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EXISTENCE OF STATIONARY SOLUTIONS IN THE CORONAL LOOP PROBLEM

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1. INTRODUCTION

OBSERVATIONS from outside the atmosphere of the Earth during the Skylab missions have revealed that the corona of the sun and other stars is highly structured. One of the features is the presence of hot plasma confined to quite persistent magnetic loops.

An extensive description of observations, physical mechanisms, modelling aspects, and most of the references have been given in [1] by Pakkert *et al.* In their model there are two independent variables: the time t and the arclength x along the loop, where $x = 0$ corresponds to the

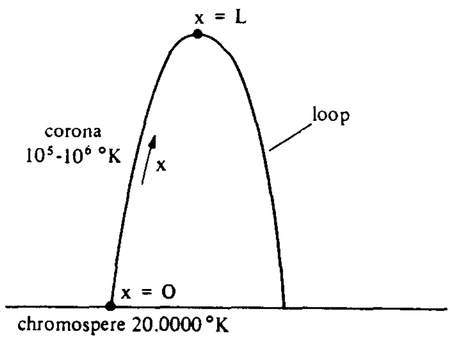

Fig. 1. A coronal loop.

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loop base, and $x = L$ to the loop summit, around which the plasma is assumed to be symmetrical. Neglecting the plasma velocity at $x = 0$, they find that the total mass M of the loop is constant and introduce a new space variable z which is given by

$$dz = \frac{\rho}{M} dx, \tag{1.1}$$

where ρ is the density. Obviously, z runs from $z = 0$ at the base to $z = 1$ at the summit. Assuming the velocities to be approximately constant with respect to time, the continuity equation decouples and is left out, whereas the momentum equation reduces to

$$\frac{\partial P}{\partial z} = -\lambda, \tag{1.2}$$

in which P is the (normalized) pressure and λ is a positive constant (to be normalized from now on: $\lambda = 1$).

The energy equation reads

$$\frac{5}{2}\frac{P}{T}\frac{\partial T}{\partial t} - \frac{\partial P}{\partial t} = \varepsilon \frac{P}{T}\frac{\partial}{\partial z}\left(T^{3/2}P\frac{\partial T}{\partial z}\right) + \alpha - P^2\chi(T). \tag{1.3}$$

Here T is the (normalized) temperature, ε is a small positive parameter, partly determined by the thermal conductivity coefficient, α is a positive constant corresponding to the heating rate of the coronal plasma, and $\chi(T)$ is the radiative loss function which depends critically on the physical ingredients of the problem: $\chi(T)$ increases quickly and monotonically until $T = \tau$, a temperature typical for the bottom part of the corona, and decreases monotonically for $T > \tau$.

The boundary conditions are first of all

$$T(0, t) = T_0, \qquad \frac{\partial T}{\partial z}(1, t) = 0, \tag{1.4}$$

with T_0 a positive constant and $0 < T_0 < \tau$. However, we simplify once more and put $T_0 = 0$ in order to make the mathematics in the next sections a little more transparent. See also remark 2.4 in Section 2.

The last condition comes from the assumption that the loop length L is constant. Via the gas law this leads to

$$\int_0^1 \frac{T}{P} dz = l \text{ (loop length condition)}, \tag{1.5}$$

where l is a positive constant (which is in fact a normalization of L).

In [1] a start has been made with the quantitative analysis of the problem, using singular perturbation methods, the perturbation parameter being ε. Here we shall focus on time-independent (stationary) solutions of (1.2)–(1.5). To this end a reformulation is useful. First we integrate (1.2) to obtain

$$P(z) = \mu - z, \tag{1.6}$$

with μ a constant, one of the unknowns to be found. Next we set

$$w = T^{5/2}, \tag{1.7}$$

and introduce

$$y = \log P = \log(\mu - z) \tag{1.8}$$

as a new space coordinate. With the time derivative gone (1.3) reduces to (absorbing 2/5 in ε)

$$\varepsilon w_{yy} + \alpha w^{2/5} - e^{2y} w^{2/5} \chi(w^{2/5}) = 0 \tag{1.9}$$

for $\log(\mu - 1) < y < \log \mu$.

The boundary conditions (1.4) (with $T_0 = 0$) become

$$w_y(\log(\mu - 1)) = w(\log \mu) = 0, \tag{1.10}$$

whereas (1.5) reduces to

$$\int_{\log(\mu-1)}^{\log \mu} w(y)^{2/5}\,dy = l. \tag{1.11}$$

So the stationary problem consists of finding a constant $\mu > 1$ and a function

$$w: [\log(\mu - 1), \log \mu] \to \mathbb{R}$$

such that (1.9)-(1.11) are satisfied.

The outline of the paper is now as follows. The mathematical results are given in Section 2. In Section 3 we set up an iteration argument which yields the existence of a decreasing solution provided ε is sufficiently small, and in Section 4 we prove the lemmas needed in Section 3.

2. THE MAIN RESULT

In the introduction the steady-state problem has been reduced to the following problem: find $\mu > 1$ and a nonnegative function $w(y)$ on $[\log(\mu - 1), \log \mu]$ such that

$$(\mathrm{P})\begin{cases} \varepsilon w''(y) + \alpha w^{2/5} - e^{2y} g(w) = 0\ (\varepsilon > 0) \qquad \text{for } y_0 = \log(\mu - 1) < y < \log \mu = y_1; & (2.1) \\ w'(y_0) = w(y_1) = 0; & (2.2) \\ \displaystyle\int_{y_0}^{y_1} w(y)^{2/5}\,dy = l, & (2.3) \end{cases}$$

where ε, α and l are given positive constants, and $g: [0, \infty) \to \mathbb{R}$ is a function satisfying

(H) $g(0) = 0$; $g(s) > 0$ for $s > 0$; $g \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$; g is Lipschitz continuous on $[0, \infty)$; $g'(s) \leq 0$ a.e. for s large enough; $g(s)$ and $h(s) = (d/ds)(sg(s))$ are $O(s^{-\beta})$ as $s \to \infty$ for some $\beta > 1$. The derivatives are taken in the distributional sense.

The main result of this paper is given by the following.

THEOREM 2.1. Let (H) be satisfied. For $\varepsilon > 0$ sufficiently small problem (P) has a decreasing solution (μ, w). As $\varepsilon \downarrow 0$ the order of μ is $\varepsilon^{-2/7}$ and the order of $w(\log(\mu - 1)) = w(y_0)$ is $\varepsilon^{-5/7}$.

