Pluripolar hulls and fine holomorphy

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Subharmonic functions

Let $U \subset \mathbb{R}^n$.
An upper semicontinuous (usc) function $v : U \to [-\infty, \infty)$ is subharmonic on U if for all $x \in U$

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Properties: for subharmonic functions v_j

1. $v_1 + v_2$ is subharmonic
2. $\{\sup v_1, v_2, \ldots\}$ is subharmonic after usc regularization
3. $\lim v_j$ is subharmonic if $v_j \downarrow$ or the limit is uniform on compact sets

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Polar sets can be much bigger than zero sets but not too big

- All countable sets are polar
- Sets of positive length or positive Lebesgue measure are non-polar

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For f holomorphic on U, $\log|f|$ is PSH and $\{\log|f|, f \text{ holomorphic}\}$ with (1,2,3) generate locally all PSH functions.

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The set

$$\{(z, w) \in \mathbb{C}^2 : |z| < 1, w = 0\}$$

is pluripolar ($\log |w|$) but not complete: a subharmonic function on $\mathbb{C}$ that equals $-\infty$ for $|z| < 1$ is identically $-\infty$

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If $E = E_D^*$, E is F_σ as well as G_δ, then E is complete pluripolar (in D) (Zeriahi 1989)

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$E = E_D^*$ and G_δ are necessary

Questions of Sadullaev (1980)

Consider the graph Γ_f of $f(x) = e^{-1/x}$, $x \in \mathbb{R}^+$, in $\mathbb{C}^2$.
If $u \in \mathrm{PSH}(\mathbb{C}^2)$ and $u|_{\Gamma_f} = -\infty$, is necessarily $u(0) = -\infty$?

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In fact these questions ask for the pluripolar hull of Γ_f

A conjecture

If Γ_f is the graph of an analytic function f on $G \subset \mathbb{C}$, and f admits an analytic extension $\tilde{f}$ to a larger domain G', then $\Gamma_f^* \stackrel{\text{def}}{=} (\Gamma_f)^*_{\mathbb{C}^2}$ contains the graph $\Gamma_{\tilde{f}}$.

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Conjecture Let f be a maximal (multivalued) holomorphic function. Then

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Support from Levenberg & Poletsky 1999:

$$\Gamma^*_{z^\alpha} = \Gamma_{z^\alpha}$$

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Same for functions on $\mathbb{C}$ with only isolated singularities (W. 2003)

Counterexamples by Edigarian & W, 2003

$$f(z) = \sum_n \frac{c_n}{z - a_n}$$

where c_n tends rapidly to 0 and $a_n \in \mathbb{C}$ has a density point d
Then Γ_f^* may contain a point over d

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Counterexamples by Zwonek (2005) and by Siciak (2005)
Blaschke products B with zeros a_j that satisfy

$$C(0,1) \subset \overline{\{a_j\}}, \quad \sum \frac{1}{\log(1 - |a_j|)} > -\infty$$

For suitable holomorphic g, Γ_{Bg}^* extends to $|z| > 1$ and contains infinitely many sheets

Counterexample by Poletsky, W (2006)

$$g(z) = \sqrt{\prod \frac{z - a_j}{z - b_j}}$$

on the complement of a large Cantor set

$$\mathcal{C} = [0, 1] \setminus \left(\cup_{j=1}^{\infty} (a_j, b_j) \right) \subset \mathbb{R}$$

with

$$\sum \frac{1}{\log(b_j - a_j)} > -\infty$$

Γ_g^* has two sheet over $\mathbb{C} \setminus \mathcal{C}$

finely holomorphic functions

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Theorem (Edigarian, El Marzguioui, & W (2010))

Let Γ_f be the graph of a finely holomorphic function f on a fine domain $G \subset \mathbb{C}$, suppose f has a fine analytic extension $\tilde{f}$ to a larger fine domain G', then $\Gamma_{\tilde{f}} \subset \Gamma_f^*$.

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Conjecture Let G be a fine domain, f finely holomorphic on G not admitting fine analytic extension, then

$$\Gamma_f^* = \Gamma_f$$

Fine topology

Definition

The fine topology on $\mathbb{C}$ is the weakest topology that makes all subharmonic functions continuous.

