Derivation shrunken variance for shrunken t-statistic

Shrinkage estimator for the true variance for variable j, σ_j^2. First write: $t_j = \hat{\sigma}^2_{j,(1)}$, for which we use the ordinary sample variance estimator (unbiased) and $t_{\text{med}} = \text{median}(t_1, \ldots, t_p)$. Shrunken estimator: $\tilde{t}_j(\lambda) = (1-\lambda)t_j + \lambda t_{\text{med}}$

$$\text{Total MSE}(\lambda) = \sum_{j=1}^{p} E[(\tilde{t}_j(\lambda) - \sigma_j^2)^2].$$

Consider the term within the square, the deviation of the estimator from the true value:

$$\tilde{t}_j(\lambda) - \sigma_j^2 = (t_j - \sigma_j^2) - \lambda(t_j - t_{\text{med}}).$$

Squaring gives:

$$t_j^2 + \lambda^2(t_j - t_{\text{med}})^2 - 2\lambda(t_j - t_{\text{med}})(t_j - \sigma_j^2). \tag{1}$$

Now we need the expectation to obtain MSE for j. For the last term, write $t_{\text{med}} = \sigma_j^2 + b_j$, where b_j is the deviation from the true value. For large p it is reasonable to assume b_j to be non-random (that is: if we would repeat the experiment we would likely obtain (nearly) the same value of b_j because $b_j = t_{\text{med}} - \sigma_j^2$ and t_{med} should be very stable as a median over many variables). Therefore,

$$E[(t_j - t_{\text{med}})(t_j - \sigma_j^2)] = E[(t_j - \sigma_j^2 - b_j)(t_j - \sigma_j^2)] = E[(t_j - \sigma_j^2)^2],$$

because $E[b_j(t_j - \sigma_j^2)] = b_j E[t_j - \sigma_j^2] = 0$, because t_j is an unbiased estimator of σ_j^2.

Likewise also take the expectation of the first two terms of (1), sum over $j = 1, \ldots, p$ to obtain

$$\text{Total MSE}(\lambda) = \sum_j E[t_j^2] + \lambda^2 \sum_{j=1}^{p} E[(t_j - t_{\text{med}})^2] - 2\lambda \sum_{j=1}^{p} E[(t_j - \sigma_j^2)^2].$$

Taking derivative with respect to λ and equating to 0 renders:

$$\lambda_{\text{opt}} = \frac{\sum_{j=1}^{p} E[(t_j - \sigma_j^2)^2]}{\sum_{j=1}^{p} E[(t_j - t_{\text{med}})^2]}.$$

For p large:

$$\sum_{j=1}^{p} E[(t_j - t_{\text{med}})^2] \approx \sum_{j=1}^{p} (t_j - t_{\text{med}})^2.$$

Moreover, numerator $E[(t_j - \sigma_j^2)^2]$ is the variance of the sample variance estimator, which is known to equal $\sigma_j^4/(n-1)^2$, and hence can be estimated by $\hat{V}(t_j) = \hat{\sigma}_j^4/(n-1)^2$. Then,

$$\hat{\lambda}_{\text{opt}} = \frac{\sum_{j=1}^{p} \hat{V}(t_j)}{\sum_{j=1}^{p} (t_j - t_{\text{med}})^2},$$

which is the given estimator, apart from constraining it at 1.