Logarithmic residues in Banach (sub)algebras generated by a single element

Problems in Operator Theory
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Joint work with

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The Logarithmic Residue Theorem

Concerned with

$$\mathrm{LogRes}(F;\Gamma) = \frac{1}{2\pi i}\int_\Gamma \frac{F'(\lambda)}{F(\lambda)}\,d\lambda$$

F scalar analytic on open neighborhood of closure of inner domain of positively oriented closed contour Γ
on which F has no zeros (so that the integral is well-defined)

Fact:
$\mathrm{LogRes}(F;\Gamma)$ equal to number of zeros of F inside Γ

In particular nonnegative integer

ISSUE:

What can be said for analytic functions F having their values in a complex Banach algebra (with unit element)**?**

Generally lack of commutativity

Two possibilities:

Left version: $$\frac{1}{2\pi i}\int_\Gamma F'(\lambda)F(\lambda)^{-1}\,d\lambda$$

Right version: $$\frac{1}{2\pi i}\int_\Gamma F(\lambda)^{-1}F'(\lambda)\,d\lambda$$

Focus on **left version**

From now on:

$\mathcal{B}$ Banach algebra (with unit element)

$$\mathsf{LogRes}(F;\mathcal{B};\Gamma) = \mathsf{LogRes}(F;\Gamma) = \frac{1}{2\pi i}\int_\Gamma F'(\lambda)F(\lambda)^{-1}\,d\lambda$$

F takes invertible values on Γ

Terminology: **logarithmic residue** of F with respect to Γ

What kind of elements in $\mathcal{B}$?

Starting point / basic observation

Scalar case $\mathcal{B} = \mathbb{C}$: nonnegative integers

Elements of the form: $1 + 1 + \cdots + 1$

Cumbersome reformulation: **sums of idempotents in** $\mathbb{C}$

General observation:

Sums of idempotents in a Banach algebra $\mathcal{B}$ are logarithmic residues in $\mathcal{B}$.

ISSUE:

Are logarithmic residues always sums of idempotents?

Often they are . . . but not always

There is a simple **counterexample** (subalgebra of $\mathbb{C}^{3\times 3}$)

Positive results: instances indicated in the abstract

In this talk:
focus on Banach (sub)algebras **generated by a single element**

Commutative (of course)

Prominent general observation in the background of what follows:

THEOREM
In a **commutative** *Banach algebra, the logarithmic residues are just the sums of idempotents.*

Specific Issue: connection with **spectral** idempotents

Key observation spectral theory:

$$F(\lambda) = \lambda e_{\mathcal{B}} - t$$

$$\begin{aligned}\mathsf{LogRes}(F;\mathcal{B};\Gamma) &= \frac{1}{2\pi i}\int_{\Gamma} F'(\lambda)F(\lambda)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma}(\lambda e_{\mathcal{B}} - t)^{-1}\,d\lambda\end{aligned}$$

Is an idempotent

Terminology: **(spectral) idempotent** associated with t (and the part of $\sigma(t)$ contained in the interior domain of Γ)

A (unital) Banach algebra $\mathcal{B}$ is called **monothetic** when it is generated by a single element

The element is question is then called the **generator** of $\mathcal{B}$

THEOREM

Suppose the Banach algebra $\mathcal{B}$ *is* **monothetic** *with generator* g. *Then* $p \in \mathcal{B}$ *is an idempotent in* $\mathcal{B}$ *if and only if* p *is a spectral idempotent associated with* g.

The proof employs standard **Gelfand Theory** for commutative Banach algebras

THEOREM

Suppose the Banach algebra $\mathcal{B}$ is **monothetic** *with generator g. Then the following sets coincide:*

(1) *The set of logarithmic residues in $\mathcal{B}$;*

(2) *The set of sums of idempotents in $\mathcal{B}$;*

(3) *The set of sums of spectral idempotents associated with g.*

(1)=(2): General observation + commutativity

(3)$\subset$(2): Obvious

(2)$\subset$(3): Previous Theorem

Move on to:

$\mathcal{B}$ Banach **subalgebra** of unital Banach algebra $\mathcal{A}$
generated by $g \in \mathcal{A}$

Notational preparations

Let $\mathcal{A}$ be unital Banach algebra, and let $\mathcal{B}$ be **any** Banach sub-algebra of $\mathcal{A}$,
not necessarily generated by a single element

NOTATION

$b \in \mathcal{B} \subset \mathcal{A}$

$\sigma_{\mathcal{B}}(b)$: spectrum of b as an element of $\mathcal{B}$
$\sigma_{\mathcal{B}}(a)$: spectrum of b as an element of $\mathcal{A}$

Standard spectral theory:

$\sigma_{\mathcal{A}}(b) \subset \sigma_{\mathcal{B}}(b)$
$\partial\sigma_{\mathcal{A}}(b) \supset \partial\sigma_{\mathcal{B}}(b)$ [$\partial =$ boundary]

