14 Implicit functions

https://www.youtube.com/playlist?list=PLQgy2W8pIli-7124huziMvr6eTmFVOCuh

In the playlist the inverse function theorem comes first. I made the videos two years after writing this part of the course notes, during the first lock down. In the course notes I had opted to do the implicit function theorem first. While doing another list of video's for Newton's method I became interested in explicit arguments for the modified Newton method used in this chapter again. I found out those arguments come out clearer when doing the inverse function theorem first.

For my first version I had decided that the implicit function theorem as well as its proof are best presented in the context of real valued functions first. Considering

$$x \to f(x;\lambda),$$

in which x is a real variable, and λ is a real parameter, the study of λ-dependent solutions of $f(x;\lambda) = 0$ is of course of fundamental importance in the first applications of calculus. There is really no good reason to postpone the theorems and proofs to the multi-variable setting where the main ideas are easily blurred by matrices and indices.

But now I first solve $f(x) = y$, assuming $f(0) = 0$, f' continuous, and $f'(0)$ invertible, via the scheme

$$g_n(y) = x_n = x_{n-1} + f'(0)^{-1}(y - f(x)), \quad x_0 = 0.$$

For small y the iterates x_n are shown to be convergent in an open interval $(-\rho,\rho)$, where ρ chosen to have

$$|f'(x) - f'(0)| < \eta, \quad \gamma\eta < 1, \quad \gamma = |f'(0)^{-1}| > 0,$$

provided the smallness condition

$$|y| \leq (\frac{1}{\gamma} - \eta)\rho$$

holds.

For each such y the limit $x = g(y)$ of the sequence x_n is the unique solution of $f(x) = y$ in $[-\rho,\rho]$. On this (same) interval the inverses of $f'(x)$ exist, as the geometric series formula for $f'(x)^{-1}$ puts no further restriction on $x = g(y)$. An essential ingredient in the proof is that the approximating functions g_n all satisfy the uniform Lipschitz continuity condition

$$|g_n(y) - g_n(\tilde{y})| \leq \frac{|y-\tilde{y}|}{\frac{1}{\gamma} - \eta},$$

en so does the limit g. Moreover, g is continuously differentiable, and

$$g'(y) = (f'(g(y))^{-1}.$$

Then I copy-paste the proof for $f : X \to Y$, X and Y Banach space, replacing the closed interval $[-\rho, \rho]$ by the closed ball

$$\bar{B}_\rho = \{x \in X : |x| \leq \rho\},$$

in which $|x|$ is the norm of $x \in X$, just as $|x|$ is the norm[1] of $x \in \mathbb{R}$. Spot the differences if you must, but I'm only typing the Banach space version.

When I'm done with inverse functions I'll have a new and fresh look at implicit functions again, in connection to equations of the form $f(x, \lambda) = 0$, with Banach spaces for $f(x, \lambda) = 0$, x and λ.

14.1 The Inverse Function Theorem Method

Let X and Y be normed spaces, let B be an open ball in X centered in $x = 0$, and let $f : B \to Y$ with $f(0) = 0$ be continuously differentiable. Finally suppose that $f'(0)^{-1}$ exists as a linear continuous map from Y back to X. The continuity of $f'(0)^{-1}$ means there exists $\gamma > 0$ such

$$|f'(0)^{-1}k| \leq \gamma\, |k| \tag{14.1}$$

for all $k \in Y$. The smallest such γ is called the norm $|f'(0)^{-1}|$ of $f'(0)^{-1}$.

Consider the Newton like scheme

$$x_n = x_{n-1} + f'(0)^{-1}(y - f(x_{n-1})), \tag{14.2}$$

which we start from $x_0 = 0$, in the hope that by choosing $|y|$ small we can keep the iterates x_n in B. Our main tool will be Theorem 12.4 and the formula

$$\begin{aligned} f(b) - f(a) &= \int_0^1 f'((1-t)a + tb)(b-a)\, dt \qquad (14.3) \\ &= \int_0^1 f'((1-t)a + tb)\, dt\, (b-a). \end{aligned}$$

We choose the first version in which

$$t \to f'((1-t)a + tb)(b-a)$$

is a continuous Y-valued function of t. Thereby we avoid the (Banach) space in which $f'((1-t)a + tb)$ lives.

