Statistical Models

Lecture 8

Time Series II

Time Series

- Definition and examples of time series
- Stationary time series
  - ⇒ general definition
- White noise and other basic stationary time series
  - ⇒ *MA*, *AR* and *ARMA* processes
- Models for nonstationarity
  - ⇒ trend and seasonality

Last time

- Estimating the (partial) autocorrelation function, mean and parameters in *ARMA* model
- Diagnostic checking

Time Series

A (univariate) time series is a collection of random variables indexed by time

$$\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$$
$$\{X_t,\ t = \ldots,-2,-1,0,1,2,\ldots\}$$

Aim: summarize structure of time series, identify structural and random part of behavior

Usual approach
- plot time series (time plot)
- extract systematic (seasonal or trend) component of time series
- model remaining time series by standard stationary time series

Stationarity

Definition: the process $(X_t)_{t=-\infty}^{\infty}$ is (weakly) stationary if EX_t and EX_tX_{t+k} do not depend on *t*, for all *k*

In particular, the autocovariance function is well defined as:

$$\gamma_x(k) = \mathrm{Cov}(X_t, X_{t+k}), k \in Z$$

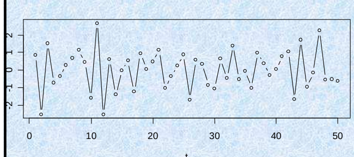
Examples of stationary time series:

MA(*q*) process
: autocovariance function shows cut-off at $\pm q$

AR(*p*) process
: autocovariance function satisfies Yule-Walker equations
: 'partial autocorrelation function' shows cut-off at $\pm p$

ARMA(*p*,*q*) process

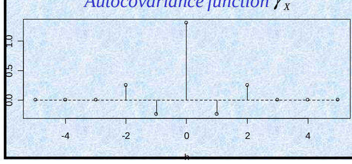
MA(2) process, example



Simulated process $(X_t)_{t=0}^{50}$

$$\beta = (1,-0.5,0.25), \sigma^2 = 1$$

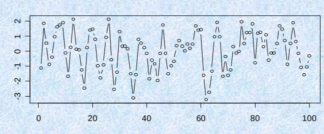
Autocovariance function γ_X



$$\gamma_X(h) = \begin{cases} 21/16 & h = 0 \\ -5/8 & h = \pm 1 \\ 1/4 & h = \pm 2 \\ 0 & |h| > 2 \end{cases}$$

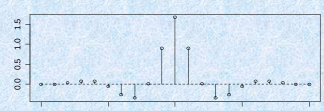
AR(2) process, example

Simulated process $(X_t)_{t=0}^{100}$ $\quad X_t = 0.75X_{t-1} - 0.4X_{t-2} + Z_t$



$$\alpha = (0.75,-0.4)^{\mathrm{T}} \quad \sigma^2 = 1$$

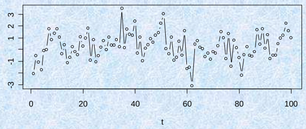
Autocovariance function γ_X



$$\gamma_X(h) = \begin{cases} 1.67 & h = 0 \\ 0.89 & h = \pm 1 \\ 0.00 & h = \pm 2 \\ -0.36 & h = \pm 3 \\ -0.27 & h = \pm 4 \\ -0.06 & h = \pm 5 \\ 0.06 & h = \pm 6 \end{cases}$$

ARMA(1,2) process, example

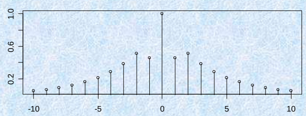
Simulated process $(X_t)_{t=0}^{100}$ $\quad X_t = 0.75X_{t-1} + Z_t - 0.5Z_{t-1} + 0.25Z_{t-2}$



$\alpha = 0.75,\ \beta = (-0.5, 0.25)^T$

$\sigma^2 = 1$

Autocorrelation function ρ_X



$$\rho_X(h) = \begin{cases} 1.000 & h = 0 \\ 0.458 & h = \pm 1 \\ 0.510 & h = \pm 2 \\ 0.383 & h = \pm 3 \\ 0.287 & h = \pm 4 \\ 0.215 & h = \pm 5 \\ 0.161 & h = \pm 6 \end{cases}$$

Time Series

A (univariate) time series is a collection of random variables indexed by time

$$\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$$
$$\{X_t,\ t = \ldots, -2, -1, 0, 1, 2, \ldots\}$$

