Lecture notes percolation (Draft, October 3, 2014)

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In these lecture notes I discuss some 'classical' percolation results, with emphasis on bond percolation on the square lattice, which are treated in the first part of the course *Percolation*. The main result in these notes (which will be extended/updated from week to week) is that the critical probability for the above mentioned model is equal to $1/2$.

The illustrations referred to in the text are provided in a separate file.

1 Some basic tools

This section concerns two very general results for independent $0-1$ valued random variables. Our state space is the set $\Omega := \{0,1\}^n$. (You could e.g. interpret this as the possible outcomes of n coinflips). Elements of Ω are typically denoted by $\omega = (\omega_1, \cdots, \omega_n)$ and called configurations. If $\omega_i = 1$ we say, somewhat informally, that the ith component of ω is (or has value) 1, or that there is a 1 at the ith position. We will often be somewhat informal in our way of writing events. For instead, instead of writing the event that the ith component has value 1 as $\{\omega \in \Omega : \omega_i = 1\}$ we usually write simply $\{\omega_i = 1\}$.

1.1 Increasing events, the notion of pivotality, and Russo's formula

Let ω be a configuration and let A be an event (i.e. a subset of Ω). We say that an index $1 \leq i \leq n$ is *pivotal* (in the configuration ω for the event A) if exactly one of the configurations ω and $\omega^{(i)}$ is in A. Here $\omega^{(i)}$ is the configuration obtained from ω by flipping the ith component of ω. (That is, $\omega_j^{(i)}$ is equal to ω_j if $j \neq i$ and equal to $1 - \omega_j$ if $j = i$).

We denote by A_i the event that i is pivotal:

$$A_i := \{\omega \in \Omega \,:\, i \text{ is pivotal in } \omega \text{ for } A\}.$$

Example 1 Consider the event that all n outcomes are 1. So we take

$$A := \{\omega \in \Omega \,:\, \omega_i = 1,\ 1 \le i \le n\}.$$

Then i is pivotal in ω if and only if $\omega_j = 1$ for all $j \neq i$. So in this example A_i is the event that all positions $\neq i$ have value 1:

$$A_i = \{\omega \,:\, \omega_j = 1 \text{ for all } j \neq i\}.$$

Note that in the above Example the pivotality of i is a property that depends only on the values at the positions $\neq i$. It is easy to see from the definition that this is always the case:

Observation 1(a): Let i be an index, ω a configuration and A and event. If i is pivotal for A in ω, then i is also pivotal for A in $\omega^{(i)}$.

A certain (large) class of events is of special importance: We say that an event A is increasing if (informally) it has a preference for 1's. Formally, A is increasing if $\omega \in A$ and $\omega' \geq \omega$ implies $\omega' \in A$. Here $\omega' \geq \omega$ means that $\omega'_i \geq \omega_i$ for all indices i.

Observation 2 Let A be an increasing event, and i an index. Then the event that A holds and i is pivotal and the event that i is pivotal and has value 1 are the same. More precisely,

$$A \cap A_i = \{\omega_i = 1\} \cap A_i.$$

Exercise: Prove this.

Now we introduce randomness. Let P_p denote the product measure with parameter p on Ω. So

$$P_p(\omega) = p^{|\{i:\omega_i = 1\}|}\,(1-p)^{|\{i:\omega_i=0\}|},\ \ \omega \in \Omega.$$

Here the notation $|V|$ is used for the number of elements in the set V.

The following simple relation (called Russo's formula, or Margulis-Russo formula) between the derivative of an increasing event and its (expected number of) pivotal indices turns out to be extremely useful:

Lemma 1.1. *Let A be an increasing events. Then*

$$\frac{d}{dp} P_p(A) = \sum_{i=1}^{n} P_p(A_i). \tag{1}$$

Note that the r.h.s. of (1) is the expectation of the number of indices that are pivotal for A.

Example 2
Let A be as in Example 1. Clearly, $P_p(A) = p^n$ and so $d/dpP_p(A) = np^{n-1}$. Let us check this with the outcome of Russo's formula: We have (see Example 1) that the probability that i is pivotal is p^{n-1}. So $\sum_{i=1}^n P_p(A_i) = np^{n-1}$ which indeed is in accordance with the above.

Exercise
Let A be the event that ω_1 and ω_2 are both 1 or ω_3 is 1. Give $P_p(A)$, $P_p(A_1)$, $P_p(A_2)$ and $P_p(A_3)$ and use this to check Russo's formula for this special case.