[1]Absolute value it is called.

14.1.1 Estimate the increments

With Theorem 12.4 we rewrite the increments as

$$x_{n+1} - x_n = x_n - x_{n-1} + f'(0)^{-1}(f(x_{n-1}) - f(x_n)) \tag{14.4}$$

$$= x_n - x_{n-1} + f'(0)^{-1} \int_0^1 f'(tx_{n-1} + (1-t)x_n)\,(x_{n-1} - x_n)\,dt$$

$$= f'(0)^{-1} \left(f'(0)(x_n - x_{n-1}) + \int_0^1 f'(tx_{n-1} + (1-t)x_n)(x_{n-1} - x_n)\,dt \right)$$

$$= f'(0)^{-1} \int_0^1 \left(f'(0)(x_n - x_{n-1}) + f'(tx_{n-1} + (1-t)x_n)(x_{n-1} - x_n)\right)\,dt$$

$$= f'(0)^{-1} \int_0^1 \left(f'(0) - f'(tx_{n-1} + (1-t)x_n)\right)(x_n - x_{n-1})\,dt,$$

and then we estimate

$$|x_{n+1} - x_n| \le \gamma \left| \int_0^1 \left(f'(0) - f'(tx_{n-1} + (1-t)x_n)\right)(x_n - x_{n-1})\,dt \right|$$

$$\le \gamma \int_0^1 \left| \left(f'(0) - f'(tx_{n-1} + (1-t)x_n)\right)(x_n - x_{n-1}) \right|\,dt$$

by first (14.1) and then the triangle inequality for integrals.

Now by assumption $f'(x)$ is a linear continuous map from X to Y, for every $x \in B$. The continuity of $x \to f'(x)$ in $x = 0$ means that for every $\eta > 0$ there exists $\rho > 0$ such that

$$|f'(x)h - f'(0)h| \le \eta\,|h| \tag{14.5}$$

for all $h \in X$ and for all $x \in X$ with $|x| \le \rho$. The equivalent statement

$$|f'(x) - f'(0)| \le \eta$$

requires the norm $|A|$ of continuous linear maps[2] A from X to Y.

Provided

$$|tx_{n-1} + (1-t)x_n| \le \rho \tag{14.6}$$

it follows that

$$|x_{n+1} - x_n| \le \gamma \int_0^1 \eta\,|x_n - x_{n-1}|\,dt = \gamma\eta\,|x_n - x_{n-1}|.$$

[2]Such maps are called bounded, $|A|$ is just the smallest possible Lipschitz constant.

We conclude that the implication

$$\left.\begin{array}{r}|x_{n-1}| \leq \rho \\ |x_n| \leq \rho\end{array}\right\} \quad \Longrightarrow \quad |x_{n+1} - x_n| \leq \gamma\eta\,|x_n - x_{n-1}| \tag{14.7}$$

holds, as the two inequalities on the left give (14.6). Choosing η with $\gamma\eta < 1$ we can thus take the sum of the iterates if all x_n satisfy $|x_n| < \rho$.

14.1.2 Estimating differences of different sequences

Recall that x_n is a y-dependent sequence. Whenever it exists we write

$$x_n = g_n(y)$$

as a function of y, and we need a non-trivial common domain for all g_n. Clearly this domain contains $y = 0$ because then we get the null-sequence for x_n. So $|x_n| = |g_n(y)| = |g_n(y) - g_n(0)|$.

Let's make this a bit more general. If we have another value of y, say $\tilde{y}$, we may write and evaluate

$$\begin{aligned} g_n(\tilde{y}) - g_n(y) = \\ \tilde{x}_n - x_n = \tilde{x}_{n-1} - x_{n-1} + f'(0)^{-1}(f(x_{n-1}) - f(\tilde{x}_{n-1})) \\ +f'(0)^{-1}(\tilde{y} - y), \end{aligned}$$

in which we recognise (14.4) with the obvious replacements. Going through the same motions the conclusion in (14.7) then has to be replaced by

$$|\tilde{x}_n - x_n| \leq \gamma\eta\,|\tilde{x}_{n-1} - x_{n-1}| + \gamma\,|\tilde{y} - y|, \tag{14.8}$$