Further analysis

K **compact** subset of the complex plane $\mathbb{C}$

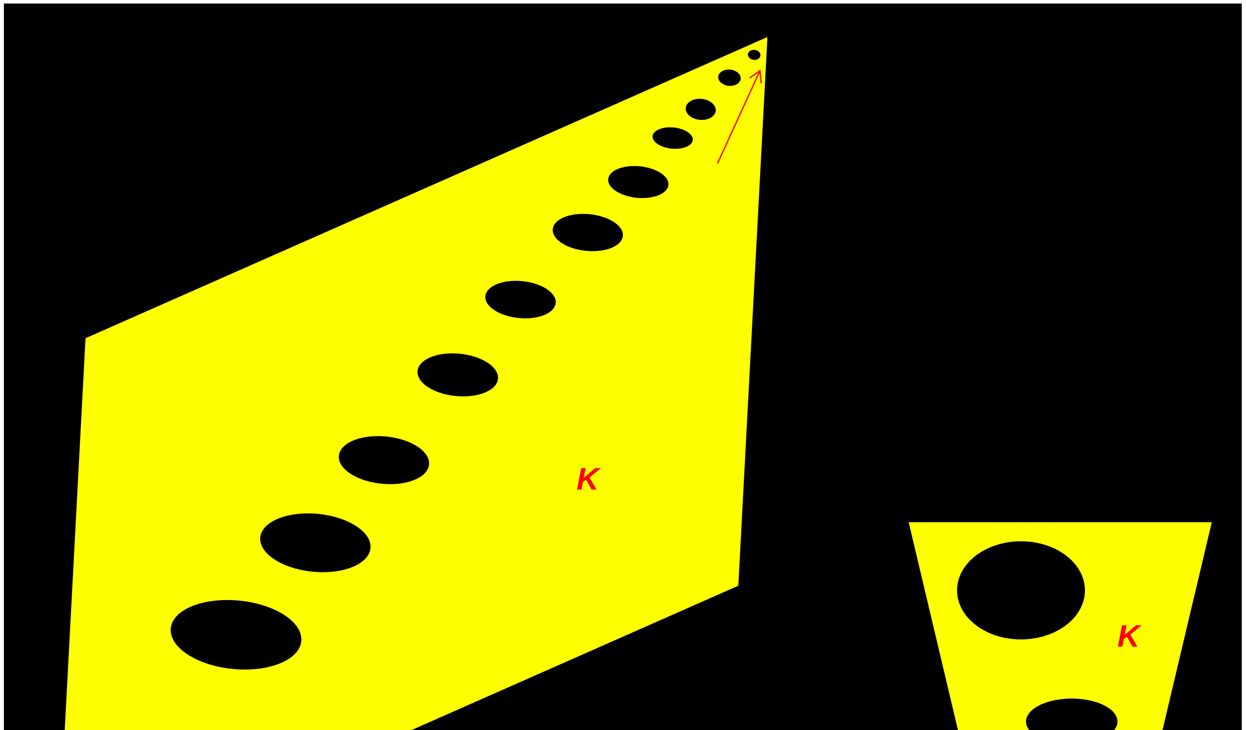
Complement $\mathbb{C} \setminus K$ is nonempty open subset of $\mathbb{C}$