Remark 2.2 We suspect the decreasing solution to be unique. Although the existence proof relies on a contraction argument, there are some technical reasons preventing us from concluding uniqueness.

Remark 2.3. The power 2/5 in (2.1) and (2.3) can be replaced by any $\beta \in (0, 1)$ without affecting the existence proof. As $\varepsilon \downarrow 0$ the orders of μ and $w(y_0)$ will then be $\varepsilon^{-\beta/(\beta+1)}$ and $\varepsilon^{-1/(\beta+1)}$, respectively.

Remark 2.4. We have taken the liberty of setting $T_0 = 0$ in the introduction (1.4). In reality T_0 is a small positive number corresponding to a normalized temperature and therefore this simplification seems reasonable from a physical point of view.

For the sake of completeness we mention that if in (2.2) the condition $w(y_1) = 0$ is replaced by $w(y_1) = w_1$(w_1 a given positive number), theorem 2.1 remains valid, but the proof is more involved, though essentially the same.

There is however some numerical evidence for the existence of oscillating solutions if $0 < T_0 < \tau$, something which is easily seen to be impossible if $T_0 = 0$, i.e. $w_1 = 0$. We hope to come back to this subject in a future paper.

3. OUTLINE OF THE PROOF

We shall construct the solution as the fixed point of a contraction. In order to do this we begin with rewriting the problem. It will be convenient to set

$$f(y, w) = \alpha w^{2/5} - e^{2y} g(w), \tag{3.1}$$

and to redefine e^{2y} by

$$e^{2y} = \begin{cases} e^{2y} & \text{for } y < \log \mu \\ \mu^2 & \text{for } y \geq \log \mu. \end{cases} \tag{3.2}$$

Whenever we write e^{2y}, it is understood to be redefined by (3.2). Of course this does not make any difference for problem (P). Note, however, that e^{2y} is now a function of both y and μ.

Let w_{ε,μ,w_0} be the unique solution of

$$(\mathrm{P}_{\varepsilon,\mu,w_0}) \begin{cases} \varepsilon w''(y) + f(y, w(y)) = 0 \qquad \text{for } y > \log(\mu - 1) = y_0; & (3.3) \\ w(y_0) = w_0 > 0; & (3.4) \\ w'(y_0) = 0. & (3.5) \end{cases}$$

Since we are looking for decreasing solutions of (P) we can restrict ourselves to solutions $w = w_{\varepsilon,\mu,w_0}$ for which the inverse w^{-1} exists. For these solutions we have

$$w'(y) = -\left\{\frac{2}{\varepsilon}\int_{w(y)}^{w_0} f(w^{-1}(s), s)\,ds\right\}^{1/2}, \tag{3.6}$$

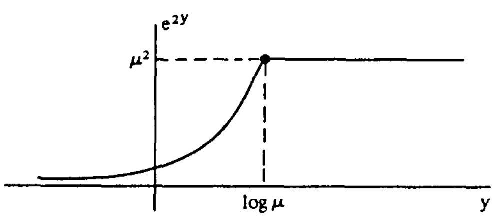

Fig. 2. e^{2y}.

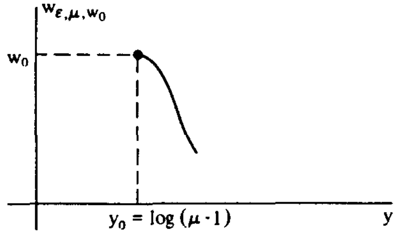

Fig. 3. w_{ε,μ,w_0}.

and hence

$$y - y_0 = -\int_{y_0}^{y} w'(\eta)\left\{\frac{2}{\varepsilon}\int_{w(\eta)}^{w_0} f(w^{-1}(s), s)\,\mathrm{d}s\right\}^{-1/2}\mathrm{d}\eta, \tag{3.7}$$

or

$$y - y_0 = \sqrt{\frac{\varepsilon}{2}}\int_{w(y)}^{w_0}\frac{\mathrm{d}w}{\sqrt{\int_{w}^{w_0} f(w^{-1}(s), s)\,\mathrm{d}s}}. \tag{3.8}$$

Observe that in (3.8) w is just an integration variable, while $w(y)$ is the solution of $(\mathrm{P}_{\varepsilon,\mu,w_0})$ and $w^{-1}(s)$ its inverse. Formula (3.8) holds for all y as long as $w(y)$ is decreasing.

In a similar way we obtain

$$\int_{y_0}^{y} w(\eta)^{2/5}\,\mathrm{d}\eta = \sqrt{\frac{\varepsilon}{2}}\int_{w(y)}^{w_0}\frac{w^{2/5}\,\mathrm{d}w}{\sqrt{\int_{w}^{w_0} f(w^{-1}(s), s)\,\mathrm{d}s}}. \tag{3.9}$$

Next we substitute

$$w = w_0(1 - z)^{5/7}, \qquad z = 1 - \left(\frac{w}{w_0}\right)^{7/5} \tag{3.10}$$

in (3.8) and (3.9). For (3.8) this yields

$$y - y_0 = \frac{5}{7}\sqrt{\frac{\varepsilon}{2}}\,w_0^{3/10}\int_0^{1-(w/w_0)^{7/5}}\frac{\mathrm{d}z}{(1 - z)^{2/7}\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon(\mu, w_0, z)}}, \tag{3.11}$$

and for (3.9)

$$\int_{y_0}^{y} w(\eta)^{2/5}\,\mathrm{d}\eta = \frac{5}{7}\sqrt{\frac{\varepsilon}{2}}\,w_0^{7/10}\int_0^{1-(w/w_0)^{7/5}}\frac{\mathrm{d}z}{\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon(\mu, w_0, z)}}, \tag{3.12}$$

where

$$\Phi_\varepsilon(\mu, w_0, z) = \frac{1}{w_0^{7/5}}\int_{w_0(1-z)^{5/7}}^{w_0} \mathrm{e}^{2w^{-1}(s)} g(s)\,\mathrm{d}s. \tag{3.13}$$

As the reader may guess, all the difficulties lie in (3.13) since Φ_ε is only well-defined if $w = w_{\varepsilon,\mu,w_0}$ is decreasing and goes to zero. What can go wrong is that $w(y)$ starts to oscillate before reaching zero. $\Phi_\varepsilon(\mu, w_0, z)$ is only defined then for $0 \le z \le z^*$, where $z^* = 1 - (w_*/w_0)^{7/5} < 1$ and w_* is the first local minimum of $w(y)$ and $\Phi_\varepsilon(\mu, w_0, z^*) = (5\alpha/7)\,z^*$. As long as the square roots in (3.11) and (3.12) exist and are positive there is no problem.