Proof. of Lemma 1.1: As is often the case, it is more convenient to prove something more general. In the above setup all indices had the same parameter p. During this proof we consider the more general case where each index i has a parameter p_i which may differ from the other parameters. Let the corresponding product measure on Ω be denoted by $P_{(p_1,\cdots,p_n)}$. More precisely,

$$P_{(p_1,\cdots,p_n)}(\omega) = \prod_{1\le i\le n:\omega_i=1} p_i \times \prod_{1\le i\le n:\omega_i=0} (1-p_i).$$

We claim that (with A as in the stament of the Lemma), for each index i:

$$\frac{\partial}{\partial p_i} P_{(p_1,\cdots,p_n)}(A) = P_{(p_1,\cdots,p_n)}(A_i). \tag{2}$$

It is easy to see (check this yourself) that this claim implies the Lemma; so we only have to prove the claim. First we write the obvious equality

$$P_{(p_1,\cdots,p_n)}(A) = P_{(p_1,\cdots,p_n)}(A \setminus A_i) + P_{(p_1,\cdots,p_n)}(A \cap A_i). \tag{3}$$

By observation 1(b) the event in the first term on the r.h.s. is completely determined by the the values ω_j, $j \neq i$. Hence its probability is a function of the p_j's, $j \neq i$. So the partial derivative with respect to p_i of the first term is 0.

Now we handle the second term in (3): By Observation 2 the event in that term is equal to the intersection of the event that that $\omega_i = 1$ and the event that i is pivotal for A. However, by Observation 1(a) these two events are independent, so we have that the second term in (3) is equal to $p_i \times P_{(p_1,\cdots,p_n)}(A_i)$. Taking the partial derivative of this expression w.r.t.

p_i, and using that (again by Observation 1(a)) the second factor in this expression is a function of the p_j's, $j \neq i$, gives (2) and hence completes the proof of the lemma. $\square$

The probability that i is pivotal is often called the *influence* of i. The above Lemma says that, for an increasing event, its derivative with respect to the parameter p is equal to the sum of the influences.

1.2 Positive correlation of increasing events

As before, P_p denotes the product distribution with parameter p.

Theorem 1.2. *Let $A, B \subset \Omega$ be increasing events.*

$$P_p(A \cap B) \geq P_p(A)P_p(B). \tag{4}$$

This result goed back to Harris (1960). Later it was extended to a larger class of probability distributions by Fortuin, Kasteleyn and Ginibre (whence the name FKG inequality).

There is a short induction proof of Theorem 1.2. Here we give a sketch of a different proof, which is longer but has the advantage of being close to intuition.

Proof. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ be independent random variables, each taking value 1 with probability p and value 0 with probability $1-p$. It is clear that the l.h.s. of (4) is equal to

$$P((X_1, \cdots, X_n) \in A,\ (X_1, \cdots, X_n) \in B), \tag{5}$$

and that the r.h.s. of (4) is equal to

$$P((X_1, \cdots, X_n) \in A,\ (Y_1, \cdots, Y_n) \in B). \tag{6}$$

The idea of the proof is to change the event in (5) step by step into the event in (6), in such a way that at each step the probability of the event decreases. (By 'decreases' we will always mean 'strictly decreases or remains the same'). It is clear that if we can do that, the probability of (5) is indeed larger than or equal to that of (6).

The first step is the following:

Claim 1 The probability in (5) is $\geq$

$$P((X_1, \cdots, X_n) \in A,\ (Y_1, X_2, \cdots, X_n) \in B) \tag{7}$$

Proof of Claim 1.

Let $a_2, \cdots, a_n \in \{0,1\}$. We will condition on the event that, for all $i = 2, \cdots, n$, $X_i = a_i$. In particular we will show that, for each choice of $a_2, \cdots a_n$, the conditional probability of the event in (5) is larger than the conditional probability of the event in (7) (which clearly completes the proof of the Claim). To do this we have to distinguish some cases:

Case (i): $(0, a_2, \cdots, a_n) \in A \cap B$. In this case it is clear (recall that A and B are increasing events) that both conditional probabilities are equal to 1.

Case (ii): $(0, a_2, \cdots, a_n) \notin A$, $(1, a_2, \cdots, a_n) \in A$, $(0, a_1, \cdots, a_n) \in B$. In this case both conditional probabilities are equal to the probability that $X_1 = 1$, which is p.

Case (ii'): Same as (ii) but with A and B exchanged. Now the first conditional probability is equal to $P(X_1 = 1)$ and the second to $P(Y_1 = 1)$ which (again) are both p.

Case (iii): $(0, a_2, \cdots, a_n) \notin A$, $(1, a_2, \cdots, a_n) \in A$, $(0, a_2, \cdots, a_n) \notin B$, $(1, a_2, \cdots, a_n) \in B$. Now the first conditional probability is equal to $P(X_1 = 1)$ (which is p) and the second is equal to $P(X_1 = 1, Y_1 = 1)$, which is $p^2 < p$.

Finally we have to consider the case where $(1, a_2, \cdots, a_n) \notin A \cap B$. It is clear that then both conditional probabilities are 0.

This proves Claim 1 and completes the first step. The second step is the following:

Claim 2 The probability in (7) is $\geq$

$$P((X_1, \cdots, X_n) \in A, \, (Y_1, Y_2, X_3, \cdots, X_n) \in B) \tag{8}$$

Proof of Claim 2.