Fact:
$\mathbb{C} \setminus K$ has **just one** unbounded component

The **bounded components** are called the **holes** of K

These are (bounded) connected open subsets of $\mathbb{C}$

Notation: $K^{\blacktriangle} =$ union of K with its holes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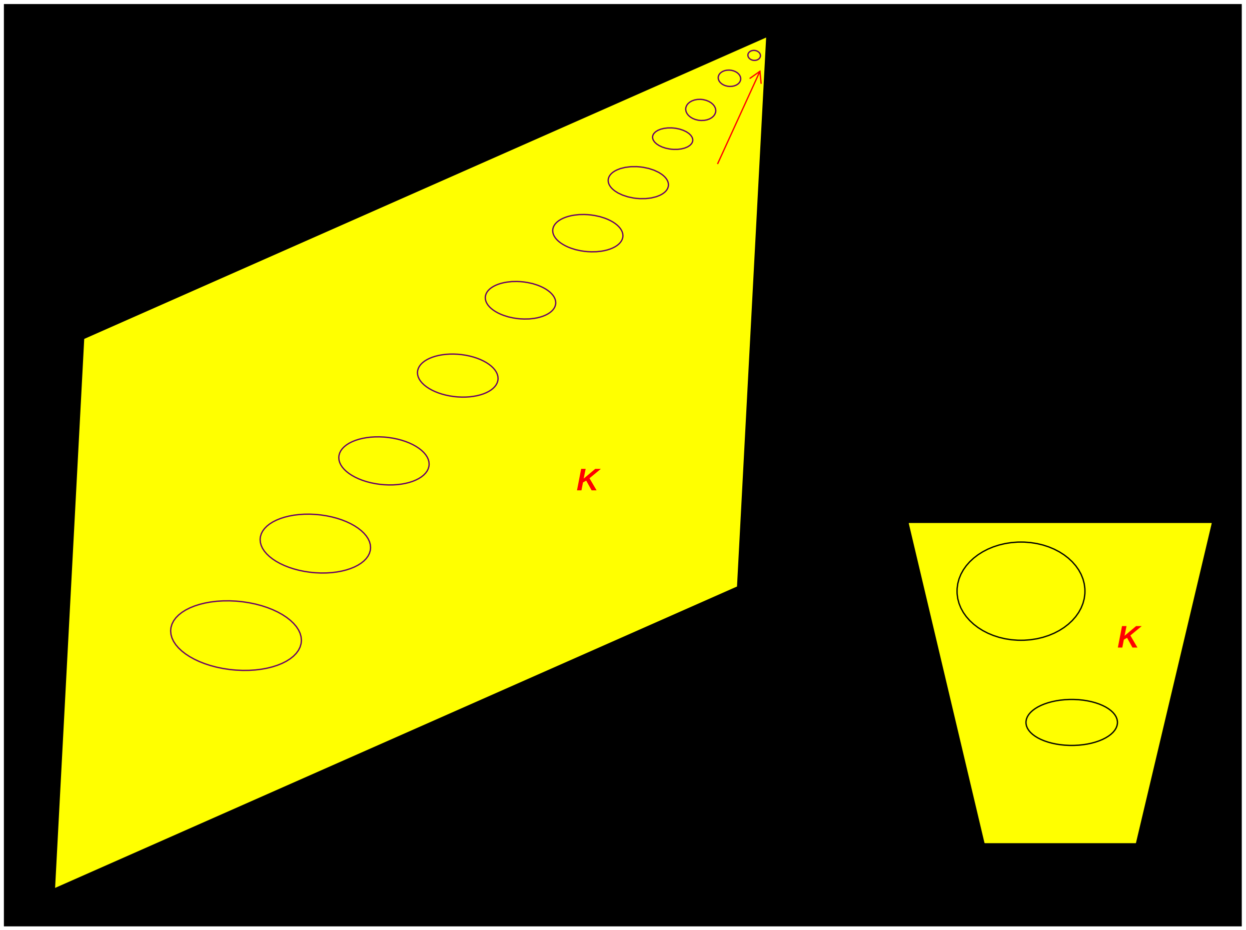
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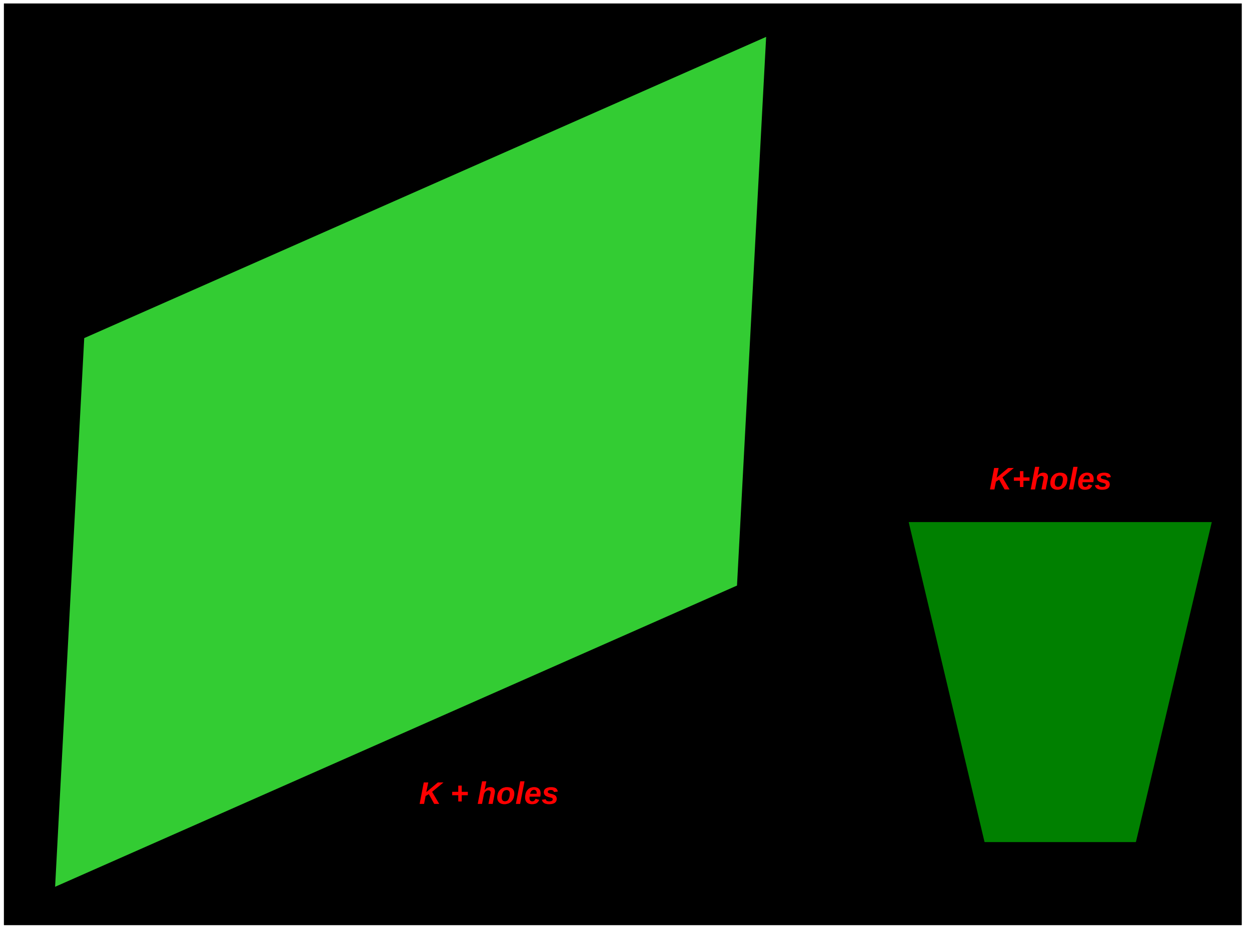
$K^{\blacktriangle}$ = union of K with its holes