PROPOSITION 3.1. Suppose that the solution $w_{\varepsilon,\mu_0,w_0}(y) = w(y)$ of $(P_{\varepsilon,\mu_0,w_0})$ is decreasing on $(\log(\mu-1), \log\mu)$. If

$$\log\frac{\mu}{\mu-1} = \frac{5}{7}\sqrt{\frac{\varepsilon}{2}}\, w_0^{3/10}\int_0^1 \frac{dz}{(1-z)^{2/7}\sqrt{(5\alpha/7)z - \Phi_\varepsilon(\mu, w_0, z)}}, \tag{3.14}$$

and

$$l = \frac{5}{7}\sqrt{\frac{\varepsilon}{2}}\, w_0^{7/10}\int_0^1 \frac{dz}{\sqrt{(5\alpha/7)\, z - \Phi_\varepsilon(\mu, w_0, z)}}, \tag{3.15}$$

then (μ, w) is a solution of (P).

Proof. Setting $w = 0$ in (3.11) and comparing with (3.14) we conclude that $w(\log\mu) = 0$. Similarly (3.12) and (3.15) yield that (2.3) holds with $y_1 = \log\mu$. ■

It follows that there is a one to one correspondence between pairs (w_0, μ) satisfying (3.14)–(3.15) and decreasing solutions of (P). Thus we are going to solve (3.14)–(3.15) instead of (P).

It will be a little bit more pleasant to work with

$$\gamma = \log\frac{\mu}{\mu-1}, \qquad \left(\mu = \frac{e^\gamma}{e^\gamma - 1}\right) \tag{3.16}$$

rather than with μ itself. Note that (3.16) is a bijection between $(1, \infty)$ and $\mathbb{R}^+$. From now on we shall write $\Phi_\varepsilon(\gamma, w_0, z)$ instead of $\Phi_\varepsilon(\mu, w_0, z)$.

Trying to define an iteration map we wish to solve $\tilde{\gamma}$ and $\tilde{w}_0$ from

$$\tilde{\gamma} = \frac{5}{7}\sqrt{\frac{\varepsilon}{2}}\, \tilde{w}_0^{3/10}\int_0^1 \frac{dz}{(1-z)^{2/7}\sqrt{(5\alpha/7)\, z - \Phi_\varepsilon(\gamma, w_0, z)}} \tag{3.17}$$

and

$$l = \frac{5}{7}\sqrt{\frac{\varepsilon}{2}}\, \tilde{w}_0^{7/10}\int_0^1 \frac{dz}{\sqrt{(5\alpha/7)\, z - \Phi_\varepsilon(\gamma, w_0, z)}}, \tag{3.18}$$

for given γ and w_0.

We need to control $\Phi_\varepsilon(\gamma, w_0, z)$ to ensure that the right-hand sides are well-defined. Computing $\tilde{\gamma}$ and $\tilde{w}_0$ is straightforward. However, we also want $\tilde{\gamma}$ and $\tilde{w}_0$ to be close to γ and w_0 so that we can repeat the process and obtain a (bounded) sequence of iterates. Finally, we want the sequence to converge. The following lemmas establish these properties.

LEMMA 3.2. There exists a constant $\lambda > 0$ such that, whenever the left-hand side exists,

$$\Phi_\varepsilon(\gamma, w_0, z) \le \frac{\lambda}{w_0^{7/5}}\left(1 + \frac{1}{\gamma}\right)^2 z \tag{3.19}$$

if w_0 is large enough (uniformly in $\gamma \in \mathbb{R}^+$, $\varepsilon \in \mathbb{R}^+$ and $z \in [0,1]$).

Hence we must keep the coefficient on the right side of (3.19) smaller than $5\alpha/7$. It immediately turns out to be of order $\varepsilon^{3/7}$ if (w_0, γ) satisfies condition (3.20) in the following lemma.

Lemma 3.3. There exist positive constants M_0, M_1, M_2, M_3 with $M_0 < M_1$ and $M_2 < M_3$, such that, for $\varepsilon > 0$ small,

$$M_0\varepsilon^{-5/7} \leq w_0 \leq M_1\varepsilon^{-5/7}, \qquad M_2\varepsilon^{2/7} \leq y \leq M_3\varepsilon^{2/7} \tag{3.20}$$

implies that $\tilde{w}_0$ and $\tilde{y}$ are well-defined by (3.17) and (3.18). Moreover, $\tilde{w}_0$ and $\tilde{y}$ satisfy the same estimate (3.20) as w_0 and y, with the same constants.

Thus, given any pair $(w_0^{(0)}, y^{(0)})$ for which (3.20) holds, we obtain a bounded sequence of iterates $(w_0^{(n)}, y^{(n)})$ $(n = 1, 2, \ldots)$, provided ε is small enough.

Lemma 3.4. There exists a positive constant L such that for $\varepsilon > 0$ small and any two pairs (w_0, y) and $(\tilde{w}_0, \tilde{y})$ satisfying (3.20), (and not necessarily (3.17) and (3.18))

$$|\Phi_\varepsilon(y, w_0, z) - \Phi_\varepsilon(\tilde{y}, \tilde{w}_0, z)| \leq L\{\varepsilon^{8/7}|w_0 - \tilde{w}_0| + \varepsilon^{1/7}|y - \tilde{y}|\}z \tag{3.21}$$

for all $z \in [0, 1]$.

We shall prove lemmas 3.2–3.4 in the next section.

