation ∇_X customary for covariant differentiation. For reasons that should be clear, covariant differentiation of scalar fields is by definition the same as differentiation of scalar fields.

5 Commutators of tangent vector fields

If X and Y are vector fields on M then the commutator of X and Y is defined as

$$[X, Y] = XY - YX$$

Verify that

$$[X, Y]^j = X^k Y^j_k - Y^k X^j_k$$

and that $[X, Y]$ is a vector field. Note that $[X, Y]$ is bilinear over de scalar fields, and verify that the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \tag{5.1}$$

holds.

Since $[X, Y] = XY - YX$ is a vector field we can write

$$\nabla_{[X,Y]}\phi = [X, Y]\phi = X(Y\phi) - Y(X\phi) = \nabla_X(\nabla_Y \phi) - \nabla_Y(\nabla_X \phi).$$

Remark: Lie derivatives

This commutator has a meaning by itself. If $\gamma(t)$ is the solution of (4.1), then the linearised flow equation transports the vector $Y(x_0)$ along $\gamma(t)$. Denoting the transported vector as $\xi(t)$, we may differentiate the difference of $\xi(t)$ and $Y(\gamma(t))$ with respect to t and evaluate the derivative in $t = 0$. This should define

$$(\mathcal{L}_X Y)(x_0) = \lim_{t\to 0} \frac{\xi(t) - Y(\gamma(t))}{t},$$

the Lie derivative of Y with respect to X in x_0, but this formula has to be handled with care because the numerator involves tangent vectors in tangent spaces that vary with t.

In coordinates $\xi(t) = \xi^i(t)x_i$ with $\xi^i(t)$ is a solution of the linearization of (4.2) around $u(t)$,

$$\dot{\xi}^i = \underbrace{(\frac{\partial X^i}{\partial u^j})}_{\text{in } (u(t)}\xi^j(t), \quad \xi^j(0) = Y^i(u_0) \tag{5.2}$$

Writing

$$\xi(t) - Y(\gamma(t)) = \xi(t) - Y(x_0) - (Y(\gamma(t)) - Y(x_0))$$

you should verify that

$$(\mathcal{L}_X Y)(x_0) = (XY)(x_0) - (YX)(x_0)$$

so that

$$[X, Y] = \mathcal{L}_X Y \tag{5.3}$$

6 Covariant differentiation of tangent vectors

Next we observe that also

$$X = X^i \frac{\partial}{\partial u^i}$$

acts covariantly on tangent fields V if we replace

$$\frac{\partial}{\partial u^i} \quad \text{by} \quad D_{u^j},$$

as defined in (3.3) through

$$D_{u^j}V := \left(\frac{\partial V^l}{\partial u^j} + V^i\Gamma^l_{ij}\right)x_l \quad \text{for} \quad V = V^i x_i.$$

The result of this action is

$$X^j\left(\frac{\partial V^l}{\partial u^j} + V^i\Gamma^l_{ij}\right)x_l$$

and is denoted as

$$\nabla_X V = (\nabla_X V)^j \frac{\partial}{\partial u^j}, \quad (\nabla_X V)^j = X^j\left(\frac{\partial V^l}{\partial u^j} + V^i\Gamma^l_{ij}\right) \tag{6.1}$$

in the notation for tangent vectors as differential operators.

The map

$$V \to \nabla_X V$$

is *not* linear over the scalar fields because

$$\nabla_X \phi V = X^j \left(\frac{\partial \phi V^l}{\partial u^j} + \phi V^i \Gamma^l_{ij} \right) x_l$$

$$= \phi X^j \left(\frac{\partial V^l}{\partial u^j} + V^i \Gamma^l_{ij} \right) x_l + X^j \frac{\partial \phi}{\partial u^j} V^l = \phi \nabla_X V + (\nabla_X \phi) V.$$

The latter term in this *Leibniz rule* destroys the tensor property of linearity over the scalar fields.

Convince yourself that in the non-abstract approach

$$\nabla_X V = X^j \left(\frac{\partial V^l}{\partial u^j} + V^i \Gamma^l_{ij} \right) x_l$$

is the tangential[5] *component of the derivative of* V *in the direction of* X and verify that

$$\nabla_X (V \cdot W) = \nabla_X V \cdot W + V \cdot \nabla_X W$$

if W is another tangent vector field on M.