In general: sequence of dependent random variables

Aim: summarize structure of time series, identify structural and random part of behavior

Usual approach

- plot time series (time plot)
- extract systematic (seasonal or trend) component of time series
- model remaining time series e.g. by standard stationary ARMA model

→ *Extraction of systematic component of time series*

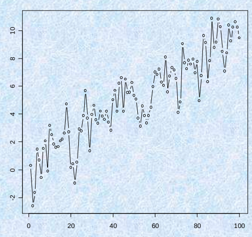
Estimation and elimination of trend

in absence of seasonality

Basic decomposition:

$X_t = m_t + Y_t$

(Y_t) stationary



Approaches to deal with this process

- Preliminary LS estimation of m_t
- Filter (X_t); taking local moving average
- Difference (X_t)

} especially useful for linear m_t

Preliminary LS estimation of trend

in absence of seasonality

Basic decomposition: $X_t = m_t + Y_t$, (Y_t) stationary

Assume the trend function belongs to a (simple) parameterized set of functions, e.g.

$$m_t = f(t;a) = a_0 + a_1 t + a_2 t^2$$

Then the LS estimator $\hat{m}_t$ of m_t is given by

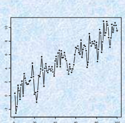
$$\hat{m}_t = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2$$

where

$$(\hat{a}_0, \hat{a}_1, \hat{a}_2) = \arg\min_{R^3} \sum_{t=1}^{n} \left(X_t - f(t;a)\right)^2$$

Elimination of trend

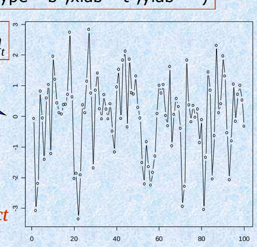
in absence of seasonality



$$\hat{m}_t = f(t;\hat{a}) = \hat{a}_0 + \hat{a}_1 t = 0.29 + 0.10t$$

```
> Yhat<-matr-predict(matr.fit)
> plot(Yhat,type="b",xlab="t",ylab="")
```

$$\hat{Y}_t = X_t - \hat{m}_t$$



Next step: try to fit e.g. AR model to this (estimated) stationary time series $(\hat{Y}_t)_{t=1}^{100}$

→ Model can be used to *predict*

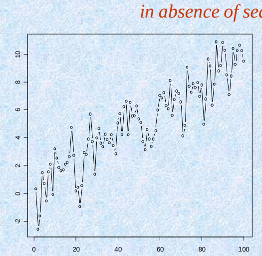
Estimation and elimination of trend

in absence of seasonality

Basic decomposition:

$X_t = m_t + Y_t$

(Y_t) stationary



Approaches to deal with this process

- Preliminary LS estimation of m_t
- Filter (X_t); taking local moving average
- Difference (X_t)

} especially useful for linear m_t

Moving average trend estimation

in absence of seasonality

Basic decomposition: $X_t = m_t + Y_t$, (Y_t) stationary

Assume the trend function is (approximately) linear

$$m_t \approx a_0 + a_1 t$$

Then define, for q a nonnegative integer $W_t = \frac{1}{2q+1}\sum_{j=-q}^{q} X_{t+j}$

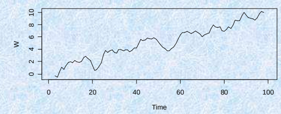
$$W_t = \frac{1}{2q+1}\sum_{j=-q}^{q} X_{t+j} = \frac{1}{2q+1}\sum_{j=-q}^{q}\left(m_{t+j} + Y_{t+j}\right) =: \hat{m}_t \approx m_t$$

$\Rightarrow$ $\hat{Y}_t = X_t - \hat{m}_t$

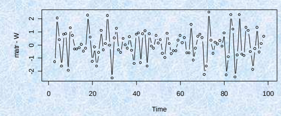
Moving average trend estimation in *R*

in absence of seasonality

Basic decomposition: $X_t = m_t + Y_t$, (Y_t) stationary




```
> W<-filter(matr,rep(1/5,5))
> par(mfrow=c(2,1))
> plot(W)
> plot(matr-W,type="b")
```