Proof of theorem 2.1 *(completion)*. We set $\xi = \varepsilon^{5/7} w_0$ and $\eta = \varepsilon^{-2/7}y$. Writing $\Phi_\varepsilon(\xi, \eta, z)$ instead of $\Phi_\varepsilon(y, w_0, z)$, lemma 3.4 yields

$$|\Phi_\varepsilon(\xi, \eta, z) - \Phi_\varepsilon(\tilde{\xi}, \tilde{\eta}, z)| \leq L\varepsilon^{3/7}\{|\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|\}z \tag{3.22}$$

for all $\xi, \tilde{\xi} \in [M_0 M_1]$, $\eta, \tilde{\eta} \in [M_2, M_3]$, $z \in [0, 1]$ and ε small, whereas (3.17) and (3.18) transform in

$$\tilde{\eta} = \tilde{\xi}^{3/10}F_\varepsilon(\xi, \eta), \tag{3.23}$$

$$1 = \tilde{\xi}^{7/10}G_\varepsilon(\xi, \eta), \tag{3.24}$$

with

$$F_\varepsilon(\xi, \eta) = \frac{5}{7\sqrt{2}}\int_0^1 \frac{\mathrm{d}z}{(1 - z)^{2/7}\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon(\xi, \eta, z)}}, \tag{3.25}$$

$$G_\varepsilon(\xi, \eta) = \frac{5}{7\sqrt{2}}\int_0^1 \frac{\mathrm{d}z}{\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon(\xi, \eta, z)}}. \tag{3.26}$$

Solving (3.23) and (3.24) we obtain

$$\tilde{\xi} = G_\varepsilon(\xi, \eta)^{-10/7} \tag{3.27}$$

$$\tilde{\eta} = F_\varepsilon(\xi, \eta)G_\varepsilon(\xi, \eta)^{-3/7} \tag{3.28}$$

which defines a map

$$T_\varepsilon\colon [M_0, M_1] \times [M_2, M_3] \to [M_0, M_1] \times [M_2, M_3]; \qquad T_\varepsilon(\xi, \eta) = (\tilde{\xi}, \tilde{\eta}). \tag{3.29}$$

It remains to show that T_ε is a contraction for ε small.

Simplifying notations we first estimate

$$|F_\varepsilon(\xi,\eta) - F_\varepsilon(\bar\xi,\bar\eta)| = \left|\frac{5}{7\sqrt{2}}\int_0^1 \left\{\frac{1}{\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon}} - \frac{1}{\sqrt{(5\alpha/7)\,z - \bar\Phi_\varepsilon}}\right\}\frac{\mathrm{d}z}{(1-z)^{2/7}}\right|$$

$$\le \frac{5}{7\sqrt{2}}\int_0^1 \frac{|\Phi_\varepsilon - \bar\Phi_\varepsilon|}{\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon}\sqrt{(5\alpha/7)\,z - \bar\Phi_\varepsilon}\{\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon} + \sqrt{(5\alpha/7)\,z - \bar\Phi_\varepsilon}\}}\frac{\mathrm{d}z}{(1-z)^{2/7}}. \tag{3.30}$$

By (3.19) and (3.20) we have for small ε that

$$\frac{5\alpha}{7}z \ge \frac{5\alpha}{7}z - \Phi_\varepsilon \ge \left\{\frac{5\alpha}{7} - \frac{\lambda\varepsilon}{\xi^{7/5}}\left(1 + \frac{1}{\varepsilon^{2/7}\eta}\right)^2\right\}z$$

$$\ge \left\{\frac{5\alpha}{7} - \frac{\lambda\varepsilon}{M_0^{7/5}}\left(1 + \frac{1}{M_2\varepsilon^{2/7}}\right)^2\right\}z \ge \left\{\frac{5\alpha}{7} - \frac{\lambda}{M_0^{7/5}}\left(1 + \frac{1}{M_2}\right)^2\varepsilon^{3/7}\right\}z, \tag{3.31}$$

which, combined with (3.30) and (3.22), yields

$$|F_\varepsilon(\xi,\eta) - F_\varepsilon(\bar\xi,\bar\eta)|$$

$$\le \frac{5}{7\sqrt{2}}\frac{L}{2\{(5\alpha/7) - (\lambda/M_0^{7/5})(1 + 1/M_2)^2\varepsilon^{3/7}\}^{3/2}}\int_0^1 \frac{\mathrm{d}z}{z^{1/2}(1-z)^{2/7}}\varepsilon^{3/7}\{|\xi - \bar\xi| + |\eta - \bar\eta|\}$$

$$\le K_1\varepsilon^{3/7}\{|\xi - \bar\xi| + |\eta - \bar\eta|\} \tag{3.32}$$

for some positive constant K_1 if ε is sufficiently small.

Similarly,

$$|G_\varepsilon(\xi,\eta) - G_\varepsilon(\bar\xi,\bar\eta)| \le K_2\varepsilon^{3/7}\{|\xi - \bar\xi| + |\eta - \bar\eta|\} \tag{3.33}$$

for some positive constant if ε is sufficiently small.

Now (3.31) implies that for ε small both F_ε and G_ε are bounded away from zero and infinity on $[M_0, M_1] \times [M_2, M_3]$ uniformly in ε. With (3.32) and (3.33) this implies that T_ε defined by (3.27), (3.28) and (3.29) is a contraction provided ε is small.

Hence any sequence of iterates converges to the unique fixed point of T_ε. By proposition 3.1 this fixed point corresponds to a decreasing solution of (P). Finally it follows from lemma 3.3 that $w_0 = w(y_0)$ is of order $\varepsilon^{-5/7}$, γ is of order $\varepsilon^{2/7}$, and hence $\mu = e^\gamma/(e^\gamma - 1)$ is of order $\varepsilon^{-2/7}$. This completes the proof of theorem 2.1. ■

Remark 3.5. In order to conclude that the decreasing solution is unique one has to deal with points outside $[M_0, M_1] \times [M_2, M_3]$ for which T_ε is well-defined. It is not clear whether the contraction property holds for those points.

4. PROOFS OF THE LEMMAS

In this section we prove the lemmas of Section 3.

Proof of lemma 3.2. Because of (3.2) and the nonnegativity of g we have

$$\Phi_\varepsilon(\gamma, w_0, z) \le \frac{\mu^2}{w_0^{7/5}}\int_{w_0(1-z)}^{w_0} g(s)\,\mathrm{d}s. \tag{4.1}$$

