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Continuous time Leech problems for rational operator functions

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Notations

$x \in \ell^2(\mathbb{C}^{n\times m})$ means $x = \begin{bmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{bmatrix}$ $\sum_{k=-\infty}^{\infty} \|x_k\|^2 < \infty$, x_k are $n \times m$ matrices.

$\ell^2_+(\mathbb{C}^{n\times m})$ consists of all x with $x_k = 0$ if $k < 0$.

$X(z) = \sum_{k=-\infty}^{\infty} z^k x_k$ for $x \in \ell^2(\mathbb{C}^{n\times m})$.

If $x \in \ell^2_+(\mathbb{C}^{n\times m})$ then $X(z)$ is analytic on $\mathbb{D}$. Here $\mathbb{D}$ is the interior of the unit circle $\mathbb{T}$.

Notations

For $x \in \ell^2_+(\mathbb{C}^{n\times m})$ the Toeplitz operator T_X is

$$T_X = \begin{bmatrix} x_0 & 0 & 0 & \cdots \\ x_1 & x_0 & 0 & \cdots \\ x_2 & x_1 & x_0 & \\ \vdots & \vdots & & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^m) \to \ell^2_+(\mathbb{C}^n).$$

$\|T_X\| = \text{ess sup}\{\|X(z)\| : |z| = 1\}.$

For $x \in \ell^2_+(\mathbb{C}^{m\times m})$ the Hankel operator H_X is

$$H_X = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ x_2 & x_3 & x_4 & \\ \vdots & \vdots & & \end{bmatrix} : \ell^2_+(\mathbb{C}^m) \to \ell^2_+(\mathbb{C}^n).$$

$\|H_X\| \leq \text{ess sup}\{\|X(z)\| : |z| = 1\}.$

The Nehari extension problem

Given $\ldots, \psi_{-2}, \psi_{-1}, \psi_0$ a sequence of $m \times m$ matrices with $\sum_{k=-\infty}^{0} \|\psi_k\| < \infty$.

Find a sequence $(\phi_k)_{k=-\infty}^{\infty}$ such that

(i) The corresponding function Φ satisfies $\|\Phi(z)\| < 1$ for $z \in \mathbb{T}$.

(ii) $\phi_k = \psi_k$ for $k \leq 0$

(iii) $\sum_{k=-\infty}^{\infty} \|\phi_k\| < \infty$.

Necessary and sufficient

$$H = \begin{bmatrix} \psi_0 & \psi_{-1} & \psi_{-2} & \cdots \\ \psi_{-1} & \psi_{-2} & \psi_{-3} & \cdots \\ \psi_{-2} & \psi_{-3} & \psi_{-4} & \\ \vdots & \vdots & & \end{bmatrix} : \ell^2_+(\mathbb{C}^m) \to \ell^2_+(\mathbb{C}^m) \text{ is a strict contraction.}$$

In that case a solution can be constructed as follows.

The Nehari extension problem

(H is a strict contraction.)

$$\text{Let} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix} = (I - H^* H)^{-1} \begin{bmatrix} I_m \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_0 \\ u_{-1} \\ u_{-2} \\ \vdots \end{bmatrix} = H \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix}.$$

Then $\Phi(z) = U(z)V(z)^{-1}$ is a solution.

The Nevanlinna-Pick interpolation problem

Given $z_1, z_2, \ldots, z_N$ different points in $\mathbb{D}$ and $y_1, y_2, \cdots, y_N$ matrices of size $m \times m$.

Find a sequence $(\phi_k)_{k=0}^{\infty}$ of $m \times m$ marices with $\sum_{k=0}^{\infty} \|\phi_k\| < \infty$ such that the corresponding function Φ on $\mathbb{D}$ satisfies

(i) $\sup\{\|\Phi(z)\| : z \in \mathbb{D}\} < 1$;

(ii) $\Phi(z_k) = y_k$ for $k = 1, \ldots, N$.

To find a necessary and sufficient condition for the existence of a solution requires some preparation.

