Introduction to Schramm-Loewner Evolutions

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These notes are, in some sense, a continuation of the notes [1] by Federico Camia, to which I will refer frequently, and are mainly based on parts of Chapter 3 of the Saint-Flour notes [6] by Wendelin Werner.

1 Summary of some basic properties of SLE

In Section 6 of [1] we saw that the search for a Loewner chain with certain stochastic properties, (which in turn came from the search for the 'scaling limit' of the percolation exploration path), led to the definition of chordal SLE_κ (Def. 6.1) in [1]).

In the present section we work in the 'reverse' direction: we start from the above mentioned definition and list and discuss the main properties of such processes. For the time being, we will only work on the above mentioned type of SLE and therefore omit the word 'chordal'.

Since SLE_κ is driven by Brownian motion, we first list the basic properties of that process. As before, we denote standard Brownian motion by $(B_t, t \geq 0)$. For each $t \geq 0$, B_t has a normal distribution with mean 0 and variance t. We list the following basic properties:

Basic properties of Brownian motion

- **Symmetry:** $(-B_t, t \geq 0)$ has the same law as $(B_t, t \geq 0)$.
- **Markov property:** Let $T \geq 0$. The process $(B_{T+t} - B_T, t \geq 0)$ has the same law as $(B_t, t \geq 0)$ and is independent of $(B_t, 0 \leq t \leq T)$. This also holds for certain random T, namely *stopping times*.
- **Scaling property:** Let $\lambda > 0$. The process $(B_{\lambda t}, t \geq 0)$ has the same law as $(\sqrt{\lambda} B_t, t \geq 0)$. In other words, $(B_{\lambda t}/\sqrt{\lambda},\ t \geq 0)$ has the same law as $(B_t,\ t \geq 0)$.

Let $\kappa > 0$ and $W_t := \sqrt{\kappa} B_t$, $t \geq 0$. Obvious analogs of the properties listed above for B clearly hold for W. In particular, if $\lambda > 0$, the process $(W_{\lambda t}/\sqrt{\lambda},\, t \geq 0)$ has the same law as $(W_t, t \geq 0)$.

Now let $(K_t, t \geq 0)$ be the SLE_κ process (as in Def. 6.1 of [1]). It is not difficult to show that the SLE process $(K_t, t \geq 0)$ driven by W 'inherits' some of the above properties for Brownian motion (as suggested in Section 6 of [1]):

Some basic properties of SLE_κ:

- **Symmetry:** The law of $(K_t, t \geq 0)$ is preserved under reflection in the imaginary axis.

- **Conformal Markov property**: Let $T \geq 0$. The process

  $$(g_T(K_{T+t} \setminus K_T) - W_T,\, t \geq 0)$$

  is again an SLE_κ process, i.e. has the same law as $(K_t, t \geq 0)$). Moreover, it is independent of $(K_t,\, 0 \leq t \leq T)$.
  This also holds if T is a stopping time (w.r.t. the Brownian motion driving K).

- **Scaling property:** Let $\lambda > 0$. The process $(K_{\lambda t},\, t \geq 0)$ has the same law as $(\sqrt{\lambda} K_t,\, t \geq 0)$. In other words, $(K_{\lambda t}/\sqrt{\lambda},\, t \geq 0)$ has the same law as $(K_t,\, t \geq 0)$.

As remarked in Section 6 of [1], it can be shown that K_t is generated by a curve. More precisely, there is a continuous curve (which may touch itself but not intersect itself) $\gamma(\cdot)$ such that, for each $t \geq 0$, K_t is the 'filling' of the curve segment $(\gamma(s), 0 \leq s \leq t)$, and we have $g_t(\gamma(t)) = W_t$. This fact is far from trivial and has been proved in [7]. Sometimes this fact helps to find more 'visual' proofs of certain results for SLE, for instance the scaling property above. However, this scaling property (and also the two other properties listed above) can also be derived by just using the Loewner equation, as we will see below:

In fact, we start quite generally, and only later use that SLE is driven by Brownian motion. Let $(K_t, t \geq 0)$ be a Loewner chain driven by some function $w(.)$. Now consider the process $(\hat{K}_t := K_{\lambda t}/\sqrt{\lambda}, t \geq 0)$. From the convention that K_t has capacity $2t$, and from what we learned earlier about capacities, it follows immediately that $\hat{K}_t$ has capacity

$$\frac{2\lambda t}{(\sqrt{\lambda})^2} = 2t,$$

so that $\hat{K}$ is also a Loewner chain. Now we will determine its driving function: It is easy to see that the family of conformal maps $\hat{g}_t, t \geq 0$ corresponding with $\hat{K}$ is given by

$$\hat{g}_t(z) = \frac{g_{\lambda t}(\sqrt{\lambda} z)}{\sqrt{\lambda}}.$$

Just check that this $\hat{g}_t$ is a map with the required properties (in particular that it has the 'correct' hydrodynamic normalization) and use that these properties are unique. From the Loewner equation for g_t we then have:

$$\begin{aligned} \frac{d\hat{g}_t(z)}{dt} &= \frac{1}{\sqrt{\lambda}} \frac{dg_{\lambda t}(\sqrt{\lambda} z)}{dt} \\ &= \frac{1}{\sqrt{\lambda}}\, \lambda\, \frac{2}{g_{\lambda t}(\sqrt{\lambda} z) - w(\lambda t)} \\ &= \frac{2}{\hat{g}_t(z) - (1/\sqrt{\lambda})\, w(\lambda t)}. \end{aligned} \tag{1}$$

Hence the driving function of $\hat{K}$ is the function $\hat{w}$ given by

$$\hat{w}(t) = \frac{w(\lambda t)}{\sqrt{\lambda}}.$$

In the special case where K is an SLE_κ process, we now get immediately from the earlier listed scaling property for Brownian motion, that the driving function of $\hat{K}$ has the same law as that of K and hence that $\hat{K}$ has the same law as K.

In a similar (but somewhat more elaborate) way the conformal Markov property for SLE follows from the Markov property of Brownian motion.

The above considerations may intuitively give the impression that two SLE processes with different parameters κ behave 'qualitatively' the same. However, as already remarked at the end of Section 6 of [1] this is not the case: the bahaviour for $\kappa < 4$ is very different from that for $\kappa > 4$. We will now point out where this difference comes from.

Let $(K_t, t \geq 0)$ be an SLE_κ process, and let x be a non-zero real number. We will study the time at which x is absorbed by K. We denote this time by T_x (which may be infinite). By the scaling property of SLE the distribution of T_x can be easily expressed in that of T_1. In particular, T_1 is finite with probability one if and only if T_x is finite with probability one. As long as 1 is not in the hull, $g_t(1)$ satisfies the Loewner equation

$$\frac{dg_t(1)}{dt} = \frac{2}{g_t(1) - W_t},$$

and vice versa: So T_1 is the time at which $g_t(1)$ hits W_t, or equivalently, $g_t(1) - W_t$ hits 0. So it is natural to study the process $(g_t(1) - W_t, t \geq 0)$. It turns out to be convenient to divide this by $\sqrt{\kappa}$, and we define

$$X_t := \frac{g_t(1) - W_t}{\sqrt{\kappa}}, \; t \geq 0.$$

Note that X starts away from 0 (namely at location $1/\sqrt{\kappa}$).

To answer the question whether X will hit 0, we study its infinitesimal increments (differentials):

$$dX_t = \frac{1}{\sqrt{\kappa}} (dg_t(1) - dW_t). \tag{2}$$

As to the term $dg_t(1)$ in (2), the Loewner equation immediately gives:

$$dg_t(1) = \frac{2dt}{g_t(1) - W_t},$$

which by the definition of X equals $2dt/(\sqrt{\kappa}X_t)$.

As to the term dW_t: Recal that W is $\sqrt{\kappa}$ times a standard Brownian motion. But by the symmetry property of Brownian motion, we can also write W_t as $-\sqrt{\kappa}B_t$, with B_t a standard Brownian motion. So we write $dW_t = -\sqrt{\kappa}dB_t$.