Since

$$\mu = \frac{e^\gamma}{e^\gamma - 1} = 1 + \frac{1}{e^\gamma - 1} \le 1 + \frac{1}{\gamma}, \tag{4.2}$$

it remains to estimate

$$\phi(z) = \int_{w_0(1-z)}^{w_0} g(s)\,ds. \tag{4.3}$$

From the assumptions on g it follows that there exists $B \in \mathbb{R}^+$ such that g is decreasing on $[B, \infty)$, and that

$$\phi(1) < A = \int_0^\infty g(s)\,ds < \infty. \tag{4.4}$$

We have

$$\phi'(z) = g(w_0(1 - z))w_0, \tag{4.5}$$

and hence, if

$$w_0 \ge 2B, \tag{4.6}$$

$$\phi'(z) \le g\left(\frac{1}{2}w_0\right)w_0 = 2g\left(\frac{1}{2}w_0\right)\frac{1}{2}w_0 \tag{4.7}$$

for $0 \le z \le 1/2$. Now observe that for $s \ge B$

$$g(s)s = g(s)B + g(s)(s - B) \le \sup_{\mathbb{R}^+} g \cdot B + A, \tag{4.8}$$

while for $s \le B$

$$g(s)s \le \sup_{\mathbb{R}^+} g \cdot B. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9) we have

$$\phi'(z) \le 2(\sup_{\mathbb{R}^+} g \cdot B + A) \qquad \text{on} \quad \left[0, \frac{1}{2}\right] \tag{4.10}$$

provided $w_0 \ge 2B$. Since obviously $\phi(0) = 0$ it follows from (4.4) and (4.10) that

$$\phi(z) \le \lambda z \qquad \text{on} \quad \left[0, \frac{1}{2}\right] \tag{4.11}$$

where

$$\lambda = 2(\sup_{\mathbb{R}^+} g \cdot B + A). \tag{4.12}$$

Since

$$\phi(z) \le A \qquad \text{on} \quad \left[\frac{1}{2}, 1\right] \tag{4.13}$$

we conclude that (4.11) is valid on [0, 1], and the lemma follows. ■

Proof of lemma 3.3. Let

$$L_1 = \frac{5}{7}\int_0^1 \frac{dz}{\sqrt{(5\alpha/7)\,z}}, \qquad L_2 = \frac{5}{7}\int_0^1 \frac{dz}{(1-z)^{2/7}\sqrt{(5\alpha/7)\,z}}. \tag{4.14}$$

(These are fixed positive finite numbers.) Suppose that for some positive numbers M_0, M_1, M_2, M_3 and for some (w_0, γ) condition (3.20) holds.

Then by lemma 3.2

$$\Phi_\varepsilon(\gamma, w_0, z) \le \frac{\lambda}{M_0^{7/5}}\left(1 + \frac{1}{M_2}\right)^2 \varepsilon^{3/7} z \tag{4.15}$$

for small ε. For the moment set

$$C(M_0, M_2) = \frac{\lambda}{M_0^{7/5}}\left(1 + \frac{1}{M_2}\right)^2. \tag{4.16}$$

Then, following (3.31),

$$\begin{aligned} L_1 &= \frac{5}{7}\int_0^1 \frac{dz}{\sqrt{(5\alpha/7)\,z}} \le \frac{5}{7}\int_0^1 \frac{dz}{\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon(\gamma, w_0, z)}} \\ &\le \frac{5}{7}\int_0^1 \frac{dz}{\sqrt{(5\alpha/7)\,z - \varepsilon^{3/7}C(M_0, M_2)z}} \\ &= \frac{5}{7}\int_0^1 \frac{dz}{\sqrt{(5\alpha/7)\,z(1 - (7/5\alpha)\,\varepsilon^{3/7}C(M_0, M_2))}} \\ &= \left(1 - \frac{7}{5\alpha}\varepsilon^{3/7}C(M_0, M_2)\right)^{-1/2} L_1 = K_\varepsilon(M_0, M_2)L_1, \end{aligned} \tag{4.17}$$

and similarly,

$$L_2 \le \frac{5}{7}\int_0^1 \frac{dz}{(1-z)^{2/7}\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon(\gamma, w_0, z)}} \le K_\varepsilon(M_0, M_2)L_2. \tag{4.18}$$

In particular, the square roots are well-defined.

Hence we can solve $\tilde{\gamma}$ and $\tilde{w}_0$ from (3.17) and (3.18) with the following estimates being a direct consequence of (4.17) and (4.18):

$$\tilde{M}_0 \varepsilon^{-5/7} \le \tilde{w}_0 \le \tilde{M}_1 \varepsilon^{-5/7}; \tag{4.19}$$

$$\tilde{M}_2 \varepsilon^{2/7} \le \tilde{\gamma} \le \tilde{M}_3 \varepsilon^{2/7} \tag{4.20}$$

where

$$\tilde{M}_0 = \left(\frac{l\sqrt{2}}{KL_1}\right)^{10/7}; \tag{4.21}$$

$$\tilde{M}_1 = \left(\frac{l\sqrt{2}}{L_1}\right)^{10/7}; \tag{4.22}$$

$$\tilde{M}_2 = \left(\frac{l\sqrt{2}}{KL_1}\right)^{3/7} \frac{L_2}{\sqrt{2}}; \tag{4.23}$$

$$\tilde{M}_3 = K\left(\frac{l\sqrt{2}}{L_1}\right)^{3/7} \frac{L_2}{\sqrt{2}}, \tag{4.24}$$

in which

$$K = K_\varepsilon(M_0, M_2) = \left(1 - \frac{7}{5\alpha}\frac{\lambda}{M_0^{7/5}}\left(1 + \frac{1}{M_2}\right)^2 \varepsilon^{3/7}\right)^{-1/2}. \tag{4.25}$$

Observe that the constants $\tilde{M}_0$, $\tilde{M}_1$, $\tilde{M}_2$ and $\tilde{M}_3$ depend on ε, M_0 and M_2. We now finish the proof of lemma 3.3 by choosing M_0, M_1, M_2 and M_3 in such a way that for $\varepsilon > 0$ sufficiently small the numbers $\tilde{M}_0$, $\tilde{M}_1$, $\tilde{M}_2$ and $\tilde{M}_3$ defined above satisfy

$$M_0 \le \tilde{M}_0 < \tilde{M}_1 \le M_1; \tag{4.26}$$

$$M_2 \le \tilde{M}_2 < \tilde{M}_3 \le M_3. \tag{4.27}$$

Clearly (4.19), (4.20), (4.26) and (4.27) prove the lemma.