7 Submanifolds in $\mathbb{R}^d$: second fundamental form

The normal part of the derivative of V in the direction of X is denoted by $\mathrm{I\!I}(X, V)$, in which $\mathrm{I\!I}$ is called the *second fundamental form* of M. Verify that it is bilinear over the smooth fields on M. Since the normal part essentially comes from the mixed derivatives x_{ij}, the *second fundamental form must be symmetric*. Moreover, if N is a normal vector field on M and N, X, V are extended smoothly[6] to the ambient space $\mathbb{R}^d$ then

$$\bar{\nabla}_X (N \cdot Y) = \bar{\nabla}_X N \cdot Y + N \cdot \bar{\nabla}_X Y, \tag{7.1}$$

in which $\bar{\nabla}$ is the (standard covariant) derivative in $\mathbb{R}^d$. On M the left hand side of (7.1) is zero, and the second term $N \cdot \bar{\nabla}_X Y$ on the right hand side only sees the normal part of $\bar{\nabla}_X Y$ which is $\mathrm{I\!I}(X, Y)$. It follows that

$$\bar{\nabla}_X N \cdot Y = -N \cdot \mathrm{I\!I}(X, Y) \quad \text{on } M. \tag{7.2}$$

This is called Weingarten's relation. Note that in the codimension 1 case $d = n + 1$ we can choose a unit normal field N and define

$$h(X, Y) = N \cdot \mathrm{I\!I}(X, Y) = -\bar{\nabla}_X N \cdot Y = h_{ij} X^i Y^j \tag{7.3}$$

[5] to M

[6] This can be done, certainly locally, why?

8 Curvature

The equality

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z + R(X,Y)Z \tag{8.1}$$

defines $R(X,Y)Z$ for tangent vector fields X, Y, Z. You may verify that $R(X,Y)Z$ is multilinear in X, Y, Z over the scalar fields on M. In the case $M = \mathbb{R}^n = \mathbb{R}^d$ you will find that $R(X,Y)Z \equiv 0$. The standard way to write $R(X,Y)Z$ in local coordinates u is

$$(R(X,Y)Z)^\alpha = R^\alpha_{ijk} Z^i X^j Y^k. \tag{8.2}$$

So Z comes first[7] and then X and Y. Using (6.1) and writing

$$\Gamma^\alpha_{ij,k} = \frac{\partial \Gamma_{ij}}{\partial u^k}$$

you should verify that[8]

$$R^\alpha_{ijk} = \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k} + \Gamma^\alpha_{ik,j} - \Gamma^\alpha_{ij,k} \tag{8.3}$$

and the zero ijk and jk cyclic sums

$$R^\alpha_{ijk} + R^\alpha_{kij} + R^\alpha_{jki} = 0 = R^\alpha_{ijk} + R^\alpha_{ikj} \tag{8.4}$$

If W is another tangent field then[9]

$$Rm(X,Y,Z,W) = R(X,Y)Z \cdot W = R^\alpha_{ijk} Z^i X^j Y^k g_{\alpha l} W^l = R_{lijk} W^l Z^i X^j Y^k, \tag{8.5}$$

which has the symmetries

$$Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0,$$

$$Rm(X,Y,Z,W) + Rm(Y,X,Z,W) = 0 = Rm(X,Y,Z,W) + Rm(X,Y,W,Z)$$

(the second one obtained from $Rm(X,Y,Z,Z) = 0$), implying

$$Rm(X,Y,Z,W) = Rm(Z,W,X,Z)$$

In the 2-dimensional case $n = 2$ the only possible nonzero entries of R_{ijk} are

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$$

In the codimension 1 case

$$R_{lijk} = h_{ik}h_{lj} - h_{ij}h_{lk}$$

consists of all the 2×2 determinants you can get from the matrix h_{ij}. Note that similarly

$$(W \cdot X)(Y \cdot Z) - (W \cdot Y)(X \cdot Z) = \underbrace{(g_{ik}g_{lj} - g_{ij}g_{lk})}_{G_{lijk}} W^l Z^i X^j Y^k, \tag{8.6}$$

[7] As if we would have prefered the notation $ZR(X,Y)$

[8] note the order ijk in the minus terms and the $j \leftrightarrow k$ relation with the plus terms

[9] $lijk$ = dead body, as if we would have prefered the notation $W \cdot ZR(X,Y)$

in which G_{lijk} has the same symmetry properties as R_{lijk} (and depends only on G_{1212} if $n = 2$).

