Homework 1, Percolation 2014 (*four* exercises).

Please give crisp and clear computations/argumentations. The solutions should be given to René Conijn (as hard-copy, or as pdf file by e-mail) before 10 a.m. Monday, October 13. His e-mail address is r.p.conijn@vu.nl

1. Let $d \geq 2$, and let $B(n)$ denote the box $[-n, n]^d$ in the d-dimensional (hyper-) cubic lattice. Suppose the edges of this box are, independently of each other, open with probability p and closed with probability $1 - p$. We interpret this box as a porous stone. Water can only flow through open edges. We put the stone in water, and define f as the fraction of vertices in the stone that will be wetted. Show that

$$E_p(f) \to \theta(p), \quad \text{as } n \to \infty,$$

where $E_p(f)$ denotes the expectation of f, and $\theta(p)$ is the probability that (for bond percolation on the d-dimensional lattice), the open cluster of the origin is infinite.
(Hint: Partition the vertices of the box in two classes: those at small distance from the boundary of the box, and those for which the distance is not small. Here 'small' has to be suitably defined).

2. Let $X_1, \cdots, X_n$ be i.i.d. $0-1$ valued random variables, each being 1 with probability p. Let A be the event that $\sum_{i=1}^n X_i \geq n - 1$. Clearly, this has probability

$$g(p) := p^n + n(1-p)p^{n-1}.$$

a) Compute, for $1 \leq i \leq n$ the probability that the index i is pivotal for the event A.

b) Use (a) to obtain the expected number of pivotal indices for A, and check that (in accordance with Russo's formula) it agrees with the derivative of g.

3. Consider bond percolation on the square lattice with parameter $p = 1/2$. Modify the proof of Theorem 4.2 in the lecture notes to show that there is a $c > 0$ such that, for all $1 \leq m \leq n$,

$$P_{\frac{1}{2}}(B(m) \leftrightarrow \partial B(n)) \geq \frac{c}{\sqrt{n/m}}).$$

4. Show, for bond percolation on the square lattice, that

$$P_{1/2}\{O \leftrightarrow (1,0)\} = 3/4.$$

Hint: First consider the event that there is an open path from O to $(1,0)$ which does not contain the edge between O and $(1,0)$. (Use duality).