First we note that $K = K_\varepsilon(M_0, M_2)$ is decreasing in M_0 and M_2. Now choose any $\rho > 1$ and let

$$M_0 = \left(\frac{l\sqrt{2}}{\rho L_1}\right)^{10/7}; \tag{4.28}$$

$$M_1 = \left(\frac{l\sqrt{2}}{L_1}\right)^{10/7}; \tag{4.29}$$

$$M_2 = \frac{L_2}{\sqrt{2}}\left(\frac{l\sqrt{2}}{\rho L_1}\right)^{3/7}; \tag{4.30}$$

$$M_3 = \rho\frac{L_2}{\sqrt{2}}\left(\frac{l\sqrt{2}}{L_1}\right)^{3/7}. \tag{4.31}$$

Then $M_0 < M_1$ and $M_2 < M_3$. Moreover, since $\rho > 1$ is fixed there exists $\varepsilon_0 > 0$, which can be computed explicitly now in terms of α, λ and ρ, such that

$$1 \le K_\varepsilon(M_0, M_2) \le \rho \qquad \text{for all} \quad 0 < \varepsilon \le \varepsilon_0. \tag{4.32}$$

Comparing (4.21)-(4.24) and (4.28)-(4.31) we see that (4.32) implies (4.26)-(4.27). This completes the proof of lemma 3.3. ■

Proof of lemma 3.4. Let M_0, M_1, M_2 and M_3 be the constants in lemma 3.3 and let (w_0, γ) and $(\tilde{w}_0, \tilde{\gamma})$ be arbitrary pairs for which (3.20) holds with $\varepsilon > 0$ as small as needed in the proof of lemma 3.3. By w we denote the solution of $(P_{\varepsilon,\mu,w_0})$ where $\mu = e^\gamma/(e^\gamma - 1)$, and by $\tilde{w}$ the solution of $(P_{\varepsilon,\tilde{\mu},\tilde{w}_0})$. We have, simplifying notation again,

$$
\begin{aligned}
|\Phi_\varepsilon(\gamma, w_0, z) - \Phi_\varepsilon(\tilde{\gamma}, \tilde{w}_0, z)| &= |\Phi_\varepsilon - \tilde{\Phi}_\varepsilon| \\
&= \left| \frac{1}{w_0^{7/5}} \int_{w_0(1-z)^{5/7}}^{w_0} e^{2w^{-1}(s)} g(s)\, ds - \frac{1}{\tilde{w}_0^{7/5}} \int_{\tilde{w}_0(1-z)^{5/7}}^{\tilde{w}_0} e^{2\tilde{w}^{-1}(s)} g(s)\, ds \right| \\
&\leq \left| \frac{1}{w_0^{7/5}} - \frac{1}{\tilde{w}_0^{7/5}} \right| \int_{w_0(1-z)^{5/7}}^{w_0} e^{2w^{-1}(s)} g(s)\, ds \\
&\quad + \frac{1}{\tilde{w}_0^{7/5}} \left| \int_{w_0(1-z)^{5/7}}^{w_0} e^{2w^{-1}(s)} g(s)\, ds + \int_{\tilde{w}_0(1-z)^{5/7}}^{w_0} e^{2\tilde{w}^{-1}(s)} g(s)\, ds \right| \\
&= J_1 + J_2.
\end{aligned} \tag{4.33}
$$

First we treat J_1:

$$
\begin{aligned}
J_1 &= \left| \frac{1}{w_0^{7/5}} - \frac{1}{\tilde{w}_0^{7/5}} \right| \int_{w_0(1-z)^{5/7}}^{w_0} e^{2w^{-1}(s)} g(s)\, ds \\
&= \left| \frac{\tilde{w}_0^{7/5} - w_0^{7/5}}{\tilde{w}_0^{7/5}} \right| \Phi_\varepsilon(\gamma, w_0, z) \\
&\leq \frac{7}{5} \max\left(w_0^{2/5}, \tilde{w}_0^{2/5} \right) |w_0 - \tilde{w}_0| \tilde{w}_0^{-7/5} \Phi_\varepsilon \\
&\leq \frac{7}{5} (M_1 \varepsilon^{-5/7})^{2/5} (M_0 \varepsilon^{-5/7})^{-7/5} |w_0 - \tilde{w}_0| C(M_0, M_2) \varepsilon^{3/7} z \\
&= \frac{7}{5} M_0^{-7/5} M_1^{2/5} C(M_0, M_2) \varepsilon^{8/7} |w_0 - \tilde{w}_0| z,
\end{aligned} \tag{4.34}
$$

where $C(M_0, M_2)$ is defined by (4.16), and where we have used (4.15).

Next we treat J_2:

$$
\begin{aligned}
J_2 &= \frac{1}{\tilde{w}_0^{7/5}} \left| \int_{w_0(1-z)^{5/7}}^{w_0} e^{2w^{-1}(s)} g(s)\, ds - \int_{\tilde{w}_0(1-z)^{5/7}}^{\tilde{w}_0} e^{2\tilde{w}^{-1}(s)} g(s)\, ds \right| \\
&\leq \frac{1}{\tilde{w}_0^{7/5}} \int_{(1-z)^{5/7}}^{1} \left| w_0 e^{2w^{-1}(w_0\xi)} g(w_0\xi) - \tilde{w}_0 e^{2\tilde{w}^{-1}(\tilde{w}_0\xi)} g(\tilde{w}_0\xi) \right| d\xi \\
&\leq \frac{1}{\tilde{w}_0^{7/5}} \int_{(1-z)^{5/7}}^{1} |w_0 g(w_0\xi) - \tilde{w}_0 g(\tilde{w}_0\xi)| e^{2w^{-1}(w_0\xi)}\, d\xi \\
&\quad + \frac{1}{\tilde{w}_0^{7/5}} \int_{(1-z)^{5/7}}^{1} \tilde{w}_0 g(\tilde{w}_0\xi) |e^{2w^{-1}(w_0\xi)} - e^{2\tilde{w}^{-1}(\tilde{w}_0\xi)}|\, d\xi \\
&= I_1 + I_2.
\end{aligned} \tag{4.35}
$$

On we go with I_1:

$$I_1 = \frac{1}{\tilde{w}_0^{7/5}} \int_{(1-z)^{5/7}}^{1} e^{2w^{-1}(w_0\xi)} |w_0 g(w_0\xi) - \tilde{w}_0 g(\tilde{w}_0\xi)| \, d\xi$$

$$\leq \frac{\mu^2}{\tilde{w}_0^{7/5}} \int_{(1-z)}^{1} |w_0 g(w_0\xi) - \tilde{w}_0 g(\tilde{w}_0\xi)| \, d\xi. \tag{4.36}$$