Put $A(z) = \sum_{k=1}^{N} y_k \left(\prod_{j=1, j\neq k}^{N} \frac{z - z_k}{z_j - z_k}\right)$ and $B(z) = \prod_{j=1}^{N} \frac{z - z_k}{1 - z z_k}$.

Let $\Psi = AB^{-1}$ and ψ_k be the k-th Fourier coefficient of Ψ.

Necessary and sufficient

The Nevanlinna-Pick interpolation problem

$$H = \begin{bmatrix} \psi_0 & \psi_{-1} & \psi_{-2} & \cdots \\ \psi_{-1} & \psi_{-2} & \psi_{-3} & \cdots \\ \psi_{-2} & \psi_{-3} & \psi_{-4} & \\ \vdots & \vdots & & \end{bmatrix} : \ell^2_+(\mathbb{C}^m) \to \ell^2_+(\mathbb{C}^m) \text{ is a strict contraction.}$$

In that case: Let $\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix} = (I - H^*H)^{-1} \begin{bmatrix} I_m \\ 0 \\ 0 \\ \vdots \end{bmatrix}$ and

$$\begin{bmatrix} u_0 \\ u_{-1} \\ u_{-2} \\ \vdots \end{bmatrix} = H \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix}.$$

Then $\Phi(z) = U(z)V(z)^{-1}$ is a solution.

The Leech problem in discrete time

Given G an $m \times p$ and K an $m \times q$ rational matrix valued bounded function on $\mathbb{D}$.
Let $T_G : \ell^2_+(\mathbb{C}^p) \to \ell^2_+(\mathbb{C}^m)$ and $T_K : \ell^2_+(\mathbb{C}^q) \to \ell^2_+(\mathbb{C}^m)$ be the corresponding Toeplitz operators.
A $p \times q$ rational matrix function X is a solution to the Leech problem if

(i) $\sup\{\|X(z)\| : z \in \mathbb{D}\} < 1$;

(ii) $GX = K$

Necessary and sufficient condition is that
$\Lambda = T_G^*(T_G T_G^*)^{-1} T_K : (\operatorname{Ker} T_G)^\perp \to \ell^2_+(\mathbb{C}^p)$ is a strict contraction.
In that case a solution is constructed as follows.

The Leech problem in discrete time

Let $\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix} = (I - \Lambda^* \Lambda)^{-1} \begin{bmatrix} I_m \\ 0 \\ 0 \\ \vdots \end{bmatrix}$ and $\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix}$.

Then $\det V(z) \neq 0$ for $z \in \mathbb{D}$ and V^{-1} is bounded on $\mathbb{D}$.

The function $X = UV^{-1}$ is such that $K = GX$ and $\|X\|_\infty < 1$. It is a solution to the Leech problem.
With rational G and K we have the minimal realization

$$\begin{bmatrix} G(z) & K(z) \end{bmatrix} = \begin{bmatrix} D_1 & D_2 \end{bmatrix} + zC(I_n - zA)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

The solution X can be expressed in A, B_1, B_2, C, D_1, D_2 and (matrix) solutions of related Lyapunov and Riccati equations.

Definitions and notations

$\mathbb{C}_+ = \{s \in \mathbb{C} : \Re s > 0\}$. $H^\infty(\mathbb{C}^{n\times m})$ consists of all functions F on $\mathbb{C}_+$ such that

$$\|F\|_\infty = \sup\{\|F(s)\| \ : \ s \in \mathbb{C}_+\} < \infty.$$

Let f_o be an integrable function on $\mathbb{R}_+$. The Laplace transform of f_o is defined as

$$(\mathfrak{L} f_o)(s) = \int_0^\infty f_o(t) e^{-st} dt \quad (s \in \mathbb{C}_+)$$

and is analytic on $\mathbb{C}_+$. For $F(s) = D + (\mathfrak{L} f_o)(s)$ we define the Wiener-Hopf operator $T_F : L^2_+(\mathbb{C}^m) \to L^2_+(\mathbb{C}^n)$ by putting

$$(T_F u)(t) = Du(t) + \int_0^t f_o(t-\tau) u(\tau) d\tau \quad (0 \le t < \infty).$$