Together the above manipulations give

$$dX_t = \frac{2}{\kappa X_t} dt + dB_t. \tag{3}$$

This can be interpreted as (the differential equation for) a Brownian motion with location-dependent drift. (If, in the right-hand-side of (3) X_t would be replaced by a constant c, we would have a Brownian motion with constant drift $2/(\kappa c)$). Note that the drift blows up when we approach 0 (and that the drift is away from 0). Equations of the form (3) have been studied widely since a long time. (In fact, it describes a $1+(4/\kappa)$ dimensional Bessel process). It is well-known that a process X satisfying (3) will hit 0 with probability 1 if $\kappa > 4$ and with probability 0 if $\kappa \leq 4$. Using this it can be shown (see the Exercise on the next page) that

Proposition 1.1. *(a) If $\kappa \leq 4$, then almost surely no point on the real line (apart from 0) will be absorbed by the SLE process. More formally, with*

probability one, $\cup_{t\geq 0} K_t \cap \mathbb{R} = \{0\}$*.*
(b) In fact, if $\kappa \leq 4$*, we even have that* K *is a simple curve.*
(c) If $\kappa > 4$ *then almost surely each point on the real line will eventually be in the hull; more precisely, with probability one,* $\mathbb{R} \subset \cup_{t\geq 0} K_t$*.*

Exercise:
(a) Show part (a) and part (c) of the proposition. (*Hint*: use the earlier made remark that, in the above context, the element 1 on the real line is not 'special').

(b) Derive part (b) of the proposition from part (a). *Hint*: Here you may use the earlier mentioned fact that SLE is generated by a continuous curve, say γ. If this curve is not simple, there is a (rational) time t such that the curve $(\gamma(t+s, s > 0)$ hits the segment $\gamma(0, t]$.

2 Computation of first-hitting probabilities

In this section we assume $\kappa > 4$. In the previous section we have seen that then a.s. each point x on the real line will eventually be in the hull. In the present section we consider the following problem: Let $a < 0 < c$. What is the probability that c is absorbed in the hull before a? In othere words: what is the probability that the SLE process hits the halfline $[c, \infty)$ before it hits the halfline $(-\infty, a]$? It turns out that this can be solved quite explicitly.

Apart from being interesting in itself, this problem is motivated by connections with critical percolation, see Section 2.4.

2.1 Notation and key ideas for the computation

It follows from the scaling property of SLE that the probability that $[c, \infty)$ is hit before $(-\infty, a]$ depends only on the ratio of a and c. Therefore we can write this probability as $F_\kappa(-a/(c-a))$, with F_κ, with F_κ a function which maps the interval $(0, 1)$ to itself. We will often omit the subscript κ.

Now let $t > 0$ and suppose that neither a nor c is in K_t. Consider the 'standard' map g_t. As we noted before (see the conformal Markov property), the process

$$(g_t(K_{t+s} \setminus K_t) - W_t,\, s \geq 0)$$

is again an SLE process (with the same κ). Moreover, the hitting of $[c, \infty)$ by the original process corresponds with hitting $[g_t(c) - W_t, \infty)$ by the new process. (And a similar statement holds for hitting $(-\infty, a]$). In particular, the original process hits $[c, \infty)$ before $(-\infty, a]$ iff the new process hits $[g_t(c) - W_t, \infty)$ before $(-\infty, g_t(a) - W_t]$. Hence, (if at time t neither c nor a is in K_t),

$$P([c, \infty) \text{ is hit before } (-\infty, a] \,|\, \mathbb{F}_t) = F\left(\frac{W_t - g_t(a)}{g_t(c) - g_t(a)}\right), \tag{4}$$

where $\mathbb{F}_t$ is the $\sigma-$field containing all information up to time t. As I explained in class, the l.h.s. of (4), considered as a random process indexed by t, is a (bounded) Martingale; in particular, it has no 'drift'. But then this also holds for the r.h.s. Therefore we will try to obtain an expression for the changes of the r.h.s. of (4), under 'infinitesimal changes' of t. This expression will contain terms of order dB_t and terms of order dt. By setting the latter (the drift terms) equal to 0, we will obtain a (differential) equation for the function F.

To carry this out we will use some (but not much) stochastic calculus. For those who have not learned stochastic calculus, I give a very short and informal intermezzo on this subject.

