Consider the problem

$$(P)\begin{cases}\Delta u + u^p = 0 & \text{in } U,\\ u = 0 & \text{on } \partial U,\end{cases}$$

where

$$U = B_R = \{x \in \mathbf{R}^n : \quad x_1^2 + ... + x_n^2 < R^2\}.$$

Theorem (Gidas-Ni-Nirenberg). *Let U be a ball, and let u be a classical solution of (P), i.e. a solution which has continuous second order derivatives in U and which is itself continuous on the closure of U. Then u is radially symmetric.*

[GNN] B. Gidas, W.M. Ni & L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. **68**, 1979, 209-243.

In view of this theorem, (P) is equivalent to

$$(P_r)\begin{cases} u''(r) + \dfrac{n-1}{r}u'(r) + u(r)^p = 0, \quad 0 \le r < R, & (1.8)\\ u(r) > 0, \quad 0 \le r < R, & (1.9)\\ u'(0) = 0, & (1.10)\\ u(R) = 0, & (1.11)\end{cases}$$

which can be solved using the techniques from the theory of ordinary differential equations. This is done in the following exercise. Note that (1.8) allows a simultaneous scaling of r and u which leaves the equation invariant.

Exercise 1.4
(i) Show that the transformation

$$X = \frac{ru'(r)}{u(r)}; \quad Y = r^2 u(r)^{p-1}; \quad t = \log r \tag{1.12}$$

transforms (1.8) into the quadratic system

$$(Q)\begin{cases} X' = \dfrac{dX}{dt} = X(2-n-X) - Y, & (1.14)\\ Y' = \dfrac{dY}{dt} = Y(2+(p-1)X). & (1.15)\end{cases}$$

(ii) Show that solutions of (1.8) satisfying (1.9) and (1.10) correspond to orbits of (Q) in the upper half plane coming out of $(0,0)$.
(iii) Show that there is only one such orbit γ.
(iv) Show that for $p = \frac{n+2}{n-2}$ all orbits of (Q) are symmetric in the line $X = \frac{2-n}{2}$.
(v) Show that γ escapes to infinity in finite time through the second quadrant along the negative X-axis, if and only if $p < \frac{n+2}{n-2}$ ($p < \infty$ if $n = 2$).
(vi) Show that (P_r) has a solution if and only if $p < \frac{n+2}{n-2}$ ($p < \infty$ if $n = 2$).
(vii) Show that this solution is unique.
(viii) Show that (P) has a radial solution for every annulus if $n > 2$ and $p = \frac{n+2}{n-2}$.

The result to bear in mind here is:

Theorem 1.5. *Let U be a ball and $p > 1$. Then (P) has a (classical) solution if and only if $p < \frac{n+2}{n-2}$ ($p < \infty$ if $n = 2$). This solution is unique.*