Observe that for small ε, (3.20) implies

$$\frac{\mu^2}{\tilde{w}_0^{7/5}} \leq \frac{(1 + (1/\gamma))^2}{\tilde{w}_0^{7/5}} \leq \frac{1}{M_0^{7/5}} \left(1 + \frac{1}{M_2}\right)^2 \varepsilon^{3/7}, \tag{4.37}$$

and set for the moment

$$\phi(w, \xi) = w g(w\xi) \qquad w \geq 0, \ \xi \geq 0. \tag{4.38}$$

Then

$$\frac{\partial \phi}{\partial w} = g(w\xi) + w\xi g'(w\xi) = \frac{d}{ds}(s g(s))|_{s = w\xi} = h(w\xi), \tag{4.39}$$

where h is as in the assumptions. Rewriting we estimate, assuming $w_0 \geq \tilde{w}_0$,

$$\int_{1-z}^{1} |w_0 g(w_0\xi) - \tilde{w}_0 g(\tilde{w}_0\xi)| \, d\xi = \int_{1-z}^{1} \left| \int_{\tilde{w}_0}^{w_0} \frac{\partial \phi}{\partial w} dw \right| d\xi$$

$$= \int_{1-z}^{1} \left| \int_{\tilde{w}_0}^{w_0} h(w\xi) \, dw \right| d\xi \leq \int_{1-z}^{1} \int_{\tilde{w}_0}^{w_0} |h(w\xi)| \, dw \, d\xi. \tag{4.40}$$

For the moment, set

$$\psi(z) = \int_{1-z}^{1} \int_{\tilde{w}_0}^{w_0} |h(w\xi)| \, dw \, d\xi. \tag{4.41}$$

Then

$$\psi(z) \leq \psi(1) = \int_0^1 \int_{\tilde{w}_0}^{w_0} |h(w\xi)| \, dw \, d\xi$$

$$= \int_0^{\varepsilon^{5/7}} \int_{\tilde{w}_0}^{w_0} |h(w\xi)| \, dw \, d\xi + \int_{\varepsilon^{5/7}}^{1} \int_{\tilde{w}_0}^{w_0} |h(w\xi)| \, dw \, d\xi$$

$$\leq \varepsilon^{5/7} \|h\|_\infty |w_0 - \tilde{w}_0| + \int_{\tilde{w}_0}^{w_0} \int_1^{\varepsilon^{-5/7}} |h(w\varepsilon^{5/7}\eta)| \varepsilon^{5/7} \, d\eta \, dw$$

$$\leq \varepsilon^{5/7} \|h\|_\infty |w_0 - \tilde{w}_0| + \varepsilon^{5/7} K \int_{\tilde{w}_0}^{w_0} \int_1^{\varepsilon^{-5/7}} (w\varepsilon^{5/7}\eta)^{-\beta} \, d\eta \, dw$$

$$\leq |w_0 - \tilde{w}_0| \left\{ \varepsilon^{5/7} \|h\|_\infty + \frac{K}{\tilde{w}_0^\beta} \varepsilon^{5(1-\beta)/7} \int_1^\infty \eta^{-\beta} \, d\eta \right\}$$

$$\leq |w_0 - \tilde{w}_0| \left\{ \|h\|_\infty + \frac{K}{M_0^\beta (\beta - 1)} \right\} \varepsilon^{5/7}, \tag{4.42}$$

where $K = \|s^\beta h(s)\|_\infty$.

For $0 \le z \le 1/2$ we have

$$\psi'(z) = \int_{\tilde{w}_0}^{w_0} |h(w(1-z))| \, \mathrm{d}w \le |w_0 - \tilde{w}_0| K w_0^{-\beta}(1-z)^{-\beta}$$

$$\le \frac{K2^\beta}{M_0^\beta} \varepsilon^{5\beta/7} |w_0 - \tilde{w}_0| \le \frac{K2^\beta}{M_0^\beta} \varepsilon^{5/7} |w_0 - \tilde{w}_0|. \tag{4.43}$$

Combining (4.42) and (4.43), it follows that

$$\psi(z) \le C\varepsilon^{5/7} |w_0 - \tilde{w}_0| z \qquad \text{for all } 0 \le z \le 1, \tag{4.44}$$

for some constant C. Hence, using (4.36), (4.37) and (4.40),

$$I_1 \le C\varepsilon^{8/7} |w_0 - \tilde{w}_0| z \tag{4.45}$$

for some (other) constant C.

It remains to estimate L_2,

$$I_2 = \frac{1}{\tilde{w}_0^{7/5}} \int_{(1-z)^{5/7}}^{1} \tilde{w}_0 g(\tilde{w}_0 \xi) |\mathrm{e}^{2w^{-1}(w_0\xi)} - \mathrm{e}^{2\tilde{w}^{-1}(\tilde{w}_0\xi)}| \, \mathrm{d}\xi$$

$$\le \frac{1}{\tilde{w}_0^{7/5}} \int_{\tilde{w}_0(1-z)}^{\tilde{w}_0} g(s) \, \mathrm{d}s \cdot \|\mathrm{e}^{2w^{-1}(w_0\xi)} - \mathrm{e}^{2\tilde{w}^{-1}(\tilde{w}_0\xi)}\|_\infty$$

$$\le \frac{\lambda}{M_0^{7/5}} \varepsilon z \|\mathrm{e}^{2w^{-1}(w_0\xi)} - \mathrm{e}^{2\tilde{w}^{-1}(\tilde{w}_0\xi)}\|_\infty \tag{4.46}$$

where we have used the same estimate as in the proof of lemma 3.2 for (4.3).

So we still have to estimate

$$\|\mathrm{e}^{2w^{-1}(w_0\xi)} - \mathrm{e}^{2\tilde{w}^{-1}(\tilde{w}_0\xi)}\|_\infty. \tag{4.47}$$

Note that so far we have not encountered terms with $|y - \tilde{y}|$. If we set

$$y(\xi) = w^{-1}(w_0\xi), \tag{4.48}$$

and

$$w = w_0\xi \tag{4.49}$$

in (3.11), we obtain

$$y(\xi) = y_0 + \frac{5}{7}\sqrt{\frac{\varepsilon}{2}} w_0^{3/10} \int_0^{1-\xi^{7/5}} \frac{\mathrm{d}z}{(1-z)^{2/7}\sqrt{(5\alpha/7)z - \Phi_\varepsilon(y, w_0, z)}}, \tag{4.50}$$

and a similar expression for $\tilde{y}(\xi) = \tilde{w}^{-1}(\tilde{w}_0\xi)$.

