Modelling of Osmotic Cell Swelling

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Equations

- Time-varying subdomain C_t.
- Diffusion of solute *inside* C_t:

$$u_t = \kappa \Delta u$$

- Neumann boundary condition on ∂C_t:

$$-\kappa \frac{\partial u}{\partial \mathbf{n}} = uV$$

- Normal velocity by osmosis and surface tension

$$V = \mathcal{P} \cdot (\chi u + \psi H)$$

Main questions

Under various assumptions on permeability $\mathcal{P}$ and surface tension coefficient ψ:

- Uniqueness of equilibrium?
- Existence of solutions?
- Stability of equilibrium?

Linear stability for small ψ

Assumptions

- Radial symmetry:

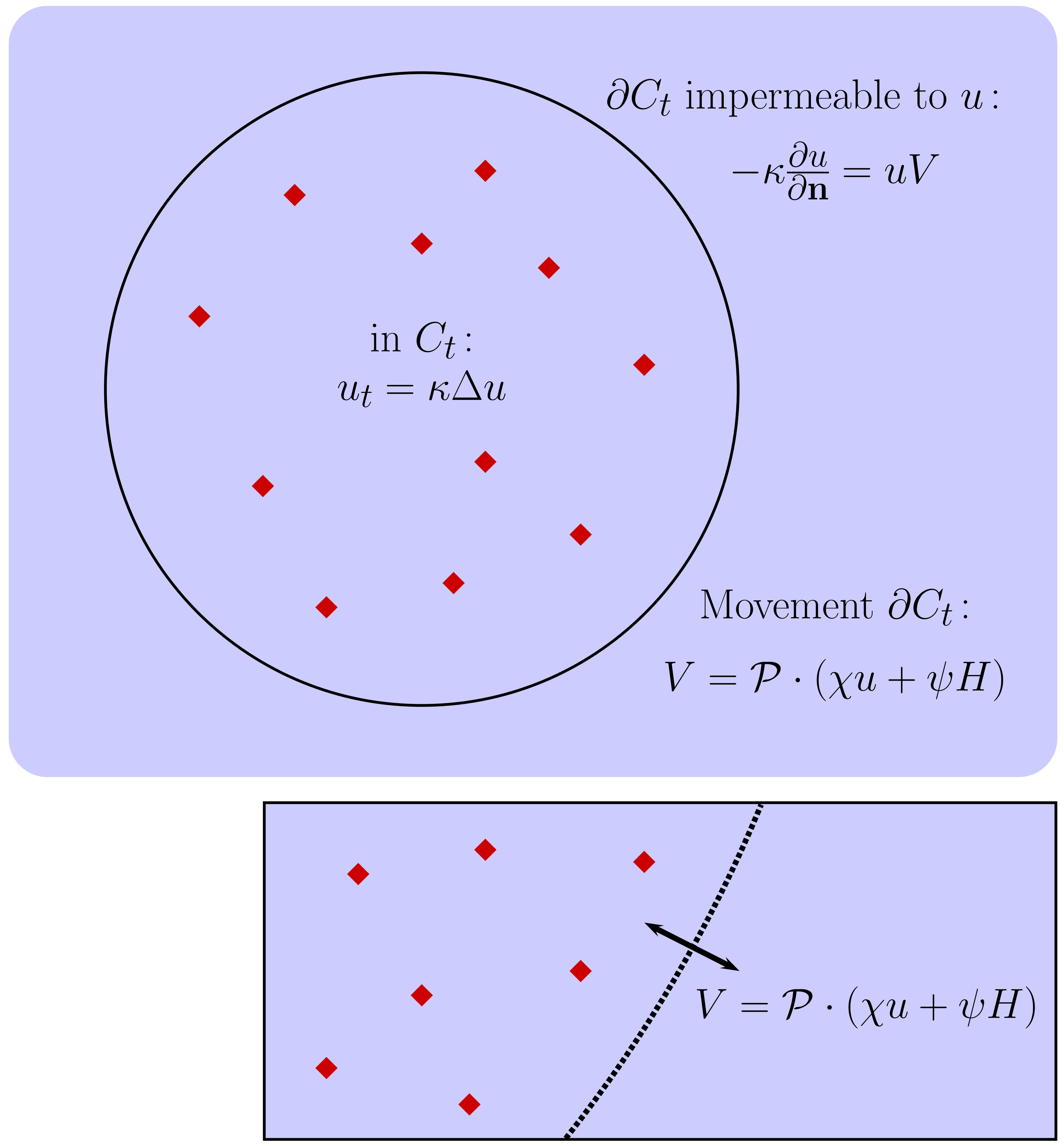
$$C_t = B(0, r(t))$$

- Permeability proportional to surface strain:

$$\mathcal{P} = \frac{1}{r(t)^2}$$

Results

- Unique equilibrium if $\psi > 0$.
- Linearization is stable for small ψ.



Consequences of gradient flow structure

- Well-developed theory on existence of solutions
- Stability of equilibrium guaranteed.

The functional for this problem has a unique critical point:

Equilibrium is globally attracting

Gradient flow approach

Assumption

- Constant permeability

Main idea

- Diffusion minimizes entropy

$$\int_{C_t} u \log u$$

with respect to the Wasserstein metric given by

$$\|\mathbf{q}\|_{L^2(C)}$$

where $\mathbf{q}$ is the flux of solute. The restriction $\operatorname{supp} u \subset C_t$ gives osmotic force.

- In addition, membrane movement minimizes surface area

$$|\partial C_t|$$

with respect to the metric given by

$$\|V\|_{L^2(\partial C_t)}$$

where V is the normal velocity of the membrane ∂C_t.

Gradient flows in metric spaces

Why a metric space

- Space of allowed pairs (u, C) has no linear structure

Problems in metric space

- No notion of velocity or gradient

Solution

- Characterization of gradient flow in Euclidean space

$$\partial_t \phi(x(t)) \leq -\frac{1}{2}|\dot{x}(t)|^2 - \frac{1}{2}|\nabla \phi(x(t))|^2$$

- *Scalar* velocity and *scalar* slope can be generalized.