Terminology:

$K^\blacktriangle$ is called the **polynomially convex hull** of K

Justification:

$$K^\blacktriangle = \bigcap_{p \text{ polynomial}} \left\{ z \in \mathbb{C} : |p(z)| \le \sup_{\kappa \in K} |p(\kappa)| \right\}$$

Recall the general relationship $\sigma_{\mathcal{B}}(b) \supset \sigma_{\mathcal{A}}(b)$

Can be sharpened for the generator g when $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ generated by the (single) element $g \in \mathcal{A}$

THEOREM

Let $\mathcal{A}$ be unital Banach algebra, let $g \in \mathcal{A}$, and let $\mathcal{B}$ be the Banach subalgebra of $\mathcal{A}$ generated by g. Then

$$\sigma_{\mathcal{B}}(g) = \sigma_{\mathcal{A}}(g)^{\blacktriangle},$$

i.e., $\sigma_{\mathcal{B}}(g)$ is obtained from $\sigma_{\mathcal{A}}(g)$ by adding its holes.

NOTATION

$b \in \mathcal{B} \subset \mathcal{A}$

$\mathcal{P}_{\mathcal{A}}(b)$: set of spectral idempotents associated with b as an element of $\mathcal{A}$

$\mathcal{P}_{\mathcal{B}}(b)$: set of spectral idempotents associated with b as an element of $\mathcal{B}$

Standard spectral theory:

$\mathcal{P}_{\mathcal{B}}(b) \subset \mathcal{P}_{\mathcal{A}}(b)$

Caveat: the inclusion can be **strict**

THEOREM

Let $\mathcal{A}$ be unital Banach algebra, let $g \in \mathcal{A}$, and let $\mathcal{B}$ be the Banach subalgebra of $\mathcal{A}$ generated by g.
Then $\mathcal{B}$ is monothetic, **hence** *the following sets coincide:*

(1) *The set of logarithmic residues in $\mathcal{B}$*;
(2) *The set of sums of idempotents in $\mathcal{B}$*;
(3) *The set of sums of spectral idempotents associated with g* **viewed as as an element of $\mathcal{B}$***.*

These (coinciding) sets are contained in the set of sums of spectral idempotents associated with g **viewed as as an element of $\mathcal{A}$**.
Indeed: $\mathcal{P}_{\mathcal{B}}(b) \subset \mathcal{P}_{\mathcal{A}}(b)$

Issue: when $\mathcal{P}_{\mathcal{B}}(b) \supset \mathcal{P}_{\mathcal{A}}(b)$ / actually equality?

Sufficient condition (easy case):
$\mathcal{B}$ inverse closed in $\mathcal{A}$ (hence $\sigma_{\mathcal{B}}(g) = \sigma_{\mathcal{A}}(g)$)

[To be returned to later]

But ... **there is more between heaven and earth** ...