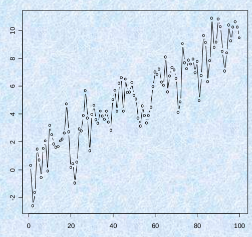
Problem with prediction: trend estimate at t+1 depends on future observations

Estimation and elimination of trend

in absence of seasonality

Basic decomposition:

$X_t = m_t + Y_t$, (Y_t) stationary



Approaches to deal with this process
- Preliminary LS estimation of m_t
- Filter (X_t); taking local moving average
- Difference (X_t)

} especially useful for linear m_t

Differencing to remove trend

in absence of seasonality

Basic decomposition: $X_t = m_t + Y_t$, (Y_t) stationary

Assume the trend function is (approximately) linear

$$m_t \approx a_0 + a_1 t$$

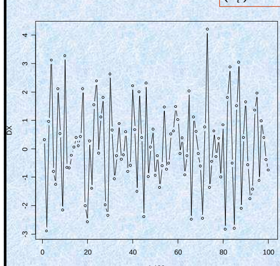
Then define $\nabla X_t = X_t - X_{t-1}$

$$\Rightarrow \nabla X_t = \nabla m_t + \nabla Y_t \approx a_1 + Y_t - Y_{t-1}$$ stationary!

Differencing trend removal in R

in absence of seasonality

Basic decomposition: $X_t = m_t + Y_t$, (Y_t) stationary



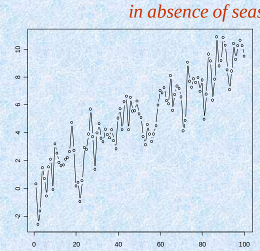

```
> DX<-diff(c(0,matr))
> plot(1:100,DX,type="b")
```

Estimation and elimination of trend

in absence of seasonality

Basic decomposition:

$X_t = m_t + Y_t$, (Y_t) stationary



Approaches to deal with this process
- Preliminary LS estimation of m_t
- Filter (X_t); taking local moving average
- Difference (X_t)

} especially useful for linear m_t

Estimation and elimination of trend and seasonal component

Basic decomposition:

$$X_t = m_t + s_t + Y_t$$
(Y_t) stationary
$$s_{t+d} = s_t, \sum_{i=1}^{d} s_i = 0$$

Three methods:

- small trend method
  trend assumed to be constant within each period
- filtering
  first take moving average such that seasonal component cancels and obtain trend estimate; then estimate the seasonal component
- differencing
  construct stationary time series by considering differences

The small-trend method

$$X_t = m_t + s_t + Y_t$$
(Y_t) stationary, mean 0
$$s_{t+d} = s_t, \sum_{i=1}^{d} s_i = 0$$

$$X_{j,k} = m_j + s_k + Y_{j,k}$$
$$X_{j,k} = X_{k+d(j-1)}$$

Assume trend is constant during each period
(compare decomposition in ANOVA models)

$$\hat{m}_j = \frac{1}{d}\sum_{k=1}^{d} X_{j,k} = \frac{1}{d}\sum_{k=1}^{d}\left(m_j + s_k + Y_{j,k}\right) = m_j + \frac{1}{d}\sum_{k=1}^{d} Y_{j,k} \approx m_j$$

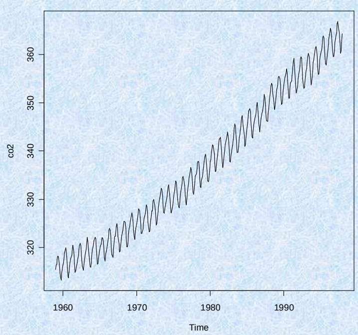
$$\hat{s}_k = \frac{1}{J}\sum_{j=1}^{J}\left(X_{j,k} - \hat{m}_j\right) = s_k + \frac{1}{J}\sum_{j=1}^{J}\left(m_j - \hat{m}_j\right) + \frac{1}{J}\sum_{j=1}^{J} Y_{j,k} \approx s_k$$

$$\hat{Y}_{j,k} = X_{j,k} - \hat{m}_j - \hat{s}_k$$ *Estimated stationary noise component*

CO_2 data

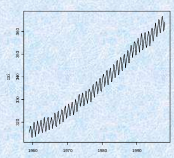
Small trend method

Atmospheric concentrations of CO_2 are expressed in parts per million (*ppm*) and reported in the preliminary 1997 SIOmanometric mole fraction scale.