The Leech problem in discrete time(2)

Given G an $m \times p$ and K an $m \times q$ rational matrix valued bounded function on $\mathbb{D}$.
Let $T_G : \ell^2_+(\mathbb{C}^p) \to \ell^2_+(\mathbb{C}^m)$ and $T_K : \ell^2_+(\mathbb{C}^q) \to \ell^2_+(\mathbb{C}^m)$ be the corresponding Toeplitz operators.
A $p \times q$ rational matrix function X is a solution to the Leech problem if

(i) $\sup\{\|X(z)\| : z \in \mathbb{D}\} < 1$;

(ii) $GX = K$

Necessary and sufficient condition is that
$\Lambda = T_G^*(T_G T_G^*)^{-1} T_K : (\operatorname{Ker} T_G)^\perp \to \ell^2_+(\mathbb{C}^p)$ is a strict contraction.

The Leech problem in continuous time

Given G an $m \times p$ and K an $m \times q$ rational matrix valued bounded function on $\mathbb{C}_+$.
Let $T_G : L^2_+(\mathbb{C}^p) \to L^2_+(\mathbb{C}^m)$ and $T_K : L^2_+(\mathbb{C}^q) \to L^2_+(\mathbb{C}^m)$ be the corresponding Wiener Hopf operators.
A $p \times q$ rational matrix function X is a solution to the Leech problem if

(i) $\sup\{\|X(s)\| : s \in \mathbb{C}_+\} < 1$;

(ii) $GX = K$

Necessary and sufficient condition is that $\Lambda = T_G^*(T_G T_G^*)^{-1} T_K$: is a strict contraction.

The Cayley transform

$$c : \mathbb{C}_+ \to \mathbb{D} \text{ and } c^{-1} : \mathbb{D} \to \mathbb{C}_+ \text{ given by}$$
$$c(s) = \frac{s-1}{s+1}, \quad c^{-1}(z) = \frac{1-z}{1+z}.$$

Solutions of discrete time Leech problem correspond in a one to one way with solutions of the contnuous time Leech problem.

$$\begin{bmatrix} G(s) \\ K(s) \\ X(s) \end{bmatrix} \longleftrightarrow \begin{bmatrix} \tilde{G}(z) = G(c^{-1}(z)) \\ \tilde{K}(z) = K(c^{-1}(z)) \\ \tilde{X}(z) = X(c^{-1}(z)) \end{bmatrix}$$

continuous time $\longleftrightarrow$ discrete time

Towards a formula in continuous time

To determine $V(z)$ in the discrete case we had

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix} = (I - \Lambda^* \Lambda)^{-1} \begin{bmatrix} I_m \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

If $(I - \Lambda^* \Lambda)^{-1} = [v_{jk}]_{j,k=0}^{\infty}$ then $v_j = v_{j0}$ $(j = 0, 1, 2. \ldots)$.
For the continuous time we want a similar formula. Assume

$$\left((I - \Lambda^* \Lambda)^{-1} g\right)(t) = \int_0^{\infty} v(t, \tau) g(\tau)\, d\tau.$$

Take $v(t) = v(t, 0)$. We define

$$\int_0^{\infty} v(t, \tau) \delta(\tau)\, d\tau \; = v(t, 0).$$

This makes sense if $v(t, \tau)$ is continuous. (So rational is sufficient!)

Theorem: Solution of continuous time Leech problem

Assume that $\Lambda = T_G^* \left(T_G T_G^*\right)^{-1} T_K$ is a strict contraction. Let X be the function defined by

$$X(s) = U(s)V(s)^{-1} \tag{1}$$

where $U(s)$ and $V(s)$ are the functions defined by

$$\begin{aligned} U(s) &= \left(\mathfrak{L}\Lambda \left(I - \Lambda^*\Lambda\right)^{-1} \delta\right)(s) \\ V(s) &= \left(\mathfrak{L} \left(I - \Lambda^*\Lambda\right)^{-1} \delta\right)(s). \end{aligned} \tag{2}$$

Here $\mathfrak{L}$ is the Laplace transform.