For submanifolds you can verify from the definitions that

$$Rm(X, Y, Z, W) = \mathrm{I\!I}(X, W)\mathrm{I\!I}(Y, Z) - \mathrm{I\!I}(X, Z)\mathrm{I\!I}(Y, W), \tag{8.7}$$

which in the codimension 1 case (7.3) reduces to

$$Rm(X, Y, Z, W) = h(W, X)h(Y, Z) - h(W, Y)h(X, Z),$$

so

$$Rm(X, Y, Z, W) = \underbrace{(h_{ik}h_{lj} - h_{ij}h_{lk})}_{R_{lijk}} W^l Z^i X^j Y^k, \tag{8.8}$$

Gauss computed this expression for R_{lijk} from $x_{ijk} = x_{ikj}$, see Chapter 10 in Schaum's Differential Geometry book by Martin Lipschutz. The Gauss curvature of a surface in $\mathbb{R}^d$ is the scalar ratio between (8.7) and (8.6). In $\mathbb{R}^3$ this is the scalar ratio between (8.8) and (8.6).

9 Geodesic curves

A smooth curve $\gamma(t) \in M$ may require several coordinate patches to describe it. For the moment we assume that it can be described by one coordinate patch. If

$$\gamma : [a, b] \ni t \to u(t) \to x(u(t)) \in M$$

is such a curve in M, then its velocity is given by

$$\dot{\gamma} = \frac{\partial x}{\partial u^1}\dot{u}^1 + \cdots + \frac{\partial x}{\partial u^n}\dot{u}^n = \sum_{i=1}^{n} \dot{u}^i \frac{\partial x}{\partial u^i} = \sum_{i=1}^{n} \dot{u}^i x_i.$$

Think of $\dot{\gamma}$ as a vector at the point $x = \gamma(t)$ in M. For every t this vector is tangent to M, and written as a linear combination of the tangent vectors obtained from the parametrization:

$$x_1 = \frac{\partial x}{\partial u^1}, \ldots, x_n = \frac{\partial x}{\partial u^n}.$$

Its length l is given by

$$l = \int_a^b |\dot{\gamma}(t)|\, dt = \int_a^b \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}\, dt = \int_a^b \sqrt{x_i \dot{u}^i \cdot x_j \dot{u}^j}\, dt$$

$$= \int_a^b \sqrt{\dot{u}^i \dot{u}^j g_{ij}(u)}\, dt$$

We will work with another quantity, called the energy, which involves an L as in Chapter 1. Since I prefer to have u in L, my u's are the γ's in the book. My $\gamma(t)$ is what is $c(t)$ in the book. The energy is defined by

$$E = \frac{1}{2}\int_a^b |\dot{\gamma}(t)|^2\, dt = \frac{1}{2}\int_a^b \dot{\gamma}(t) \cdot \dot{\gamma}(t)\, dt = \frac{1}{2}\int_a^b x_i \dot{u}^i \cdot x_j \dot{u}^j\, dt$$

$$= \frac{1}{2}\int_a^b \dot u^i \dot u^j g_{ij}(u)\,dt = \int_a^b L(u(t),\dot u(t))\,dt,$$

in which

$$L = L(u,p) = \frac{1}{2}p^i p^j g_{ij}(u). \qquad (9.1)$$

Playing with the estimate

$$\int_a^b |\dot\gamma(t)|\,dt = \int_a^b 1\,|\dot\gamma(t)|\,dt \le \sqrt{\int_a^b 1^2\,dt}\sqrt{\int_a^b |\dot\gamma(t)|^2\,dt}$$

and reparametrization of γ to make $|\dot\gamma|$ constant you should easily conclude that minimizers of l are minimizers of E and vice versa if we keep $[a,b]$ fixed.

The Euler-Lagrange equations for E involve the derivatives of g_{ij} and come out as

$$\ddot u^i + \Gamma^i_{\alpha\beta}\dot u^\alpha \dot u^\beta = 0 \qquad (9.2)$$

and are called the geodesic equations. Indeed,

$$\Gamma^i_{\alpha\beta} = \frac{1}{2}g^{ik}\left(g_{\alpha k,\beta} + g_{\beta k,\alpha} - g_{\alpha\beta,k}\right),$$

the symbols computed in (3.2). You should repeat this calculation without looking at the notes above. What is the conservation law for this system?