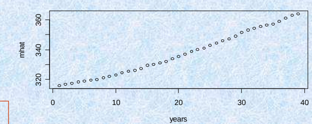


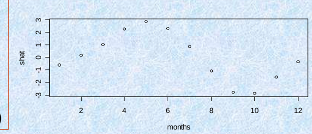
CO_2 data

Small trend method



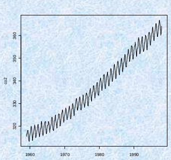
```
> co2m<-matrix(co2,
+ byrow=TRUE,ncol=12)
> mhat<-apply(co2m,1,
+ mean)
> shat<-apply(
+ co2m-mhat,2,mean)
> par(mfrow=c(2,1))
> plot(mhat,xlab="years")
> plot(shat,xlab="months")
```





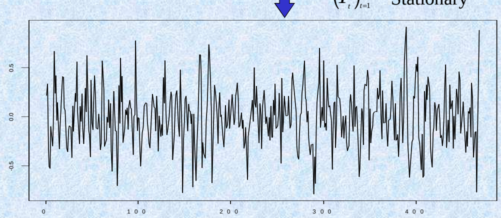
CO_2 data

Small trend method



```
> Yhat<-co2m-outer(mhat,shat,"+")
> plot(as.vector(t(Yhat)),xlab="months",
+ ylab="",type="l")
```

$(\hat{Y}_t)_{t=1}^{468}$ "Stationary"



Estimation and elimination of trend and seasonal component

Basic decomposition:

$$X_t = m_t + s_t + Y_t$$
(Y_t) stationary
$$s_{t+d} = s_t, \sum_{i=1}^{d} s_i = 0$$

Three methods:

- small trend method
  trend assumed to be constant within each period
- filtering
  first take moving average such that seasonal component cancels and obtain trend estimate; then estimate the seasonal component
- differencing
  construct stationary time series by considering differences

Filtering X_t

$X_t = m_t + s_t + Y_t$
(Y_t) stationary
$s_{t+d} = s_t, \sum_{i=1}^{d} s_i = 0$

Suppose $d=2q+1$

$$\hat{m}_t = \frac{1}{d}\sum_{j=-q}^{q} X_{t+j} = \frac{1}{d}\sum_{j=-q}^{q}\left(m_{t+j} + s_{t+j} + Y_{t+j}\right)$$
$$= \frac{1}{d}\sum_{j=-q}^{q} m_{t+j} + \frac{1}{d}\sum_{j=-q}^{q} Y_{t+j} \approx \frac{1}{d}\sum_{j=-q}^{q} m_{t+j} = m_t$$

in case of a linear trend

$$V_k = \frac{1}{n-2q}\sum_j \left(X_{k+jd} - \hat{m}_{k+jd}\right)$$

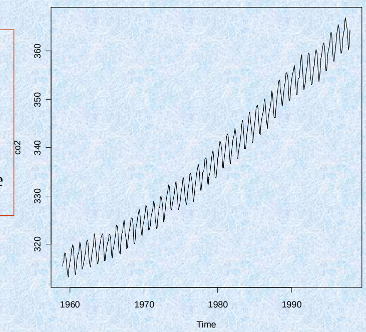
$$\hat{s}_k = V_k - \frac{1}{d}\sum_{i=1}^{d} V_i$$ (to ensure that the sum equals zero)

$\Rightarrow$ $\hat{Y}_t = X_t - \hat{m}_j - \hat{s}_k$ $\Rightarrow$ *Estimated stationary noise component*

CO_2 data

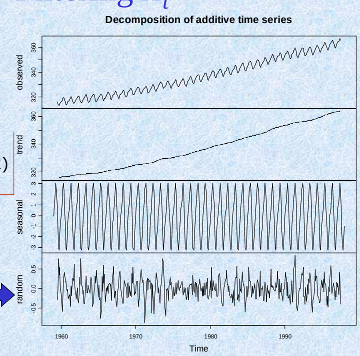
Filtering method

Atmospheric concentrations of CO_2 are expressed in parts per million (*ppm*) and reported in the preliminary 1997 SIOmanometric mole fraction scale.