2.2 A short (and very informal) introduction to Stochastic Calculus / Ito's formula

As I pointed out in class, in 'ordinary' calculus we are used to write (when f is a 'smooth' function):

$$df(t) = f'(t)dt. \tag{5}$$

In principle we *could* add the term

$$\frac{1}{2}f''(t)(dt)^2,$$

but don't do that because, 'when we sum things up, the contribution of such terms vanishes'. Let us recall where this vanishing comes from: Divide the interval $(t, t+s)$ in n sub-intervals

$$(t + (i-1)s/n, t + is/n), 1 \leq i \leq n.$$

The sum of the squares of the lengths of the subintervals is

$$\sum_{i=1}^{n}(s/n)^2 = s^2/n, \tag{6}$$

which goes to 0 as $n \to \infty$; this essentially explains the above mentioned vanishing.

Now suppose we 'start' with a standard Brownian motion $(B_t, t \geq 0)$ and consider the above function, but now applied to the Brownian motion. That is, we consider the function (or process)

$$t \to f(B_t), t \geq 0.$$

If we repeat the above considerations, $f'(t)dt$ is replaced by $f'(B_t)dB_t$ and the analog of (6) becomes

$$\sum_{i=1}^{n}\left(B_{(t+is/n)} - B_{(t+(i-1)s/n)}\right)^2. \tag{7}$$

The terms in this summation are independent, and each term has (by the scaling property of Brownian motion) the same distribution as

$$\left(\frac{B_s}{\sqrt{n}}\right)^2 = \frac{(B_s)^2}{n}.$$

In other words, the distribution of (7) is the same as that of the average of n independent copies of $(B_s)^2$. Hence (by the law of large numbers) (7) converges in probability to the expectation of $(B_s)^2$, which is s. Therefore, the 'second-order' term now doesn't vanish but becomes $\frac{1}{2}f''(B_t)dt$. Summarizing, instead of (5) we get

$$d(f(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt \ , \tag{8}$$

which is known as (a form of) *Ito's formula*.

2.3 The computation

Now we proceed from the situation at the end of Subsection 2.1. Define

$$Z_t := \frac{W_t - g_t(a)}{g_t(c) - g_t(a)}.$$

Recall that we want to study the 'infinitesimal increments' of $F(Z_t)$. Therefore we first do this for Z_t. This appears to be easy: Simple use of the 'quotient rule' for differentiation gives

$$dZ_t = \frac{(g_t(c) - g_t(a))\,(dW_t - dg_t(a)) - (W_t - g_t(a))\,(dg_t(c) - dg_t(a))}{(g_t(c) - g_t(a))^2}. \tag{9}$$

To work this out further, we apply the Loewner equation, which gives us immediately

$$dg_t(c) = \frac{2dt}{g_t(c) - W_t},$$

and its analog for a instead of c. Plugging that into (9) and doing some elementary algebra (and using the definition of Z_t again) gives:

$$dZ_t = \frac{dW_t}{g_t(c) - g_t(a)} + \frac{2dt}{(g_t(c) - g_t(a))^2}\left(\frac{1}{Z_t} - \frac{1}{1 - Z_t}\right). \tag{10}$$

Recall that we are eventually interested in the drift term of $dF(Z_t)$. In the spirit of the ideas in Subsection 2.2 we write, somewhat informally,

$$dF(Z_t) = F'(Z_t)dZ_t + \frac{1}{2}F''(Z_t)(dZ_t)^2. \tag{11}$$

This tells us (again somewhat informally) that the drift terms (the terms involving dt) in $dF(Z_t)$ consists of $F'(Z_t)$ times the the drift terms in dZ_t, together with $1/2F''(Z_t)$ times the drift terms in the square of dZ_t. The former gives, by (10), of course a contribution

$$F'(Z_t)\frac{2dt}{(g_t(c)-g_t(a))^2}\left(\frac{1}{Z_t}-\frac{1}{1-Z_t}\right).$$