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3. Local basis at a:

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$r > 0$, h subharmonic on $B(a, r)$, and $h(a) = 1$

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   $r > 0$, h subharmonic on $B(a, r)$, and $h(a) = 1$
4. Every polar set is f-closed and consists of f-isolated points

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e.g. $h(z) = \sum_{n=2}^{\infty} \frac{2^{-n}}{\log n} \log(|z - 1/n|)$

$h(0) = -1/2$, hence $\{h > -1\}$ is a fine neighborhood of 0 in $\mathbb{C} \setminus \{1/2, 1/3, 1/4, \ldots\}$

f-holomorphic functions

Definition

A function f on an f-domain D is called f-differentiable at $z_0 \in D$ if $\exists f'(z_0) \in \mathbb{C}$ s.t. $\forall \varepsilon > 0 \; \exists V \subset D$, an f-neighborhood of z_0, s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad (z \in V)$$

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The function f is f-holomorphic on D iff f-differentiable on D and f' is f-continuous on D.

finely holomorphic functions

Theorem

Let f be a complex valued function on an f-domain D. Equivalent are

1. *f is f-holomorphic on D.*
2. *Every point $z_0 \in D$ admits an f-neighborhood $V \subset D$ s.t. f is uniform limit of rational functions on V.*

Example

$$f(z) = \sum_{n=1}^{\infty} \frac{4^{-n^2}}{z - 1/n}$$

converges uniformly on $E = \bigcap_j \{|z - 1/j| \geq 3^{-j^2}\}$

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$$h(0)=\sum_{n=3}^{\infty}\frac{-1}{n^{3/2}}\log n\geq\int_2^{\infty}\frac{-1}{x^{3/2}}\log x\,dx>-1$$

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If $z \notin E$ then $\exists n_0,\ |z - 1/n_0| < 3^{-n_0^2}$ and

$$h(z) < \frac{-n_0^2 \log 3}{n_0^{3/2}} \leq \frac{-3^2 \log 3}{3^{3/2}} \leq -19/10$$

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If $z \notin E$ then $h(z) \leq -19/10$

$\{h > -3/2\}$ is an f-nbhd of 0 on which f converges:

$$(0, f(0)) \in \Gamma_f^*$$

Fine holomorphy on Cantor-type sets

$$f(z) = \frac{z-1}{z} \prod_{j=1}^{\infty} \frac{z-a_j}{z-b_j} = \frac{z-1}{z} \prod_{j=1}^{\infty} \left(1 + \frac{b_i - a_i}{z - b_i}\right)$$

defined on the complement of a large Cantor set

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Need to show f converges on a fine open subset of $\mathcal{C}$

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$$E_N = \bigcap_{j \geq N} \{|z - b_j| \geq e^{-jc_j/2}\} \setminus \{b_1, b_2, \ldots, b_{N-1}\}$$

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Lemma

For N sufficiently large, the set E_N contains a fine neighborhood E of a nonempty subset of $\mathcal{C}$. The product converges uniformly on compact subsets of E.

"proof" of the lemma

$$F_N = \bigcup_{i=1}^{\infty} [a_i, b_i] \cup \bigcup_{n=N}^{\infty} \{z \in \mathbb{C} : \ |z - b_n| \leq e^{-nc_n/2}\}.$$

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u is finely continuous and $\tilde{E} = \{u > 0\}$ is a finely open set
$F_N^c \subset E_N \cup \{b_1, \ldots, b_{N-1}\}$, and $u \equiv 0$ on F_N, the product converges uniformly on compact sets in the finely open set $E = \tilde{E} \setminus \{b_1, \ldots, b_{N-1}\}$.

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$$u(z) := g_{F_N} - g_{J_N},$$

with g_X the Green function of X with pole at infinity
u is finely continuous and $\tilde{E} = \{u > 0\}$ is a finely open set
$F_N^c \subset E_N \cup \{b_1, \ldots, b_{N-1}\}$, and $u \equiv 0$ on F_N, the product converges uniformly on compact sets in the finely open set $E = \tilde{E} \setminus \{b_1, \ldots, b_{N-1}\}$.
E is a fine neigborhood of certain points of $\mathcal{C}$

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$$\Gamma_f^* \supset \Gamma_{\tilde{f}} \qquad \Gamma_g^* \supset \Gamma_{\tilde{g}}$$

More on Γ^*

Let $\Omega = (\mathbb{C} \setminus \mathcal{C}) \times \mathbb{C})$

$$\Gamma^*_f \cap \Omega = \Gamma_{\tilde{f}} \cap \Omega \quad \Gamma^*_g \cap \Omega = \Gamma_{\tilde{g}} \cap \Omega$$

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If, e.g., $c_j = (j+2)!$, then

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If $|w - f(z)| > \delta > 0$

$$|w - f_n(z)||Q_n(z)| \geq \delta e^{-nc_n}, \quad n \geq N$$

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