Preparations / background material

K nonempty compact subset of $\mathbb{C}$

Terminology: K *polynomially convex* if $K^{\blacktriangle} = K$

In other words:
K has no holes / the complement of K in $\mathbb{C}$ is connected

Spectral idempotents correspond to partitions of the spectrum in (two) closed, hence compact, subsets

Needed:
discussion of **polynomial convexity** of the disjoint union of (two) compact sets

THEOREM

Let L and M be disjoint nonempty compact subsets of $\mathbb{C}$. Then the (disjoint) union $L \cup M$ is polynomially convex **if and only if** *so are both L and M.*
So $(L \cup M)^{\blacktriangle} = L \cup M$ **if and only if** *$L^{\blacktriangle} = L$ and $M^{\blacktriangle} = M$.*

Intuitively plausible

Rigorous arguments involve two results from **classical advanced plane topology:**

Schoenflies/Kuratowski Theorem

Let Γ be a Jordan curve in the complex plane. Then there exists a homeomorphism ψ from $\overline{\mathbb{D}}$ onto the closure $\overline{\Gamma_+} = \Gamma_+ \cup \Gamma$ of Γ such that $\psi[\mathbb{D}] = \Gamma_+$ and $\psi[\mathbb{T}] = \Gamma$.

$\overline{\mathbb{D}}$: **open unit disc**, $\overline{\mathbb{T}}$: **unit circle**, Γ_+: **interior domain Γ**

Alexander Addition Theorem

Let $U_-, U_+ \subset \mathbb{C}$ be nonempty and open, and let

$$F : \overline{\mathbb{D}} \to (U_- \cup U_+)$$

be a continuous map such that

$$F[\mathbb{T}_-] \subset U_-, \qquad F[\mathbb{T}_+] \subset U_+,$$

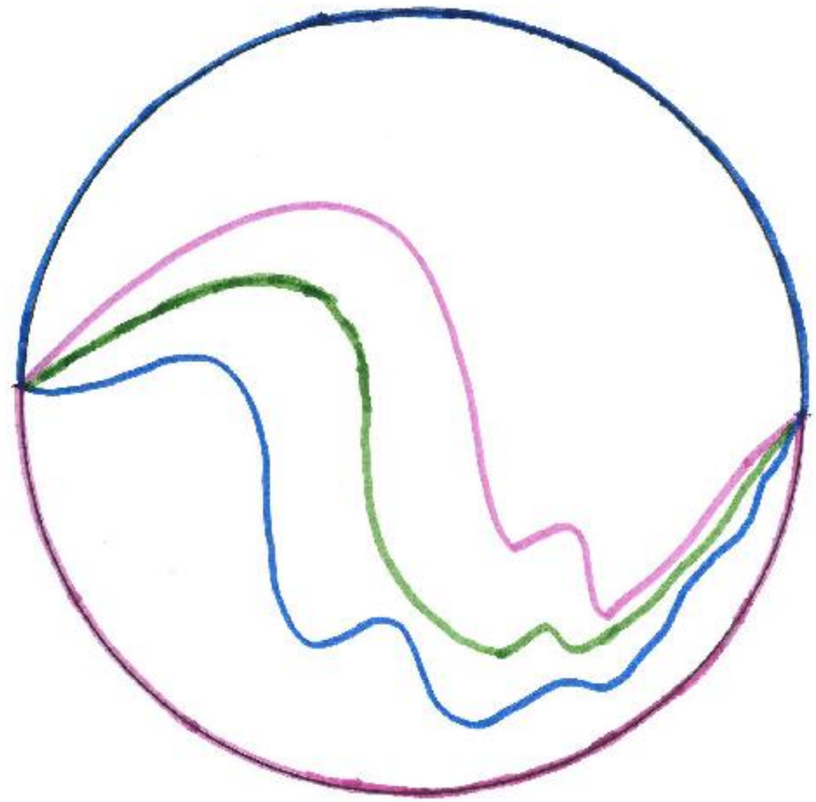
hence both $F(-1)$ and $F(+1)$ are in the intersection $U_- \cap \; U_+$ of U_- and U_+ which is therefore nonempty.
Then there exists a continuous map

$$f : [-1, +1] \to (U_- \cap U_+)$$

such that $f(-1) = F(-1)$ and $f(+1) = F(+1)$.

$\mathbb{T}_-$ / $\mathbb{T}_+$: part of $\mathbb{T}$ in **lower** / **upper** half plane

Alexander Addition Theorem



Next: **polynomially convex hulls** of disjoint unions

THEOREM

Let L and M be disjoint nonempty compact subsets of $\mathbb{C}$. Then $(L \cup M)^{\blacktriangle} = L^{\blacktriangle} \cup M^{\blacktriangle}$.