Filtering X_t

```
> library(ts)
> m<-decompose(co2)
> plot(m)
```



"Stationary" $(\hat{Y}_t)_{t=1}^{468}$ $\Rightarrow$

Estimation and elimination of trend and seasonal component

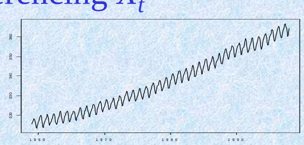
Basic decomposition: $X_t = m_t + s_t + Y_t$
(Y_t) stationary
$s_{t+d} = s_t, \sum_{i=1}^{d} s_i = 0$

Three methods:

- small trend method
  trend assumed to be constant within each period
- filtering
  first take moving average such that seasonal component cancels and obtain trend estimate; then estimate the seasonal component
- $\Rightarrow$ differencing
  construct stationary time series by considering differences

Differencing X_t

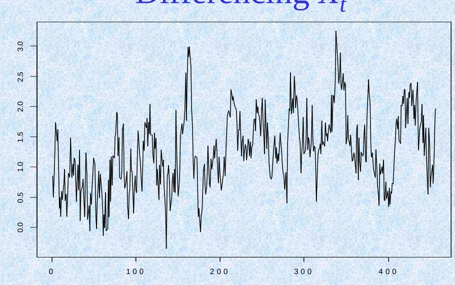
$X_t = m_t + s_t + Y_t$
(Y_t) stationary
$s_{t+d} = s_t, \sum_{i=1}^{d} s_i = 0$



$$\nabla_d X_t = X_t - X_{t-d} = m_t - m_{t-d} + s_t - s_{t-d} + Y_t - Y_{t-d}$$
$$= \underbrace{m_t - m_{t-d}}_{\text{trend}} + \underbrace{Y_t - Y_{t-d}}_{\text{stationary process}}$$

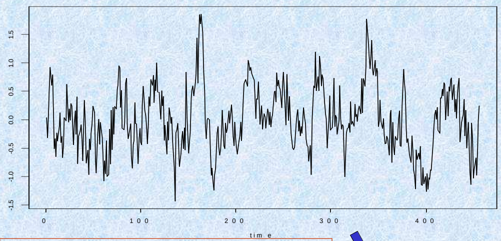
Trend can now be estimated by one of the standard methods

Differencing X_t




```
> DX<-diff(co2m) # difference taken for each column
> plot(as.vector(t(DX)),type="l",xlab="time",ylab="")
```

Differencing X_t *detrending*



```
> Z<-as.vector(t(DX))
> t<-1:length(Z)
> Z.fit<-lm(Z~t) # extract linear trend
> Y<-Z-predict(Z.fit)
> plot(t,Y,type="l",xlab="time",ylab="")
```

'Stationary process'

Where are we?

Aim: summarize structure of time series, identify structural and random part of behavior

Usual approach
- plot time series (time plot)
- extract systematic (seasonal or trend) component of time series
- model remaining time series by standard stationary time series

Having detrended and de-seasonalized, an approximately stationary time series is extracted that can be modelled using a basic *ARMA* model

Choose a specific ARMA model
Estimate parameters of this model
Diagnose the model

Estimation of the ACF of stationary time series

Let X_t be a stationary time series

Recall: $\mu_t = EX_t$ *mean function*

$\gamma_X(h) = \mathrm{Cov}(X_t, X_{t+h})$ *autocovariance function*

$\rho_X(h) = \mathrm{Corr}(X_t, X_{t+h}) = \gamma_X(h)/\gamma_X(0)$ *autocorrelation function*

These functions can be estimated by their sample counterparts:

$$\hat{\mu}_t \equiv \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\hat{\gamma}_X(h) = \frac{1}{n}\sum_{i=1}^{n-h}\left(X_{t+h} - \overline{X}_n\right)\left(X_t - \overline{X}_n\right)$$

$$\hat{\rho}_X(h) = \hat{\gamma}_X(h)/\hat{\gamma}_X(0)$$

Choice for *MA*, *AR* or *ARMA*

Let X_t be a stationary time series

Estimate the autocorrelation function by its *correllogram* $\hat{\rho}_x(h)$

If correllogram shows clear cut-off at certain lag q, choose *MA(q)*

Estimate the partial autocorrelation function *PACF* by the *partial correllogram*

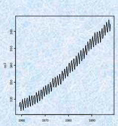
If partial correllogram shows clear cut-off at certain lag p, choose *AR(p)*

