Toeplitz operators and H_∞-calculus.

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Before we start

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- Courses at DISC/Mastermath
- IWOTA 2014

Functions of operators.

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- If $f : \mathbb{C} \mapsto \mathbb{C}$ is analytic, then we can define $f(A)$ via the contour integral

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All these functions have in common:

1. If $f(s) = 1$, then $f(A) = I$.
2. If $f(s) = (s - a)^{-1}$, then $f(A) = (sI - A)^{-1}$.
3. If $f(s) = f_1(s) f_2(s)$, then $f(A) = f_1(A) f_2(A)$.

Aim

What we would like to do is to set up a Functional Calculus for unbounded A's and $f : \mathbb{C}^+ \mapsto \mathbb{C}$ which are bounded on the right half plane $\mathbb{C}^+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$.

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More precise

- A infinitesimal generator of a C_0-semigroup, $(T(t))_{t\geq 0}$;
- $f \in H_\infty$;
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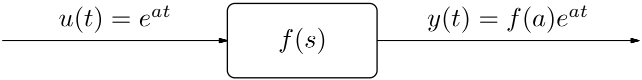
New idea using systems theory.

Basic idea

When we have system with transfer function f, then putting e^{at} in (as input), gives as output $f(a)e^{at}$.



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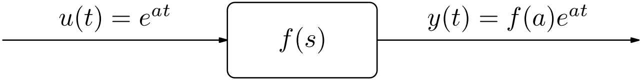

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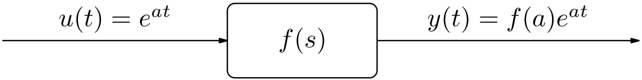

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Idea: Extend this to C_0-semigroups. They are "e^{At}", and thus generalisations of e^{at}.

Problems

- A C_0-semigroup is defined for $t \geq 0$ only.
- In the above figure $a > \omega_0$, but we want $a < \omega_0$ (stable systems).

Set-up

We flip the functions, i.e.,
$f \in H_\infty^- := \{f : \mathbb{C}_- \mapsto \mathbb{C} \mid f \text{ analytic and } \sup_{s\in\mathbb{C}_-} |f(s)| < \infty\}$.

Definition For $f \in H_\infty^-$ we define the Toeplitz operator M_f as

$$M_f G = \mathcal{L}^{-1}\left(\Pi(f\hat{G})\right), \qquad G \in L^2((0,\infty);X),$$

where $\hat{G}$ is the Fourier tranform of G, Π is the projection from $L^2(i\mathbb{R};X)$ to $H^2(X)$, and $\mathcal{L}^{-1}$ is the inverse Laplace transform.

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Lemma M_f has the following properties:

1. M_f is bounded linear operator on $L^2((0,\infty);X)$ with norm bounded by $\|f\|_\infty$;
2. $M_{f_1 f_2} = M_{f_1} M_{f_2}$, for $f_1, f_2 \in H_\infty^-$;
3. If K is a bounded operator on X, then $KM_f = M_f K$

More results

Let σ_τ, $\tau \geq 0$ denote the shift, i.e.,

$$(\sigma_\tau(G))\,(t) = G(t+\tau), \quad t \geq 0$$

Lemma For $\tau \geq 0$ there holds

$$(\sigma_\tau(M_f G)) = M_f(\sigma_\tau(G)).$$

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Lemma For $\tau \geq 0$ there holds

$$(\sigma_\tau(M_f G)) = M_f(\sigma_\tau(G)).$$

Now we take $G(t) = T(t)x_0$, where we assume that $(T(t))_{t\geq 0}$ is an exponentially stable semigroup. Define for $x_0 \in X$

$$\mathcal{O}_f x_0 = M_f(T(\cdot)x_0).$$

Toward functional caluculus

From the above Lemma we see that for $\tau > 0$

$$\begin{aligned}\sigma_\tau(\mathcal{O}_f x_0) &= \sigma_\tau\left(M_f T(\cdot)x_0\right) = M_f(\sigma_\tau(T(\cdot)x_0)) \\ &= M_f(T(\cdot+\tau)x_0) = M_f(T(\cdot)T(\tau)x_0) = \mathcal{O}_f T(\tau)x_0.\end{aligned}$$

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Now G. Weiss proved that there exists an operator C bounded from $D(A)$ to X such that

$$\left(\mathcal{O}_f x_0\right)(t) = CT(t)x_0, \quad x_0 \in D(A).$$

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Thus

$$M_f(T(\cdot)x_0) = f(A)T(\cdot)x_0.$$

Some observations.

If f is the Fourier transform of an L^1-function h with support in $(-\infty, 0)$, then

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If $C : D(A) \mapsto X$ is such that $CT(\cdot)x_0 \in L^2((0,\infty); X)$ for all $x_0 \in X$, then for $x_0 \in D(A^2)$

$$\begin{aligned} M_f(CT(\cdot)x_0) &= CM_f(T(\cdot)x_0) \\ &= C\mathcal{O}_f(x_0) = Cf(A)T(\cdot)x_0. \end{aligned}$$

Bounded H_∞ calculus

Theorem Assume that for $C : D(A) \to X$ there exists $0 < m_1 \leq m_2 < \infty$ such that for all $x_0 \in X$

$$m_1 \|x_0\|^2 \leq \int_0^\infty \|CT(t)x_0\|^2 dt \leq m_2 \|x_0\|^2.$$

Then

$$\|f(A)\| \leq \sqrt{\frac{m_2}{m_1}} \|f\|.$$

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Proof For $x_0 \in D(A^2)$ there holds

$$\begin{aligned} m_1 \|f(A)x_0\|^2 \leq & \int_0^\infty \|CT(t)f(A)x_0\|^2 dt = \|Cf(A)T(t)x_0\|^2_{L^2} \\ & \leq \|CM_f(T(\cdot)x_0)\|^2_{L^2} = \|M_f(CT(\cdot)x_0)\|^2_{L^2} \\ & \leq \|f\|^2 \|CT(\cdot)x_0\|^2_{L^2} \leq \|f\|^2 m_2 \|x_0\|^2. \end{aligned}$$

Von Neumann's inequality.

Assume now that A is dissipative. That is

$$\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle \leq 0, \quad x_0 \in D(A).$$

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We define C as

$$\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle = -\|Cx_0\|^2.$$

It is not hard to show that this implies that

$$\int_0^\infty \|CT(t)x_0\|^2 dt = \|x_0\|^2.$$

Thus by the previous result we find von Neumann's inequality.

$$\|f(A)\| \leq \|f\|.$$

Looking back

What we have done...

- We have set-up a functional calculus using Toeplitz operators.
- All (and more) can be found in *Toeplitz operators and H_∞ calculus* by HZ, Journal of Functional Analysis 263 (2012) 167–182.
- More to be gained from this approach??

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Thanks for your attention and
Happy retirement, André!