[disjointness essential, both ways]

But . . . $L^{\blacktriangle}$ and $M^{\blacktriangle}$ need not be disjoint

EXAMPLE

Let $L = \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ and $M = \{z \in \mathbb{C} \,\big|\, |z| = 2\}$. Then L and M are disjoint compact subsets of $\mathbb{C}$, and

$$L^{\blacktriangle} = \{z \in \mathbb{C} \,\big|\, |z| \leq 1\}, \qquad M^{\blacktriangle} = \{z \in \mathbb{C} \,\big|\, |z| \leq 2\}.$$

Also $(L \cup M)^{\blacktriangle} = \{z \in \mathbb{C} \big| |z| \leq 2\}$, corroborating that $(L \cup M)^{\blacktriangle} = L^{\blacktriangle} \cup M^{\blacktriangle}$. **But $L^{\blacktriangle}$ and $M^{\blacktriangle}$ are not disjoint.**

Observations on polynomial convexity up to now: known

But not so easy to find in the literature

Great help from:

Jan van Mill

Alexander J. Izzo, Bowling Green, Ohio, U.S.A.

Eugene Bilokopytov, Edmonton, Alberta, Canada

Still to be addressed: disjointness of $L^{\blacktriangle}$ and $M^{\blacktriangle}$

CROWBAR THEOREM

Let $K = L \cup M$ be the disjoint union of the nonempty compact subsets L and M of $\mathbb{C}$.
Then the polynomially convex hulls $L^\blacktriangle$ and $M^\blacktriangle$ are disjoint **if and only if** *the boundary of each hole of K lies in* **just one** *of the sets L and M.*

NB

[The boundary of a hole of K must lie in K which is here the union of L and M. The point is that we consider the situation where ∂K lies in just one of those (disjoint) sets.]

In that case the following statements are true:

CROWBAR THEOREM continued

If the boundary of each hole of K lies in **just one** *of the sets L and M. Then*

(1) *Each hole of L is a hole of K having its boundary in L*;

(2) *Each hole of M is a hole of K having its boundary in M;*

(3) *Each hole of K is* **either** *a hole of L* **or** *a hole of M* **(exclusive either/or)**;

(4) $L^{\blacktriangle}$ *and* $M^{\blacktriangle}$ *allow for the representations*

$$L^{\blacktriangle} = L \cup \bigcup_{\substack{H \text{ hole of } K \\ \partial H \subset L}} H, \qquad M^{\blacktriangle} = M \cup \bigcup_{\substack{H \text{ hole of } K \\ \partial H \subset M}} H;$$

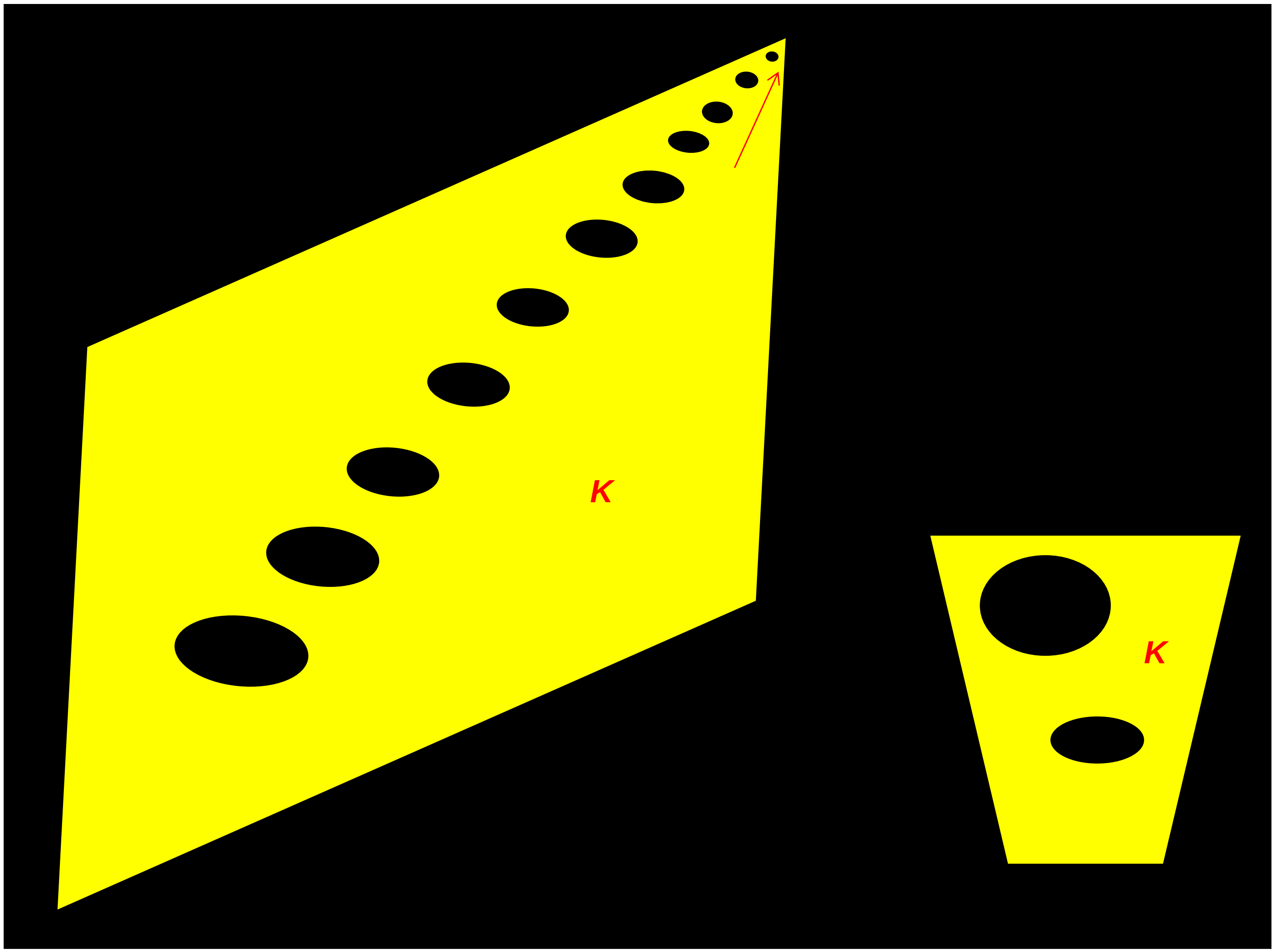
(5) $K^{\blacktriangle} = L^{\blacktriangle} \cup M^{\blacktriangle}$, *and this is a disjoint union.*