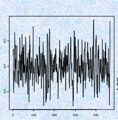
Otherwise choose ARMA model

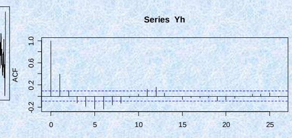
(choice of p and q e.g. based on *AIC*)

Choice of model

Residuals CO2-data, using small trend method

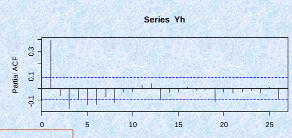







```
> Yh<-as.vector(t(Yhat))
> par(mfrow=c(2,1))
> acf(Yh)
> pacf(Yh)
```

Correllogram and partial correllogram



We will illustrate AR(3)

Estimation of the parameters in *ARMA* model

Maximum Likelihood, Least Squares, Substitution (in *AR*-model, using Yule Walker equations). Yule-Walker: solve

$$\hat{\gamma}_X(h) = \begin{cases} \sum_{j=1}^{p} \alpha_j \hat{\gamma}_X(h-j) & h > 0 \\ \sum_{j=1}^{p} \alpha_j \hat{\gamma}_X(j) + \hat{\sigma}^2 & h = 0 \end{cases}$$

```
> YhAR3<-ar(Yh,method="yule-walker",order.max=3)
> YhAR3
Call: ar(x = Yh, order.max = 3, method = "yule-walker")
Coefficients:
      1        2        3
 0.4052   0.0082   -0.1669
Order selected 3  sigma^2 estimated as  0.06883
```

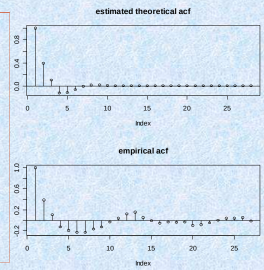
Related:

```
> YhAR32<-ar(Yh,method="mle",order.max=3)
> YhAR33<-arima(Yh,c(3,0,0))
```

Comparing ACF’s

Fitted model: $Y_t = 0.4052Y_{t-1} + 0.0082Y_{t-2} - 0.1669Y_{t-3} + Z_t$
where $(Z_t)_t$ is white noise with variance 0.06883

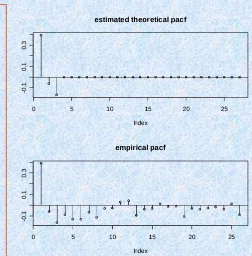
```
> empacf<-acf(Yh,plot=FALSE)
> plot(ARMAacf(ar=
+ c(0.4052,0.0082,-0.1669),
+ lag.max=27),type="h",
+ ylab="",main="estimated
+ theoretical acf")
> points(ARMAacf(ar=c(0.4052,
+ 0.0082,-0.1669),lag.max=27))
> abline(0,0)
> plot(empacf$acf,type="h",
+ ylab="",main="empirical acf")
> points(empacf$acf)
> abline(0,0)
```



Comparing PACF’s

Fitted model: $Y_t = 0.4052Y_{t-1} + 0.0082Y_{t-2} - 0.1669Y_{t-3} + Z_t$
where $(Z_t)_t$ is white noise with variance 0.06883

```
> emppacf<-pacf(Yh,plot=FALSE)
> plot(ARMAacf(ar=
+ c(0.4052,0.0082,-0.1669),
+ lag.max=27,pacf=TRUE),
+ type="h",ylab="",main=
+ "estimated theoretical pacf")
> points(ARMAacf(ar=
+ c(0.4052,0.0082,-0.1669),
+ lag.max=27,pacf=TRUE))
> abline(0,0)
> plot(emppacf$acf,type="h",
+ ylab="",main="empirical pacf")
> points(empacf$acf)
> abline(0,0)
```



Diagnostic checks of model

Most important aspect: residuals!

Basic ingredient of our time series models is a white noise process. This process can be approximated by decomposing the observed time series according to the model formulation, with trend, seasonal component and further parameters estimated

Residuals can be plotted versus time
ACF of residuals can be plotted

Time Series

summary

- Definition and examples of time series
- Stationary time series
  ⇒ general definition
- White noise and other basic stationary time series
  ⇒ *MA*, *AR* and *ARMA* processes
- Models for nonstationarity
  ⇒ trend and seasonality
- Estimating the (partial) autocorrelation function, mean and parameters in *ARMA* model
- Diagnostic checking