As to the contribution of the latter: (again informally), if we take the square of the r.h.s. of (10) and then replace, in the spirit of Subsection 2.2, $(dW_t)^2$ by κdt (the factor κ arises of course because W_t can be regarded as a standard Brownian motion times $\sqrt{\kappa}$), we get the contribution

$$\frac{1}{2}F''(Z_t)\,\frac{\kappa dt}{(g_t(c)-g_t(a))^2}.$$

Together with the former contribution this gives, for the drift term in $dF(Z_t)$:

$$\frac{2}{(g_t(c)-g_t(a))^2}\left(F'(Z_t)(\frac{1}{Z_t}-\frac{1}{1-Z_t})+\frac{\kappa}{4}F''(Z_t)\right)dt. \tag{12}$$

It is intuitively obvious (and can be shown from the equation (10)) that Z_t 'can take all values between 0 and 1)'. Therefore (12) and the requirement that $F(Z_t)$ has no drift gives the following differential equation for F:

$$\frac{\kappa}{4}F''(z)+\left(\frac{1}{z}-\frac{1}{1-z}\right)F'(z)=0,\ 0<z<1. \tag{13}$$

Another intuitively obvious result (which can be shown rigorously by doing some extra work on (10)) is that $F(z)$ tends to 0 as $z\to 0$ and it tends to 1 as $z\to 1$.

The above differential equation with boundary conditions has a unique solution. More explicitly, we have

Theorem 2.1. *For all $\kappa > 4$,*

$$F(z)=c(\kappa)\int_0^z\frac{dx}{x^{4/\kappa}(1-x)^{4/\kappa}},\ z\in(0,1), \tag{14}$$

where

$$c(\kappa)=\left(\int_0^1\frac{dx}{x^{4/\kappa}(1-x)^{4/\kappa}}\right)^{-1}.$$

Remark: In the step leading to (11) we assumed certain smoothness properties of F. This smoothness can be proved from further exploiting the equation (10) and an elegant interpretation of $F(z)$ in terms of the process Z_t.

2.4 Connections with critical percolation

Earlier in this course it was made plausible that the scaling limit of an exploration path in critical site percolation on the triangular lattice corresponds with an SLE process. What should be the parameter κ of that process? The computation in the previous subsection (resulting in (14), together with Smirnov's results give the answer, as we will see now: Let the points a and c as in the previous subsection. Now consider critical site percolation on the triangular lattice, with mesh δ, in the upper half-plane. Using the Cardy-Smirnov theorem (as we saw in Section 5.7 of [3]) we can, in principle, compute the probability (in the limit, as $\delta \to 0$) that there is an open path from the segment $[a, 0]$ to the segment $[c, \infty)$. I write 'in principle' because the result is in terms of a conformal map from the half-plane to the domain inside an equilateral triangle (see Theorem 5.46 in [3]). Fortunately, such a map is explicitly known, and the resulting computation yields exactly (14) with $\kappa = 6$. Recalling the correspondence between crossing probabilities and hitting probabilities of the percolation exploration path, this means that, in the scaling limit (as $\delta \to 0$), the probability that the exploration path (starting at O) hits the segment $[c, \infty)$ before $(-\infty, a]$ is the same as the probability of the corresponding event for SLE_6.

This makes plausible that the percolation exploration path converges in distribution, as the mesh of the lattice tends to 0, to the trace of an SLE_6 path. This has indeed been proved in the literature. (An outline was given in [5], see also the introduction of [4]; a detailed proof was given in [2]). A full treatment would take far too much time in this course.

References

[1] F. Camia, *Notes for the Course on Percolation and SLE*, April 2007.

[2] F. Camia and C.M. Newman, *Critical Percolation Exploration Path and SLE(6): a Proof of Convergence*, *Probability Theory and Related Fields* **139**, 473-519 (2007).

[3] G.R. Grimmett, *Probability on Graphs*, Cambridge University Press (2010).

[4] G.F. Lawler, O. Schramm and W. Werner, One-arm exponent for critical 2D percolation, *Electron. J. Probab.* **7:** 13 (2001).

[5] S. Smirnov, Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits, *Comptes Rendus Acad. Sci. Sér. I Math.* **333**, 239-244 (2001).

[6] W. Werner, *Random planar curves and Schramm-Loewner evolutions*, in: Lecture Notes in Math., Springer, 2004.

[7] S. Rohde and O. Schramm, Basic properties of SLE, *Ann. Math.* (2) 161, pp. 883-924 (2005)