This is essentiall the same as that of Claim 1: Now let $a_1, a_3, \cdots, a_n \in \{0,1\}$ and $b_1 \in \{0,1\}$. We will condition on the event that $Y_1 = b_1$ and that, for all $i \neq 2$, $X_i = a_i$. We show that, for each choice of b_1 and the a_i's, the conditional probability of the event in (7) is at least that of the event in (8). Again one has to consider similar cases as in the proof of Claim 1, and each case is handled practically the same as before. We only show one case here, namely the one corresponding with case (iii) in the proof of Claim 1:

$(a_1, 0, a_3, \cdots, a_n) \notin A$, $(a_1, 1, a_3, \cdots, a_n) \in A$, $(b_1, 0, a_3, \cdots, a_n) \notin B$, $(b_1, 1, a_3, \cdots, a_n) \in B$. Again, (as in case (iii) of Claim 1, and for essentially the same reason), the first conditional probability is p and the second is p^2. The other cases also correspond with similar cases as in the proof of Claim 1, and are handled in practically the same way. This completes the proof of Claim 2.

More generally, the kth step, $1 \leq k \leq n$, is described and handled in practically the same way.

This completes the (sketch of the) proof of Theorem 1.2.

□

2 The critical probability for bond percolation on the square lattice is at least $1/2$

The main result of this section was first proved by Harris around 1960. In the late seventies a new proof, based on box-crossing inequalities, was obtained by Russo (1978) and by Seymour and Welsh (1978). These box-crossing results turn out to be very important for many other purposes. Below we give a recent (slightly more elegant, in some sense weaker, but strong enough for our current purpose) version of the RSW box-crossing results, following Bollobás and Riordan (2006).

2.1 Box crossing inequalities

The key issue in this subsection is to give lower bounds for the probability of crossing certain rectangles in terms of probabilities of crossing other (easier to cross) rectangles.
First a trivial (and very general: it holds for every probability space) inequality and some notation.

Lemma 2.1. *Let A_1 and A_2 be two events that have the same probability. Then*

$$P(A_1) \geq \frac{P(A_1 \cup A_2)}{2}.$$

Remark: Later we will introduce a sharper, less trivial, version, but for our present purpose Lemma 2.1 will do.

Back to percolation: For any rectangle R we denote by $H(R)$ the event that there is a horizontal open crossing of R (that is, an open path that lies inside R and crosses R from left to right). Similarly, $V(R)$ deontes the event that there is an open vertical crossing of R. Let $h_p(n, m)$ be the probability that there is an open horizontal crossing of a given $n \times m$ box:

$$h_p(n, m) = P_p\left(H([0, n] \times [0, m])\right).$$

The trivial Lemma 2.1 above, together with other simple observations, has the following consequence:

Lemma 2.2. *Let R be the square $[0, 2n]^2$, and S the square $[0, n]^2$ (see Figure 1). Let P_1 be a (deterministic) top-down crossing of S and let P_1' be its image under reflection in the line $y = n$. Note that the 'concatenation' of P_1 and P_1' forms a top-down crossing of R, and divides R in three parts: the top-down-crossing just mentioned, the part $R_r(P_1)$ to the right of it, and the part $R_l(P_1)$ to the left of it. Let $A(P_1)$ be the event that there is an open path in $R_r(P_1)$ from the right-hand side of R to P_1. We have*

$$P_p(A(P_1) \geq \frac{P_p(H(R))}{2}. \tag{9}$$

Proof. Let $A(P_1')$ be the event that there is an open path in R from its right-hand side to P_1'. It is easy to see that $H(R) \subset A(P_1) \cup A(P_1')$. Hence, using Lemma 2.1,

$$P_p(A(P_1)) \geq P_p(H(R))/2.$$

□

It is not difficult to give a lower bound for horizontally crossing a $3n$ by n rectangle in terms of the probabiliities of similar events for n by n squares and $2n$ by n rectangles:

$$h_p(3n, n) \geq h_p(2n, n)^2 \, h_p(n, n).$$

For convenience we will mainly work with rectangles with even width, and often write the above result as

Lemma 2.3.

$$h_p(6n, 2n) \geq h_p(4n, 2n)^2 \, h_p(2n, 2n).$$

Proof. See Figure 2 and use FKG. □

In practically the same way one can prove that

Lemma 2.4.

$$h_p(4n, 2n) \geq h_p(3n, 2n)^2 \, h_p(2n, 2n).$$

But, can we give lower bounds (of, say, $h_p(3n, 2n)$) in terms of crossing probabilities of squares only? That appears to be more tricky! Most of the work is in the following intermediate result:

Lemma 2.5. *Let R and S be as in Lemma 2.2 (see also Fig. 1). Let $X(R)$ be the event that there is an open vertical crossing P_1 of S and an open path P_2 in R from the right-hand side of R to P_1. Then*

$$P_p(X(R)) \geq P_p(H(R))P_p(V(S))/2.$$

Proof. Let π be a (deterministic) vertical crossing of S. Let $E(\pi)$ be the event that π is the left-most open vertical crossing of S. It is clear that (with the notation of Lemma 2.2 and its proof)

$$E(\pi) \cap A(\pi) \subset X(R).$$

Moreover, the event $E(\pi)$ depends only on the edges in the path π and in $R_l(\pi)$, while the event $A(\pi)$ depends only on the edges in $R_r(\pi)$. Hence these events are independent. Finally, if π_1 and π_2 are different vertical crossings of S, the events $E(\pi_1)$ and $E(\pi_2)$ are disjoint. Hence,

$$P_p(X(R)) \geq \sum_{\pi} P_p(E(\pi))P_p(A(\pi)) \geq \frac{P_p(H(R))}{2} \sum_{\pi} P_p(E(\pi)), \tag{10}$$

where we sum over all vertical crossings π of S, and where the second inequality follows from Lemma 2.2. Lemma 2.5 now follows from the fact that the last summation in the r.h.s. of (10) equals $P_p(V(S))$. □

From Lemma 2.5 we can easily obtain a lower bound for $h_p(3n, 2n)$ of the form announced above Lemma 2.5:

Lemma 2.6.

$$h_p(3n, 2n) \geq h_p(2n, 2n)^2 h_p(n, n)^3/4.$$

Proof. Consider the squares $R = [0, 2n]^2$ and $R' = [-n, n] \times [0, 2n]$, and the square $S = [0, n]^2$ in their intersection. (See Fig. 3). In Fig. 3 the events $X(R)$, and its reflected analog for R' hold. If, in addition, $H(S)$ holds, then we have an open horizontal crossing of $R \cup R'$ (which is a $3n$ by $2n$ rectangle). Hence, using FKG,

$$h_p(3n, 2n) \geq P_p(X(R))^2\, P_p(H(S)).$$

The desired result now follow from Lemma 2.5, and by noting that $P_p(V(S)) = P_p(H(S)) = h_p(n, n)$ and $P_p(H(R)) = h_p(2n, 2n)$. □

Combining this lemma with Lemma 2.4 and Lemma 2.3 we can also give a lower bound for $h_p(6n, 2n)$ in terms of $h_p(n, n)$ and $h_p(2n, 2n)$.

Now consider a rectangle $R = [0, n+1] \times [0, n]$ and its dual rectangle R^d (see Fig. 4). Note that R and R' are iosomorphic. There is either a horizontal open crossing of R or a vertical closed (dual) crossing of R'. From these symmetry properties we get that $h_{1/2}(n, n+1)$ is exactly $1/2$. This, together with the above Lemma's gives:

Proposition 2.7. *There is a $\varepsilon > 0$ such that $h_{1/2}(6n, 2n) > \varepsilon$ for all $n \geq 1$.*

We are now quite close to showing that $\theta(1/2) = 0$ (and hence that $p_c \geq 1/2$). First the follolwing consequence of Proposition 2.7. Recall that $B(n)$ denotes the set of vertices $\{(x, y) : |x|, |y| \leq n\}$. Now let, for integers $0 \leq n \leq m$, $A(n, m)$ denote the set of vertices $\{(x, y) \in \mathbb{Z}^2 : n \leq |x| \leq m,\ n \leq |y| \leq m\}$. A set of this form is called an *annulus.* We say, somewhat informally, that '$A(n, m)$ has an open circuit' if there is an open circuit $\mathcal{C}$ in $A(n, m)$ such that $B(n)$ is contained in the union of C and its interior.

Corollary 2.8.

$$\inf_n P_{1/2}\left(A(n, 3n) \text{ has an open circuit }\right) > 0.$$

Proof. Note that $A(n, 3n)$ is the union of four $2n$ by $6n$ rectangles as indicated in Figure 5. If each of these rectangles has an open crossing 'in the long direction', then the annulus has an open circuit. By the FKG inequality and Proposition 2.7 this has probability larger than ε^4, with ε as in the Proposition. □

2.2 $p_c \geq 1/2$ and other consequences of Corollary 2.8

From Corollary 2.8 we easily get our main result of this section, the following theorem.

Theorem 2.9. *For bond percolation on the square lattice,*

$$\theta(1/2) = 0 \text{ and hence } p_c \geq \frac{1}{2}.$$

Proof. There are infinitely many disjoint annuli of the form in Corollary 2.8. For instance, to be specific, take the annuli $A(3^k, 3^{k+1})$, $k \geq 1$ and even. Now consider perolation with parameter $1/2$. According to the corollary there is an $\alpha > 0$ such that each of the above mentioned annuli has probability larger than α to have an open circuit. Hence (since for disjoint annuli these events are independent), with probability 1 there is at least one (and, in fact infinitely many) of these annuli that have an open circuit. However, by symmetry ($p = 1/2$ and duality) an analogous result holds for closed circuits in the dual lattice. We conclude that with probability 1 there is a closed circuit in the dual lattice that has the vertex 0 in its interior. But if there is such a circuit, 0 can not be in an infinite open cluster. Hence $\theta(1/2) = 0$. $\square$

By refining the proof of the Theorem a little, we get the following result which can be interpreted as a bound for the so-called critical one-arm exponent:

Theorem 2.10. *There is a $\delta > 0$ such that for all $n \geq 1$*

$$P_{1/2}(O \leftrightarrow \partial B_n) \leq n^{-\delta}.$$

Proof. Look at the proof of Theorem 2.9. It is easy to see that the number of annuli in the dual lattice of the form $A(3^{k+1}, 3^k) + (1/2, 1/2)$, $k \geq 1$ and even, that are in the interior of $\partial B(n)$, is of order $\log n$. More precisely, there is a $c > 0$ such that for each n the number of such annuli is at least $c \log n$. The probability of the event $\{O \leftrightarrow \partial B(n)\}$ is smaller than or equal to the probability that none of those annuli has a closed circuit, which in turn is at most $(1-\alpha)^{c \log n}$. This can be written as

$$n^{c \log(1-\alpha)},$$

so the desired result holds with $\delta = -c \log(1-\alpha)$. (Note that $\alpha \in (0,1)$ and hence that $\log(1-\alpha) < 0$). $\square$

3 The critical probability for the square lattice is at most $1/2$

The strategy followed here is essentially that in the original proof by Harry Kesten (1980). There are more 'modern' proofs, which put the result in a more general framework (see Russo (1982) and Bollobás and Riordan (2006)) but, in my opinion, the proof described here is the shortest and most natural self-contained proof. First we have to do some preliminary work: We need a so-called *finite-size criterion* for percolation.

3.1 A finite-size criterion

Let $h_p(n,m)$ be the crossing probability defined in Subsection 2.1. We will need a result of the following form:

Theorem 3.1. *Let $p \in [0,1]$.*
If there is an $n > 4$ with $h_p(3n,n) > 25/26$, then $\theta(p) > 0$.

Proof. We need a small modification of the events $H(R)$ and crossing probabilities $h_p(n,m)$ introduced in subsection 2.1: We define, for a rectangle R, $\hat{H}(R)$ as the event that there is an open path in R that starts from the left-side of R, ends on the right side of R and does not visit the upper or the lower side of R. Further, we define, for all positive integers n, m,

$$\hat{h}_p(n,m) = P_p\left(\hat{H}([0,n] \times [0,m])\right).$$

It is easy to see (check this; use Figure 6) that for each positive integer k

$$1 - \hat{h}_p(4k,k) \leq 5(1 - \hat{h}_p(2k,k)). \tag{11}$$

Further note that the events $\hat{H}([0,4k] \times [0,k])$ and $\hat{H}([0,4k] \times [k,2k])$ are independent, and that both of them are contained in $\hat{H}([0,4k] \times [0,2k])$. (Note that the above independence does not hold if we replace $\hat{H}$ by H. This explains why we introduced $\hat{H}$). Hence,

$$1 - \hat{h}_p(4k,2k) = 1 - P_p\left(\hat{H}([0,4k] \times [0,2k])\right) \leq (1 - \hat{h}_p(4k,k))^2. \tag{12}$$

Combining this with (11) we get

$$1 - \hat{h}_p(4k,2k) \leq (5(1 - \hat{h}_p(2k,k)))^2. \tag{13}$$

Now we are ready to prove the theorem. In the rest of this proof we denote the number $1/26$ by α. Suppose $n > 4$ and $h_p(3n,n) > 25/26 = 1-\alpha$.

Then, clearly (use a picture), we also have $\hat{h}_p(2(n+2),(n+2)) > 1-\alpha$. (This is where we used $n > 4$). So we have $\hat{h}_p(2m,m) > 1-\alpha$ (where we took $m = n+2$), and hence

$$1 - \hat{h}_p(2m,m) < \alpha. \tag{14}$$

Applying (13) we get

$$1 - \hat{h}_p(4m,2m) \leq \Big(5(1-\hat{h}_p(2m,m))\Big)^2 < 25\alpha(1-\hat{h}_p(2m,m)). \tag{15}$$

Since $\alpha < 1/25$ this is again smaller than α, so (14) still holds with m replaced by $2m$. So we can iterate the above and get, for all $k \geq 0$,

$$1 - \hat{h}_p(2^{k+1}m, 2^k m) < \alpha\,(25\alpha)^k. \tag{16}$$

Now let, for $k \geq 0$, R_k be the reactangle $[0, 2^{k+1}m] \times [0, 2^k m]$ if k is even, and $[0, 2^k m] \times [0, 2^{k+1}m]$ if k is odd. If each of these rectangles has an open crossing 'in the long direction', then (see Figure 7) there is an infinite open cluster. Hence, by (16) and FKG,

$$P_p(\exists \text{ an infinite open cluster }) \geq \prod_{k \geq 0} \left(1 - \alpha(25\alpha)^k\right),$$

which is larger than 0 because $\sum_k (25\alpha)^k < \infty$. □

Exercise 3.1 Give a modification of the last part (below (16) without using FKG.