(i) Then X is a solution to our Leech problem. To be precise, X is a function in $H^\infty(\mathbb{C}^{p\times q})$ satisfying $G(s)X(s) = K(s)$ and $\|X\|_\infty < 1$.

(ii) Moreover, $V(s)$ is an invertible outer function, that is, both $V(s)$ and $V(s)^{-1}$ are functions in $H^\infty(\mathbb{C}^{q\times q})$.

In fact, the function

$$\Theta(s) = V(\infty)^{\frac{1}{2}} V(s)^{-1}$$

is the outer spectral factor for the function $I - X^*(-\overline{s})X(s)$, that is

$$I - X(-\overline{s})^* X(s) = \Theta(-\overline{s})^* \Theta(s).$$

Realizations

Given is the minimal realization

$$\begin{bmatrix} G(s) & K(s) \end{bmatrix} = \begin{bmatrix} D_1 & D_2 \end{bmatrix} + C(sI - A)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}$$

We assume $T_G T_G^* - T_K T_K^*$ strictly positive or equivalently
$\Lambda = T_G (T_G T_G^*)^{-1} T_K$ strictly contractive.
Then $R_0 = D_1 D_1^* - D_2 D_2^*$ is strictly positive.
Denote by Δ the solution of the Lyapunov equation

$$A\Delta + \Delta A^* + B_1 B_1^* - B_2 B_2^* = 0$$

Put $\Gamma = B_1 D_1^* - B_2 D_2^* + \Delta C^*$.
Let Q be the stabilizing solution of the Riccati equation

$$A^* Q + QA + (C - \Gamma^* Q)^* R_0^{-1} (C - \Gamma^* Q) = 0.$$

That is: Q is strictly positive and $A_0 = A - \Gamma R_0^{-1}(C - \Gamma^* Q)$ is stable.

Realizations

Let

$$g_o(t) = C_1 e^{tA} B \quad \text{with } A \text{ stable and } t \geq 0.$$

Then

$$(\mathfrak{L} g_o)(s) = C_1 (sI - A)^{-1} B.$$

Therefore with $G(s) = D_1 + C_1(sI - A)^{-1}B$ and $f \in L_+(\mathbb{C}^p)$:

$$(T_G f)(t) = D_1 f(t) + \int_0^\infty C_1 e^{(t-\tau)A} B f(\tau) d\tau.$$

Theorem: Realization of the solution X .

Let G and K be be given by the minimal (stable) realization of the previous slide. Furthermore, assume that $T_G T_G^* - T_K T_K^*$ is strictly positive.
Then a solution X to Leech problem with data G and K is given by the following stable state space realization:

$$\begin{aligned} X(s) &= D_1^\dagger D_2 + C_x\big(sI - \mathbf{A}\big)^{-1}\big(B_2 - B_1 D_1^\dagger D_2\big) \\ C_x &= D_1^\dagger C + \big(I - D_1^\dagger D_1\big) B_1^* Q\big(I - \Delta Q\big)^{-1} \\ \mathbf{A} &= A - B_1 C_x \\ D_1^\dagger &= D_1^*(D_1 D_1^*)^{-1}. \end{aligned} \tag{3}$$

The operator $\mathbf{A}$ is stable, and $\|X\|_\infty < 1$.
Finally, the McMillan degree of X is less than or equal to the McMillan degree of $\begin{bmatrix} G & K \end{bmatrix}$.

Theorem: realization of $V(s)$.

Recall $X(s) = U(s)V(s)^{-1}$.
Put $C_0 = R_0^{-1}(C - \Gamma^* Q)$ and accept that $I - \Delta Q$ is invertible.
Assume that $\Lambda = T_G(T_G T_G^*)^{-1} T_K$ strictly contractive. Consider the function $V(s)$ defined by

$$V(s) = \left(\mathfrak{L}\,(I - \Lambda^*\Lambda)^{-1}\,\delta\right)(s) \tag{4}$$

where $\delta(t)$ is the Dirac delta function.
Then a state space realization for $V(s)$ is given by

$$\begin{aligned} V(s) &= (I + D_2^* R_0^{-1} D_2) + (D_2^* C_0 + B_2^* Q)\left(sI - A_0\right)^{-1}\mathbb{B}, \\ \mathbb{B} &= (I - \Delta Q)^{-1}\left(B_2 - B_1 D_1^\dagger D_2\right)\left(I + D_2^* R_0^{-1} D_2\right). \end{aligned} \tag{5}$$

Because A_0 is stable, V is a function in $H^\infty(\mathbb{C}^{q\times q})$.