as long as the iterates are in

$$\bar{B}_\rho = \{x \in X : |x| \leq \rho\}.$$

14.1.3 Recurrence inequalities

Putting

$$d_n = |\tilde{x}_n - x_n|, \quad \delta = |\tilde{y} - y|$$

we have the inequality recurrence

$$d_n \leq \gamma(\eta d_{n-1} + \delta), \tag{14.9}$$

which starts from $d_0 = 0$. It's an easy exercise to show that (14.9) then implies

$$d_n < \bar{d}$$

for all $n \in \mathbb{N}$, $\bar{d}$ being the unique (positive) solution of $\bar{d} = \gamma(\eta\bar{d} + \delta)$. Thus

$$|g_n(\tilde{y}) - g_n(y)| < \frac{\gamma\delta}{1-\gamma\eta} = \frac{\gamma}{1-\gamma\eta}\,|\tilde{y} - y|, \tag{14.10}$$

and in particular

$$|x_n| = |g_n(y)| < \frac{\gamma}{1-\gamma\eta}\,|y| \le \rho. \tag{14.11}$$

The implication

$$|y| \le \frac{1-\gamma\eta}{\gamma}\,\rho = (\frac{1}{\gamma} - \eta)\,\rho \quad \Longrightarrow \; |x_n| < \rho \quad \text{for all} \quad n \in \mathbb{N}$$

now gives us a closed y-ball $\bar{B}_r$ with radius

$$r = (\frac{1}{\gamma} - \eta)\,\rho \tag{14.12}$$

centered in $y = 0$ on which the functions g_n are well defined as maps from $\bar{B}_r$ to the open ball B_ρ. Recall that f is a map from $\bar{B}_\rho$ to $\bar{B}_r$.

14.1.4 Uniform convergence to a Lipschitz continuous function

By (14.10) these functions are all Lipschitz continuous with Lipschitz constant

$$L = \frac{\gamma}{1-\gamma\eta} = \frac{1}{\dfrac{1}{\gamma} - \eta} = \frac{\rho}{r}\,, \tag{14.13}$$

and by (14.7) the increment estimate

$$|g_{n+1}(y) - g_n(y)| \le \gamma\eta\,|g_n(y) - g_{n-1}(y)| \tag{14.14}$$

holds for all $n \in \mathbb{N}$ starting from

$$|g_1(y) - g_0(y)| = |f'(0)^{-1}(y) - 0| = |f'(0)^{-1}(y)| \le \gamma\,|y|.$$

It follows that g_n is a Cauchy sequence in $C(\bar{B}_r, \bar{B}_\rho)$.

The limit g of the sequence g_n therefore exists in $C(\bar{B}_r, \bar{B}_\rho)$ as a Lipschitz continuous function with the same Lipschitz constant L given by (14.13). The convergence is in the maximum norm, which is equivalent to saying that $g_n \to g$ uniformly on $\bar{B}_r$.

14.1.5 Which inverse?

Clearly (14.11) and (14.12) imply

$$|g(y)| \le \frac{\gamma}{1-\gamma\eta}\,|y| = \frac{\rho}{r}\,|y|$$

for all $y \in \bar{B}_r$, so g maps $\bar{B}_r$ to $\bar{B}_\rho$, and B_r to B_ρ.

Writing (14.2) as

$$g_n(y) = g_{n-1}(y) + f'(0)^{-1}(y - f(g_{n-1}(y))),$$

we take the limit to find

$$g(y) = g(y) + f'(0)^{-1}(y - f(g(y))),$$

so

$$f'(0)^{-1}(y - f(g(y))) = 0. \tag{14.15}$$

Observer that $g(y)$ may be on the boundary of B_ρ, but only if y is on the boundary B_r. In all cases it follows that

$$y - f(g(y)) = 0$$

because $f'(0)^{-1}$ is the inverse of the invertible (linear) map $A_0 = f'(0)$.

Finally consider y and $\tilde{y} \neq y$ in $\bar{B}_r$. Then

$$f(g(y)) = y \neq \tilde{y} = f(g(\tilde{y})),$$

so $g(y) \neq g(\tilde{y})$. This shows that g is injective on $\bar{B}_r$. We have constructed an injective Lipschitz continuous function or map

$$f:\ \bar{B}_r \to \bar{B}_\rho$$

with $f(B_r) \subset B_\rho$, and $g \circ f$ is the identity on $\bar{B}_\rho$. The Lipschitz constant $\dfrac{\rho}{r}$ is consistent with the ratio between the radii of the balls.