In order to control (4.47) we first estimate

$$|y(\xi) - \tilde{y}(\xi)| \le |y_0 - \tilde{y}_0| + \frac{5}{7}\sqrt{\frac{\varepsilon}{2}} |w_0^{3/10} - \tilde{w}_0^{3/10}| \int_0^1 \frac{\mathrm{d}z}{(1-z)^{2/7}\sqrt{(5\alpha/7)z - \Phi_\varepsilon}}$$

$$+ \frac{5}{7}\sqrt{\frac{\varepsilon}{2}} \tilde{w}_0^{3/10} \int_0^1 \left| \frac{1}{\sqrt{(5\alpha/7)z - \Phi_\varepsilon}} - \frac{1}{\sqrt{(5\alpha/7)z - \tilde{\Phi}_\varepsilon}} \right| \frac{\mathrm{d}z}{(1-z)^{2/7}}$$

$$= |y_0 - \tilde{y}_0| + B_1 + B_2. \tag{4.51}$$

For B_1 we have from (4.18), (4.32), the mean value theorem and (3.20)

$$B_1 \leq \rho L_2 \sqrt{\frac{\varepsilon}{2}} \frac{3}{10} (M_0 \varepsilon^{-5/7})^{-7/10} |w_0 - \tilde{w}_0| = \frac{3\rho L_2}{10 M_0^{7/10} \sqrt{2}} \varepsilon |w_0 - \tilde{w}_0| \tag{4.52}$$

whereas

$B_2 \leq$

$$\frac{5}{7}\sqrt{\frac{\varepsilon}{2}}\,\tilde{w}_0^{3/10} \int_0^1 \frac{|\Phi_\varepsilon - \tilde{\Phi}_\varepsilon|}{\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon}\,\sqrt{(5\alpha/7)\,z - \tilde{\Phi}_\varepsilon}\,\{\sqrt{(5\alpha/7)\,z - \Phi_\varepsilon} + \sqrt{(5\alpha/7)\,z - \tilde{\Phi}_\varepsilon}\}} \frac{dz}{(1 - z)^{2/7}}$$

(using a similar estimate as in (4.17), together with (4.32))

$$\leq \frac{5}{7}\sqrt{\frac{\varepsilon}{2}} (M_1 \varepsilon^{-5/7})^{3/10} \frac{1}{2\rho^3} \int_0^1 \frac{|\Phi_\varepsilon - \tilde{\Phi}_\varepsilon|}{(5\alpha/7\, z)^{3/2} (1 - z)^{2/7}} dz$$

$$= C\varepsilon^{2/7} \int_0^1 \frac{|\Phi_\varepsilon(y, w_0, z) - \Phi_\varepsilon(\tilde{y}, \tilde{w}_0, z)|}{z^{3/2}(1 - z)^{2/7}} dz \tag{4.53}$$

for some constant C.

At this point the reader may get the impression that we are back at where we started off. However, let us collect all the estimates so far for $|\Phi_\varepsilon - \tilde{\Phi}_\varepsilon|$ and insert them in (4.53). From (4.34), (4.45) and (4.46) it follows that

$$|\Phi_\varepsilon(y, w_0, z) - \Phi_\varepsilon(\tilde{y}, \tilde{w}_0, z)| \leq C\{\varepsilon^{8/7}|w_0 - \tilde{w}_0| + \varepsilon\|e^{2y(\xi)} - e^{2\tilde{y}(\xi)}\|_\infty\} z, \tag{4.54}$$

for some constant C. Substitution in (4.53) yields

$$B_2 \leq C\{\varepsilon^{10/7}|w_0 - \tilde{w}_0| + \varepsilon^{9/7}\|e^{2y(\xi)} - e^{2\tilde{y}(\xi)}\|_\infty\} \tag{4.55}$$

for yet another constant C involving

$$\int_0^1 \frac{z}{z^{3/2}(1 - z)^{2/7}} dz \tag{4.56}$$

which is finite. Combining (4.55) with (4.51) and (4.52) we find, absorbing higher powers of ε in the lower ones, and taking the supremum over ξ,

$$\|y(\xi) - \tilde{y}(\xi)\|_\infty \leq |y_0 - \tilde{y}_0| + C\{\varepsilon|w_0 - \tilde{w}_0| + \varepsilon^{9/7}\|e^{2y(\xi)} - e^{2\tilde{y}(\xi)}\|_\infty\}. \tag{4.57}$$

Now observe that because of (3.2)

$$|e^{2y(\xi)} - e^{2\tilde{y}(\xi)}| \leq 2\max(\mu^2, \tilde{\mu}^2)|y(\xi) - \tilde{y}(\xi)|. \tag{4.58}$$

As in (4.37) we have that

$$2\max(\mu^2, \tilde{\mu}^2) \leq 2\left(1 + \frac{1}{M_2}\right)^2 \varepsilon^{-4/7}, \tag{4.59}$$

which together with (4.58) implies that in (4.57) the last term on the right hand-side is of lower order than the left-hand side if $\varepsilon > 0$ is sufficiently small. We obtain

$$\|y(\xi) - \tilde{y}(\xi)\|_\infty \leq C\{|y_0 - \tilde{y}_0| + \varepsilon|w_0 - \tilde{w}_0|\} \tag{4.60}$$

and consequently, as in (4.58) and (4.59),

$$\|e^{2y(\xi)} - e^{2\tilde{y}(\xi)}\|_\infty \leq C\{\varepsilon^{-4/7}|y_0 - \tilde{y}_0| + \varepsilon^{3/7}|w_0 - \tilde{w}_0|\}. \tag{4.61}$$

Finally, we observe that

$$
\begin{aligned}
|y_0 - \tilde{y}_0| &= |\log(e^{\gamma} - 1) - \log(e^{\tilde{\gamma}} - 1) \\
&\le \max\left(\frac{e^{\gamma}}{e^{\gamma} - 1}, \frac{e^{\tilde{\gamma}}}{e^{\tilde{\gamma}} - 1}\right)|\gamma - \tilde{\gamma}| = \max(\mu, \tilde{\mu})|\gamma - \tilde{\gamma}| \\
&\le C\varepsilon^{-2/7}|\gamma - \tilde{\gamma}|, \qquad (4.62)
\end{aligned}
$$

as in (4.59). Substitution of (4.62) in (4.61) yields

$$\|e^{2y(\xi)} - e^{2\tilde{y}(\xi)}\|_{\infty} \le C\{\varepsilon^{-6/7}|\gamma - \tilde{\gamma}| + \varepsilon^{3/7}|w_0 - \tilde{w}_0|\} \qquad (4.63)$$

which inserted in (4.54) completes the proof. ■

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