9.1 A special metric with radial symmetry

A nice example is a surface M which is described by a single set of coordinates $u \in \mathbb{R}^2$ with a metric

$$g_{ij}(u) = g(|u|)\delta_{ij} \qquad (9.3)$$

in which $u \to g(|u|)$ is smooth and positive[10]. You can write the geodesic equations as in the book (2.1.27). In a special case the example is related to stereographic projection through

$$u^1 = \frac{x^1}{1-x^3}, \quad u^2 = \frac{x^2}{1-x^3},$$

which you may prefer as

$$u = \frac{x}{1-z}, \quad v = \frac{y}{1-z}$$

without indices.

- Verify that large circles on $x^2+y^2+z^2$ correspond to circles in the uv-plane. Hint: describe the large circles as $z = ax+by$ and avoid goniometric functions.
- The large circles not contained in this description are the vertical great circles which correspond to lines through the origin in the uv-plane. Assuming unit speed for both the vertical great circles and lines through the origin derive the formula for $g(|u|)$.

[10] implying $0 = g'(0) = g'''(0) = g'''''(0) = \cdots$

We return to (9.3) with general $g(|u|)$.

- Why are geodesics through the origin straight lines?
- Take a geodesic line parametrized by t such that $t = 0$ corresponds to $(0, 0)$ and that the speed in $(0, 0)$ is equal to 1. Use the conservation law to derive a first order equation for $R(t) = |u(t)|$ and solve it.
- Examine how long it takes for the geodesic curve to reach infinity. What is the condition on $g(|u|)$ to reach infinity in finite time? This should involve some integral with g. Do the same in dimension $n > 2$? Is there a difference?
- Can you cook up an example for which the geodesic cannot cross $|u| = 1$? Can you classify these examples?
- Incidentally, what is the Gauss curvature for metrics of the form (9.3) in $\mathbb{R}^2$?

9.2 The Jacobi equations

Consider the Lagrangian (9.1).

- Show that the Jacobi equations (1.4) for (9.1) are

$$\ddot{\eta}^i + 2\Gamma^i_{jk}\dot{u}^j\dot{\eta}^k + \Gamma^i_{jk,l}\dot{u}^j\dot{u}^k\eta^l = 0 \qquad (9.4)$$

Both $\dot{u}^i(t)$ and $\eta^i(t)$ define vector fields along $\gamma(t) = x(u(t))$ in $M \in \mathbb{R}^d$ tangent to M through

$$\dot{\gamma}(t) = \dot{u}^i(t)x_i(u(t)) \quad \text{and} \quad \eta^i(t)x_i(u(t))$$

The Jacobi equations are much more transparent if we work with the tangential parts D_tV of the time derivatives of such vector fields

$$V(\gamma(t)) = V^i(t)x_i(u(t))$$

- Derive that

$$D_tV = (D_tV)^j x_j \quad \text{with} \quad (D_tV)^j = \dot{V}^j + V^\alpha\Gamma^j_{\alpha\beta}\dot{u}^\beta$$

- Apply this to $V(\gamma(t)) = V^i(t)x_i$ with $V^i(t) = \dot{u}^i(t)$ and derive that the geodesic equation (9.2) may be written as

$$D_t\dot{\gamma} = 0, \quad \dot{\gamma} = \dot{u}^i x_i$$

  This should be easy and will tell you that the covariant t-derivative of the t-derivative of $\gamma(t)$ is zero.

- You can also apply the rule for D_tV to $V(\gamma(t)) = V^i(t)x_i$ with $V^i(t) = \eta^i(t)$ and $V^i(t) = (D_t\eta)^i(t)$. You should be able to derive that (9.4) may be written as

$$(D_t^2\eta)^i + \dot{u}^\alpha R^i_{\alpha\beta k}\eta^\beta\dot{u}^k = 0, \quad \text{i.e.} \quad D_t^2\eta + R(\eta, \dot{\gamma})\dot{\gamma} = 0$$

  In the latter formula $\eta = \eta^i(t)x_i$. This is harder but it gives an equation that is obviously more informative.