3.2 Proof of $p_c \leq 1/2$

In this subsection we use the following generalisation of Proposition 2.7, which can be proved in practically the same way as Proposition 2.7.

Proposition 3.2. *For all $k > 0$ there is an $\delta_k > 0$ such that $h_{1/2}(kn,n) > \delta_k$ for all $n \geq 1$.*

Now we have done enough preliminary work to start the proof that $p_c \leq 1/2$. The strategy will roughly be as follows: Let us suppose that $p_c > 1/2$. Let H_n denote the event that there is an open horizontal crossing of the box $[0,8n] \times [0,2n]$. We will show that if $p \in (1/2, p_c)$ and n is very large, then the expected number of pivotal edges for the event H_n is also very large. But then, according to Russo's formula, $\frac{d}{dp}P_p(H_n)$ is large on the entire interval $(1/2, p_c)$. However, since $P_p(H_n)$ is bounded (namely, between 0 and 1) this gives a contradiction.

Now we carry this out more explicitly. Let $N(H_n)$ denote the number of pivotal edges for the event H_n.

Proposition 3.3. *There is a constant $C_2 > 0$ such that for all $p \in (1/2, p_c)$ and all $n \geq 1$*

$$E_p(N(H_n)) > C_2 \log n.$$

Proof. We start by three 'observations':

Observation 1 See Figure 8. Let π be a 'deterministic' vertical dual crossing of some box R, and let π' be a (also 'deterministic') horizontal path, starting from the right side of R and ending 'near' π. By the latter we mean that the end vertex of π' has distance $1/2$ to the midpoint of some edge (denoted by e in Figure 8) of which the dual edge is in π. Now suppose we have a configuration (assignment of states, open or closed' to the edges) in which π is the leftmost, closed vertical dual crossing of R and in which π' is open. In such configuration the above mentioned edge e is pivotal for the event $H(R)$. (This is so because if we make e open, there is no longer a vertical closed dual crossing of R: we cannot avoid e by making a 'leftgoing' detour beacuse π was the leftmost such crossing, and neither by making an 'rightgoing' detour because that is blocked by π').

Observation 2 See Figure 9. Let, again, π be a 'deterministic' vertical dual crossing of some box R and π' a horizontal path, starting from the right side of R and ending 'near' π. Let $NE(\pi, \pi')$ be the edges in the 'North-East' region indicated in Figure 9. Let $LL(\pi, \pi')$ be the event that π is the leftmost closed, dual, vertical crossing of R and that π' is the lowest open horizontal path with the above mentioned property. This event is independent of the edge values in $NE(\pi, \pi')$.

Observation 3 See Figure 10. Let, again, π and π' be a dual path, respectively path, with the properties in the second sentence of Observation 2. Let $v \in \mathbb{Z}^2$ be the endpoint of π'. Let, for $k \geq 1$, $A_k = A_k(\pi, \pi')$ be the annulus $v + A(3^k, 3^{k+1})$. Let, for those k for which the north-east corner of A_k is inside R, $C_k = C_k(\pi, \pi')$ be the event that there is an open path in $A_k \cap NE(\pi, \pi')$ that starts at π and ends 'near' π. If $p \geq 1/2$, we have

$$P_p(C_k) \geq \eta, \tag{17}$$

where $\eta > 0$ is the infimum in Corollary 2.8.

We continue with the proof of the Proposition: During the proof we let R_n denote the box $[0, 8n] \times [0, 2n]$. Let $p \in (1/2, p_c)$ and n a positive integer. Let π be a vertical dual crossing of the box $[0, 6n] \times [0, 2n]$. From now on we will often call that box *the left part of* R_n. Let π' be a horizontal path in the lower half of R_n, which starts at the right side of R_n and ends 'near' π. Let the events $LL(\pi, \pi')$ and $C_k(\pi, \pi'$ be as in Observation 2. Note that if both these events occur then, by Observation 1, there is a pivotal edge inside A_k. Hence

$$E_p\left(N(H_n) \,|\, LL(\pi, \pi')\right) \geq E_p\left(\sum_k I(C_k(\pi, \pi') \,|\, LL(\pi, \pi')\right), \qquad (18)$$

where the summation is over all even k for which the north-east corner of A_k is inside R_n and where $I(\cdot)$ denotes the indicator function. The right-hand-side of (18) is of course equal to

$$\sum_k P_p\left(C_k(\pi, \pi') \,|\, LL(\pi, \pi')\right),$$

which by Observation 2 equals

$$\sum_k P_p(C_k(\pi, \pi')),$$

which by Observation 3 is at least

$$\sum_k \eta.$$

Since the number of terms in the summation is of order $\log n$ (here we use that π and π' are located in the left part, respectively lower half, of R_n), this last expression is at least $\eta c \log n$ for some constant $c > 0$ which does not depend on n or p. Using this we have

$$\begin{aligned} E_p(N(H_n)) &\geq \sum_{\pi,\pi'} P_p(LL(\pi, \pi'))\, E_p\left(N(H_n) \,|\, LL(\pi, \pi')\right) \\ &\geq \eta c \log n \sum_{\pi,\pi'} P_p(LL(\pi, \pi')), \qquad (19) \end{aligned}$$

where the sum is over all π, π' with the properties mentioned a few lines above (18). Using (as before) that the event that π is the leftmost closed vertical dual crossing of R_n is independent of the edge values in the region to the right of π, it follows that, for fixed π, the sum over π' in (19) is at least the product of

$$P_p(\pi \text{ is the leftmost closed vertical dual crossing of } R_n)) \qquad (20)$$

and

$$P_p(\exists \text{ an open horizontal crossing of the lower half of } R_n). \qquad (21)$$