Back to where we were: $\mathcal{P}_{\mathcal{B}}(g) = \mathcal{P}_{\mathcal{A}}(g)$?

Necessary and sufficient condition:

THEOREM

Let $\mathcal{A}$ be unital Banach algebra, let $g \in \mathcal{A}$, and let $\mathcal{B}$ be the Banach subalgebra of $\mathcal{A}$ generated by g. Then $\mathcal{P}_{\mathcal{B}}(g) = \mathcal{P}_{\mathcal{A}}(g)$ **if and only if** *all holes of $\sigma_{\mathcal{A}}(g)$ are simply connected.*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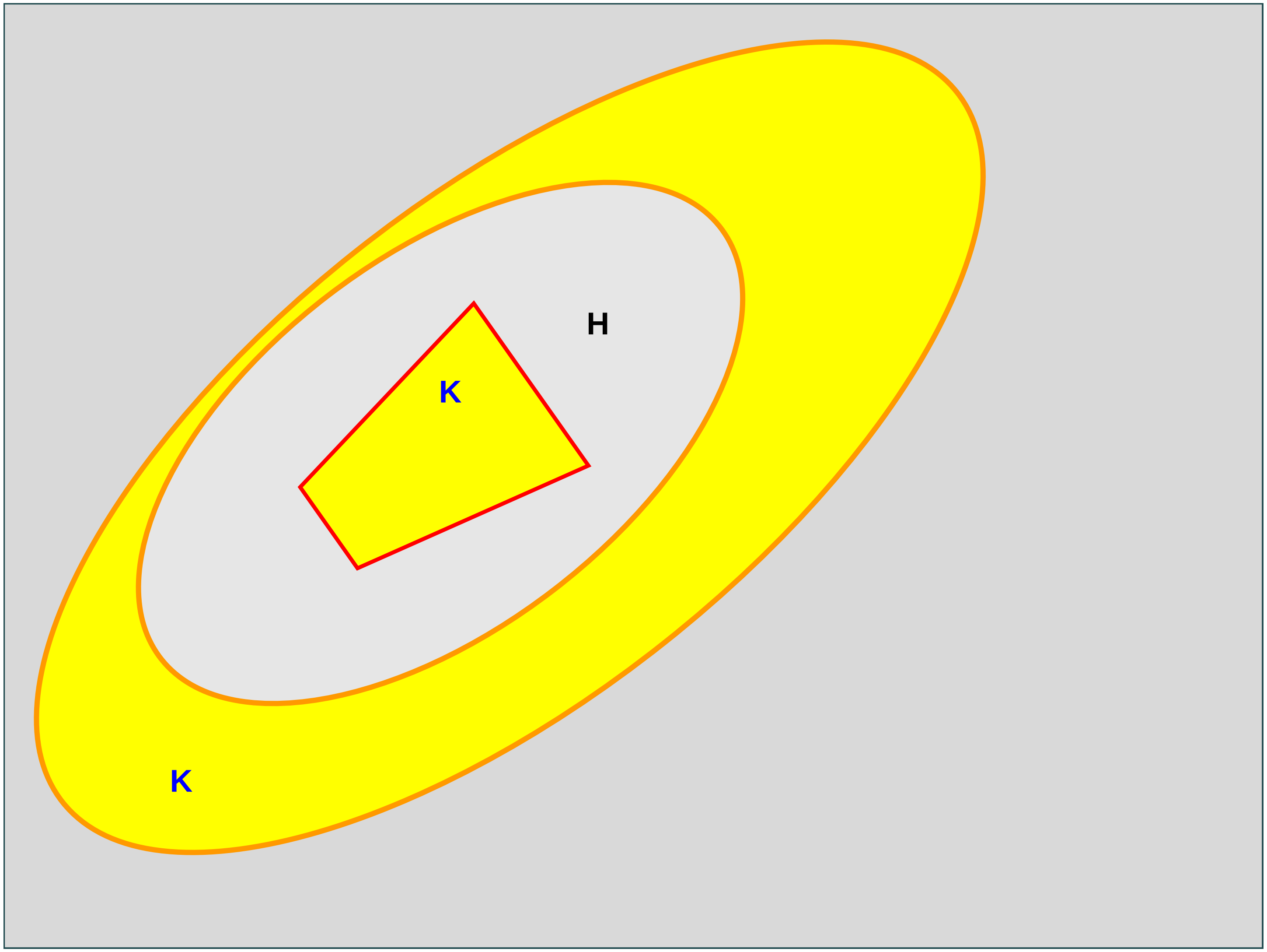
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Immediate corollary:

COROLLARY

Let $\mathcal{A}$ be unital Banach algebra, let $g \in \mathcal{A}$, and let $\mathcal{B}$ be the Banach subalgebra of $\mathcal{A}$ generated by g.

Suppose all holes of $\sigma_{\mathcal{A}}(g)$ are simply connected.

Then the following sets coincide:

(1) *The set of logarithmic residues in $\mathcal{B}$;*

(2) *The set of sums of idempotents in $\mathcal{B}$;*

(3) *The set of sums of spectral idempotents associated with g* **viewed as as an element of $\mathcal{B}$.**

(4) *The set of sums of spectral idempotents associated with g* **viewed as as an element of $\mathcal{A}$.**

In fact:

THEOREM

Let $\mathcal{A}$ be unital Banach algebra, let $g \in \mathcal{A}$, and let $\mathcal{B}$ be the Banach subalgebra of $\mathcal{A}$ generated by g.
Then all holes of $\sigma_{\mathcal{A}}(g)$ are simply connected

if and only if

the following sets coincide:

(1) *The set of logarithmic residues in $\mathcal{B}$*;

(2) *The set of sums of idempotents in $\mathcal{B}$*;

(3) *The set of sums of spectral idempotents associated with g viewed as as an element of $\mathcal{B}$.*

(4) *The set of sums of spectral idempotents associated with g viewed as as an element of $\mathcal{A}$.*

The inverse closed case

Recall that the situation where $\sigma_{\mathcal{B}}(g)$ is **inverse closed** in $\mathcal{A}$
hence $\sigma_{\mathcal{B}}(g) = \sigma_{\mathcal{A}}(g)$
was mentioned as an easy case.

THEOREM
Let $\mathcal{A}$ be unital Banach algebra, let $g \in \mathcal{A}$, and let $\mathcal{B}$ be the Banach subalgebra of $\mathcal{A}$ generated by g.
Then $\mathcal{B}$ is inverse closed in $\mathcal{A}$ **if and only if** *$\sigma_{\mathcal{A}}(g)$ has no holes,*
equivalently: *the complement of $\sigma_{\mathcal{A}}(g)$ in $\mathbb{C}$ is connected*

Known?

Connected complements

A homeomorphic image in $\mathbb{C}$ of the closed interval $[0,1]$ is called an **arc**.

Alexander Duality Theorem
The complement in $\mathbb{C}$ of an arc is connected.

A homeomorphic image in $\mathbb{C}$ of a nonempty interval on the real line will be called a **trail**.

Alexander Duality Theorem (generalized)
The complement in $\mathbb{C}$ of a compact subset of a trail is connected.

A result of another type:

THEOREM
The complement in $\mathbb{C}$ of an at most countable subset of $\mathbb{C}$ is connected.

COROLLARY *Let n be a positive integer, let G be an $n \times n$ matrix, and let $\mathbb{C}^{n\times n}[G]$ be the Banach subalgebra of $\mathbb{C}^{n\times n}$ generated by G.* **Then the following sets are identical:**
(1) *The set of logarithmic residues in $\mathbb{C}^{n\times n}[G]$;*
(2) *The set of sums of idempotents in $\mathbb{C}^{n\times n}[G]$;*
(3) *The set of sums spectral idempotents associated with G viewed as element of the Banach algebra $\mathbb{C}^{n\times n}[G]$;*
(4) *The set of sums spectral idempotents associated with G viewed as element of the Banach algebra $\mathbb{C}^{n\times n}$.*

A slightly less complete version of his matrix result, **from a 2001 BES paper***, has been the trigger for the investigation carried out last year that led to the material presented in this lecture.*

Thank you

for

your attention