Theorem: Realization of $U(s)$

Assume that $\Lambda = T_G(T_G T_G^*)^{-1} T_K$ strictly contractive.
Consider the function $U(s)$ defined by

$$U(s) = \left(\mathfrak{L}\Lambda\,(I - \Lambda^*\Lambda)^{-1}\,\delta\right)(s). \tag{6}$$

Then a state space realization for $U(s)$ is given by

$$\begin{aligned} U(s) &= D_1^* R_0^{-1} D_2 + (D_1^* C_0 + B_1^* Q)(sI - A_0)^{-1}\mathbb{B} \\ \mathbb{B} &= (I - \Delta Q)^{-1}\left(B_2 - B_1 D_1^\dagger D_2\right)\left(I + D_2^* R_0^{-1} D_2\right). \end{aligned} \tag{7}$$

Because A_0 is stable, U is a function in $H^\infty(\mathbb{C}^{p\times q})$.

Theorem: Realization of $V(s)^{-1}$

Assume that $\Lambda = T_G (T_G T_G^*)^{-1} T_K$ strictly contractive.
Consider the function $V(s) = \big(\mathfrak{L}\,(I - \Lambda^*\Lambda)^{-1}\,\delta\big)(s)$.
Recall

$$
\begin{aligned}
V(s) &= D_v + C_v(sI - A_0)^{-1}\mathbb{B} \\
D_v &= (I + D_2^* R_0^{-1} D_2), \qquad C_v = D_2^* C_0 + B_2^* Q \\
\mathbb{B} &= \big(I - \Delta Q\big)^{-1}\Big(B_2 - B_1 D_1^{\dagger} D_2\Big) D_v
\end{aligned}
$$

Then a state space realization for $V(s)^{-1}$ is given by

$$
\begin{aligned}
V(s)^{-1} &= D_v^{-1} - D_v^{-1} C_v (I - \Delta Q)^{-1}\big(sI - \mathbf{A}\big)^{-1}\big(B_2 - B_1 D_1^{\dagger} D_2\big), \\
\mathbf{A} &= A - B_1 D_1^{\dagger} C - B_1\big(I - D_1^{\dagger} D_1\big) B_1^* Q (I - \Delta Q)^{-1}.
\end{aligned}
$$

Moreover, the operator $\mathbf{A}$ is stable. In particular, $V(s)$ is an invertible outer function, that is, both $V(s)$ and its inverse $V(s)^{-1}$ are functions in $H^\infty(\mathbb{C}^{q\times q})$.

The operator T_R

$$R(s) = G(s)G(-\bar{s})^* - K(s)K(-\bar{s})^*.$$

Then

$$R(s) = C(sI - A)^{-1}\Gamma + R_0 - \Gamma^*(sI + A^*)^{-1}C^*.$$

with

$$\begin{aligned} R_0 &= D_1 D_1^* - D_2 D_2^*, \\ \Gamma &= B_1 D_1^* - B_2 D_2^* + \Delta C^*, \end{aligned}$$

'Here Δ is the unique solution of the Lyapunov equation

$$A\Delta + \Delta A^* + B_1 B_1^* - B_2 B_2^* = 0.$$

The Wiener-Hopf operator T_R plays an important role in the proofs.

$$(T_R y)(t) = R_0 y(t) + \int_0^t Ce^{A(t-\tau)}\Gamma y(\tau)d\tau + \int_t^\infty \Gamma^* e^{A^*(\tau-t)}C^* y(\tau)d\tau$$

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Thank you for your attention