By Proposition 3.2 the probability in (21) is at least δ_4. Further, the sum over π of (20) clearly equals

$$P_p(\exists \text{ closed vertical dual crossing of } [0, 6n] \times [0, 2n]),$$

which (using duality) equals $1 - h_p(6n, 2n)$, which (by Theorem 3.1 and because $p \geq 1/2$) is at least $\frac{1}{26}$.

So the summation over π, π' in (19) is at least $\frac{\delta_4}{26}$ and hence

$$E_p(N(H_n)) \geq c\eta \frac{\delta_4}{26} \log n.$$

Taking $C_2 = c\eta\frac{\delta_4}{26}$ this completes the proof of Proposition 3.3. □

From this Proposition the main result in this section follows easily:

Theorem 3.4. *(*Kesten (1980)*).*

$$p_c \leq 1/2,$$

and hence, by Theorem 1.9,

$$p_c = 1/2.$$

Proof. Let C_2 as in Proposition 3.3 If $p_c > 1/2$ we can choose an n satisfying $(p_c - 1/2)C_2 \log n > 1$. By Russo's formula and Proposition 3.3 we then have

$$P_{p_c}(H_n) \geq P_{1/2}(H_n) + (p_c - 1/2) \inf_{p \in (1/2, p_c)} E_p(N(H_n)) \geq (p_c - 1/2)C_2 \log n > 1,$$

which is impossible. Hence $p_c \leq 1/2$. □

4 Connection probabilities at criticality

In this section we study, for the critical case (that is, for $p = 1/2$), the asymptotic behaviour of the probability that O has an open path to some vertex at large distance from O. Recall that Theorem 2.10 gives an upper bound, in the form of a power of n, for $P_{1/2}(O \leftrightarrow \partial B(n))$. A lower bound in the form of a power law of n is obtained as follows: Let, for each even integer $k \geq 0$, A_k denote the event that there is an open horizontal crossing of the box $[0, 2^{k+1}] \times [0, 2^k]$. Similarly, let for each odd integer $k \geq 1$, A_k denote the event that there there is an open vertical crossing of the box $[0, 2^k] \times [0, 2^{k+1}]$. Let $\hat{k} = \hat{k}(n)$ be the smallest k with $2^k > n$. Clearly, there is a $c > 0$ such that, for all n, $\hat{k}(n) < c \log n$. Also note that if all the events $A_1, \cdots A_{\hat{k}}$ occur, and the edge with endpoints 0 and $(1, 0)$ as well as the edge with endpoints $(1, 0)$ and $(2, 0)$ are open, then there is an open path from 0 to the boundary of $B(n)$. (This is a similar sitution as in Figure 7 at the end of the proof of Theorem 3.1). By Proposition 2.7 there is an $\varepsilon > 0$ such that each of the events A_k mentioned above has probability $> \varepsilon$. Hence (using the FKG inequality), we get the following power-law lower bound:

$$P_{1/2}(O \leftrightarrow \partial B(n)) \geq \frac{1}{4}\varepsilon^{\hat{k}} \geq \frac{1}{4}\varepsilon^{c \log n} = \frac{1}{4}n^{c \log \varepsilon}.$$

A considerably better power-law lower bound can be obtained easily by using a correlation-like inequality which we present in the following subsection (and which is also useful in many other percolation arguments).

4.1 Another basic tool: the BK inequality

We have used several times results of the form that the probability that there is an open path from a to b and an open path from u to v is larger than or equal to the probability that there is an open path from a to b times the probability that there is an open path from u to v. (Here u, v, a and b are vertices in, for instance, the square lattice). This was a direct consequence of the FKG inequality (see Section 1).

It turns out that it is useful to have an *upper* bound for the probability that there exist *disjoint* open paths from a to b and from u to v. (In this context, we say that two paths are disjoint if they have no edge in common).