14.1.6 Existence of the candidate derivative of the inverse map

In view of

$$y = f(x) = f(g(y))$$

for all $y \in B_r$ the derivative of g in y is expected to be $f'(g(y))^{-1}$. We therefore examine the y-dependence of $f'(g(y))^{-1}$ before we show that it actually is the derivative.

Decomposing the map

$$y \to f'(g(y))^{-1}$$

as

$$y \xrightarrow{g} x \xrightarrow{f'} A \to A^{-1} \tag{14.16}$$

we have the Lipschitz continuity of the first step $y \to g(y)$ at our disposal, with Lipschitz constant given L by (14.13).

By assumption the second step $x \to f'(x)$ is continuous, which we used to have the integrals in (14.3) well defined. The continuity of f' in $x = 0$ was used in (14.5), also to define L, via the bound (14.1). This latter bound is equivalent to the statement

$$|A_0^{-1}| \le \gamma$$

for $A_0 = f'(0)$, and we can take $\gamma = |A_0^{-1}|$ if we like.

For the third step, which we will call the inversion map, we write

$$A = f'(g(y)) = f'(x)$$

to examine the obviously desired invertibility of A. For the map

$$A \to A^{-1}$$

to be well defined in this context we only have the assumed invertibility of the continuous linear operator A_0 at our disposal, so we write

$$A = A_0 + (A - A_0) \quad \text{for} \quad X \xrightarrow{A} Y,$$

which simply means for all $h \in X$ that

$$Ah = A_0 h + (A - A_0)h, \quad \text{in which} \quad (A - A_0)h = Ah - A_0 h.$$

Then

$$A_0^{-1}A = A_0^{-1}A_0 + A_0^{-1}(A - A_0) = \underbrace{A_0^{-1}A_0}_{I} - \underbrace{A_0^{-1}(A_0 - A)}_{H},$$

because A_0^{-1} is a left inverse of A_0.

Now if there exists $\theta < 1$ such that

$$|Hh| \le \theta\,|h| \quad \text{for all} \quad h \in X, \tag{14.17}$$

we can define the map K by[3]

$$Kh = h + Hh + HHh + \cdots, \tag{14.18}$$

[3]We write $Kh = \cdots$, and not $Kx = \cdots$.

and clearly

$$|Kh| \le |h| + \theta\,|h| + \theta^2\,|h| + \cdots = \frac{|h|}{1-\theta} \tag{14.19}$$

implies that K is Lipschitz continuous with Lipschitz constant $\frac{1}{1-\theta}$. Moreover

$$HKh = Hh + HHh + HHHh + \cdots = KHh,$$

so

$$K(I-H)h = h = (I-H)Kh \quad \text{for all} \quad h \in X,$$

meaning that $I - H = A_0^{-1}A$ is a bijection between X and X with inverse

$$K = (I-H)^{-1}.$$

So

$$KA_0^{-1}A = I = A_0^{-1}AK. \tag{14.20}$$

The first equality in (14.20) says that KA_0^{-1} is a left inverse of A. The second equality in (14.20)implies

$$A_0 = A_0A_0^{-1}AK = AK, \quad \text{whence} \quad I = A_0A_0^{-1} = AKA_0^{-1},$$

because A_0^{-1} is a right inverse of A_0. This says that KA_0^{-1} is also a right inverse of A. Thus

$$A^{-1} = KA_0^{-1}, \tag{14.21}$$

provided that for some $\theta < 1$ we have

$$|A_0^{-1}(A_0 - A)h| = |Hh| \le \theta\,|h| \quad \text{for all} \quad h \in X, \tag{14.22}$$

which holds if

$$|(A_0 - A)h| \le \frac{\theta}{\gamma} \quad \text{for all} \quad h \in X.$$

Now recall that $A_0 = f'(0)$ and $A = f'(x)$, and that in (14.5)

$$|f'(x)h - f'(0)h| \le \eta\,|h|$$

was introduced to later obtain (14.7), in which we needed $\gamma\eta < 1$. With the same assumption we now take $\theta = \gamma\eta$ in (14.22) to conclude that

$$A^{-1} = KA_0^{-1}$$

exists for all $y \in B_r$ because $x = g(y)$ is in B_ρ. Recall ρ was chosen at the very beginning to have (14.5), and that the Lipschitz constant

$$L = \frac{\gamma}{1-\gamma\eta} = \frac{1}{\dfrac{1}{\gamma} - \eta} = \frac{\rho}{r}\,,$$

for g in (14.13) gets larger with larger ρ, but so does the radius r on which we find the inverse function g.

14.1.7 Explicit local existence and continuity of the inversion map

Now note that

$$|A - A_0| \leq \frac{\theta}{\gamma}$$

sufficed to conclude that

$$A^{-1} = KA_0^{-1}$$

in (14.20) exists with

$$|A^{-1}| \leq \frac{\gamma}{1-\gamma\eta} = \frac{\gamma}{1-\theta},$$

and moreover that

$$A^{-1} - A^{-1} = KA_0^{-1} - A_0^{-1} = (K - I)A_0^{-1},$$

so

$$|A^{-1} - A_0^{-1}| \leq |K - I|\,|A_0^{-1}| \leq \frac{\gamma^2\eta}{1-\gamma\eta} = \frac{\gamma\theta}{1-\theta}.$$

Taking $\gamma = |A_0^{-1}|$ the implication

$$|A - A_0| \leq \frac{\theta}{|A_0^{-1}|} \quad \Longrightarrow \quad \begin{aligned} |A^{-1}| &\leq \frac{|A_0^{-1}|}{1-\theta} \\ |A^{-1} - A_0^{-1}| &\leq \frac{\theta|A_0^{-1}|}{1-\theta} \end{aligned} \tag{14.23}$$

now holds whenever it makes sense[4] and implicitly says that then the inverse of A is well defined.

Now let $\varepsilon > 0$ and choose θ as in

$$\varepsilon = \frac{\theta|A_0^{-1}|}{1-\theta} \iff \theta = \frac{\varepsilon}{\varepsilon + |A_0^{-1}|}$$

to define

$$\delta = \frac{\theta}{|A_0^{-1}|} = \frac{\varepsilon}{|A_0^{-1}|}\frac{1}{\varepsilon + |A_0^{-1}|} \quad \text{and} \quad M = \frac{|A_0^{-1}|}{1-\theta} = \varepsilon + |A_0^{-1}|.$$

Then

$$|A - A_0| \leq \frac{\varepsilon}{|A_0^{-1}|}\frac{1}{\varepsilon + |A_0^{-1}|} \quad \Longrightarrow \quad \begin{aligned} |A^{-1}| &\leq \varepsilon + |A_0^{-1}| \\ |A^{-1} - A_0^{-1}| &\leq \varepsilon \end{aligned}$$

is an explicit existence, continuity and boundedness statement for the map

$$A \to A^{-1},$$

[4] Meaning that $\theta < 1$.

but the estimate for A^{-1} is equivalent with the obvious estimate

$$|A^{-1}| \leq |A^{-1} - A_0^{-1}| + |A_0^{-1}| \leq \varepsilon + |A_0^{-1}|.$$

So we record

$$|A - A_0| \leq \frac{\varepsilon}{|A_0^{-1}|} \frac{1}{\varepsilon + |A_0^{-1}|} \qquad \Longrightarrow \qquad |A^{-1} - A_0^{-1}| \leq \varepsilon \tag{14.24}$$

as the explicit existence and continuity statement to remember. So

$$\frac{\varepsilon}{|A_0^{-1}|} \frac{1}{\varepsilon + |A_0^{-1}|}$$

is the explicit δ for ϵ in the existence and continuity statement with non-strict inequalities for the inversion map $A \to A^{-1}$ in every A_0 for which A_0^{-1} exists.