As the results in Section 1, the tool we present now is relevant in a much more general context than percolation theory. Again, we work on $\Omega = \{0, 1\}^n$, and P_p is the product distribution on Ω with parameter p. Before we state the main definition and results, we introduce some notation:

Let $\omega \in \Omega$ and $K \subset \{1, \cdots, n\}$. We use the notation $[\omega]_K$ for the set of all elements of Ω that 'agree with ω on K'. More formally,

$$[\omega]_K := \{\alpha \in \Omega \,:\, \alpha_i = \omega_i \text{ for all } i \in K\}.$$

Now let $A, B \subset \Omega$. We define $A \Box B$ as the set of all $\omega \in \Omega$ with the property that there are disjoint subsets $K, L \subset \{1, \cdots n\}$ such that, informally speaking, the ω values on K guarantee that ω is in A, and the ω values on L guarantee that ω is in B. Formally, the definition is:

$$A \Box B := \{\omega \in \Omega \,:\, \exists \text{ disjoint} K, L \subset \{1, \cdots, n\} \text{ s.t. } [\omega]_K \subset A \text{ and } [\omega]_L \subset B\}.$$

Theorem 4.1. *For all n and all $A, B \subset \{0,1\}^n$,*

$$P_p(A \Box B) \leq P_p(A) P_p(B). \tag{22}$$

This theorem was proved for increasing events by Van den Berg and Kesten (1985) (whence the name BK inequality), who conjectured that it holds for all events. Between then and (about) 1995 the result for increasing events was extended to some other special classes of events, but there was not much hope for a proof for the general case. Then, unexpectedly, a young mathematician who was at that time in the final stage of his PhD work, David Reimer, obtained a proof for the general case. The main idea in Reimer's proof (for which he received a George Polya price) was to 'replace' the problem by a (linear-)algebraic problem, in a very clever and elegant way.

In this course we will use the inequality only for increasing events. That special case can be proved in a similar way (but a bit more tricky) as the FKG inequality in Section 1.2, namely by a step-by-step procedure. (Now at each step the monotonicity is opposite to that in the proof of FKG: at each step the probability of (the version at that step of) $A \Box B$ does not *decrease* but *increase* (or remains the same). We omit the details here.

Remark: To illustrate the BK inequality in a percolation-like setting, let G be a finite graph of which the edges are independently open with probability p and closed with probability $1 - p$. Let, as in the introduction in the beginning of this subsection, a, b, u and v be vertices of G. Let A be the event $\{a \leftrightarrow b\}$ (i.e. the event that there is an open path from a to b) and B the event $\{u \to v\}$. By taking n equal to the number of edges of G, and taking 0 for 'closed' and 1 for 'open', we can translate this in terms of the general context, and it is easy to see (check this yourself) that, for A and

B as above, $A\square B$ is the event that there are disjoint open paths from a to b and from u to v. (Where, by 'disjoint' we mean here that the two paths have no edges in common).

4.2 Back to connection probabilities at criticality

We will now use the inequality in the previous subsection to give a lower bound for the probability (in the critical case) that 0 has an open path to the boundary of $B(n)$.

Theorem 4.2. *For all n,*

$$P_{\frac{1}{2}}(O \leftrightarrow \partial B(n)) \geq \frac{1}{2\sqrt{n}}.$$

Proof. Fix an $n > 0$ and consider the rectangle $R := [0, 2n] \times [0, 2n-1]$. We have seen before (using symmetry and duality) that the probability $P_{1/2}(H(R))$ that there is a horizontal open crossing of R is exactly $1/2$. Let l and r denote the left side and the ride side of R respectively, and let m be the vertical line segment that divides R in two halves. Now suppose there is an open horizontal crossing π of R. This path intersects m, and it is clear that each vertex v on $\pi \cap m$ has disjoint open paths to l and r. Also note that each path from v to r or l intersects $\partial B(v, n)$, the $2n \times 2n$ square centered at v. Hence,

$$H(R) \subset \bigcup_{v \in l} \{\exists \text{ two disjoint open paths from } v \text{ to } \partial B(v, n)\}. \tag{23}$$

Hence, since the number of vertices on m is $2n$, and by using the BK inequality (and translation invariance), we have

$$\frac{1}{2} = P_{\frac{1}{2}}(H(R)) \leq 2n P_{\frac{1}{2}}(O \leftrightarrow \partial B(n))^2,$$

from which the desired result follows immediately. □

In fact it is believed that, for critical percolation on the square lattice and other 'nice' planar lattices, $P_{1/2}(O \leftrightarrow \partial B(n))$ behaves like $n^{-5/48}$, in the sense that

$$-\log(P_{p_c}(O \leftrightarrow \partial B(n)) / \log n \rightarrow \frac{5}{48}, \quad \text{as } n \rightarrow \infty.$$

So far this has only been proved for site percolation on the triangular lattice (using SLE processes, which will be introduced later in this course). Such behaviour is called 'power law behaviour' and the corresponding exponent (here 5/48) is called a 'critical exponent. It is believed that, at and near criticality, the asymptotic behaviour of several other functions can also be described in terms of power laws. For instance, it is believed that, as p approaches p_c from above, $\theta(p)$ behaves like $(p-p_c)^\beta$, where the exponent β essentially depends only on the dimension of the lattice. It has been proved that in sufficiently high dimensions (> 19 is sufficient) the critical exponents are exactly the same as for the binary tree. For instance, β is then exactly 1. For dimension 2 it is believed that $\beta = 5/36$. Again, this has so far only been proved for site percolation on the triangular lattice.

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