14.1.8 Continuity of the candidate derivative of the inverse map

We return to the chain

$$y \xrightarrow{g} x \xrightarrow{f'} A \to A^{-1} = (f'(g(y))^{-1}$$

in (14.16) and recall that

$$f(0) = 0, \quad g(0) = 0, \quad A_0 = f'(0) = f'(g(0)).$$

Now let $\varepsilon > 0$. Then

$$|f'(g(y)) - f'(0)| = |A - A_0| \leq \frac{\varepsilon}{|A_0^{-1}|} \frac{1}{\varepsilon + |A_0^{-1}|} = \frac{\varepsilon}{|f'(0)^{-1}|} \frac{1}{\varepsilon + |f'(0)^{-1}|}$$

will suffice to have existence of $A^{-1} = (f'(g(y))^{-1}$ with

$$|f'(g(y))^{-1} - f'(0)^{-1}| = |A^{-1} - A_0^{-1}| \leq \varepsilon,$$

so let

$$\eta = \frac{\varepsilon}{|f'(0)^{-1}|} \frac{1}{\varepsilon + |f'(0)^{-1}|} = \tilde{\varepsilon}$$

and assume for convenience that η is smaller than the radius of the original open ball B all this started with in Section 14.1. Let $\rho > 0$ be such that (14.5) holds. The Lipschitz constant for g in (14.13) becomes

$$L = \frac{1}{\dfrac{1}{\gamma} - \eta} = \gamma + \varepsilon = |A_0^{-1}| + \varepsilon = |f'(0)^{-1}| + \varepsilon,$$

so with the restriction

$$|y| \leq \frac{\rho}{|f'(0)^{-1}| + \varepsilon}$$

on y we have existence of $f'(g(y))^{-1}$ and

$$|f'(g(y))^{-1} - f'(0)^{-1}| \leq \varepsilon.$$

Thus

$$\frac{\rho}{|f'(0)^{-1}| + \varepsilon}$$

is the non-explicit δ for ε in the $\varepsilon - \delta$ statement for existence near and continuity in $y = 0$ of

$$y \to f'(g(y))^{-1}.$$

It is non-explicit because it contains ρ which is the $\tilde{\delta}$ for $\tilde{\varepsilon}$ in the $\tilde{\varepsilon} - \tilde{\delta}$ statement for continuity of $x \to f'(x)$ in $x = 0$. If we assume that f' is Lipschitz continuous we can make it explicit of course.

14.1.9 Differentiability of the inverse map

Consider $y = 0$ first, and use Theorem 12.4 to write

$$y = f(g(y)) = f(g(y)) - f(0) = f(g(y)) - f(g(0)) = \int_0^1 f'(tg(y))g(y)\, dt$$

$$f'(0)g(y) + \int_0^1 (f'(tg(y)) - f'(0))g(y)\, dt,$$

whence

$$f'(0)g(y) = y - \int_0^1 (f'(tg(y)) - f'(0))g(y)\, dt,$$

so

$$g(y) = f'(0)^{-1}y - f'(0)^{-1} \int_0^1 (f'(tg(y)) - f'(0))g(y)\, dt. \tag{14.25}$$

This is not a formula for $g(y)$ because the 'remainder' term

$$R(y) = f'(0)^{-1} \int_0^1 (f'(tg(y)) - f'(0))g(y)\, dt.$$

depends on $g(y)$. But by now we know so much of $g(y)$ that it should be a piece of cake to show that it has the right property.

Indeed, let $\varepsilon > 0$. Then we have

$$|R(y) \leq |\gamma| \Big| \int_0^1 (f'(tg(y)) - f'(0))g(y)\, dt \Big|$$

$$\leq \gamma \int_0^1 \big|(f'(tg(y)) - f'(0))g(y)\big|\, dt$$

$$\leq \gamma \int_0^1 |(f'(tg(y)) - f'(0))|\, |g(y)|\, dt$$

$$\leq \gamma L \int_0^1 |(f'(tg(y)) - f'(0))|\, dt\, |y| < \varepsilon\, |y|$$

if we choose $\delta > 0$ such that

$$|f'(tg(y)) - f'(0)| \leq \frac{\varepsilon}{\gamma L}$$

if $|tg(y)| < \delta L$. Then

$$0 < |y| < \delta \qquad \Longrightarrow \qquad |R(y)| < \varepsilon\, |y|,$$

which completes the proof for $y = 0$.

The proof for the other y in B_r is similar, but needs the previous section to be re-done starting from $x_0 \in X$ and $y_0 \in Y$ with $y_0 = f(x_0)$. To be done, but not a big deal.