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Linear Stability of Osmotic Cell Swelling

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Abstract. In this thesis, a simple mathematical model for osmotic cell swelling is developed. Equilibria and linear stability are examined for the radial symmetric case. The implications of including surface tension of the membrane are investigated from a mathematical point of view.

Contents

Introduction

Observations of osmotic effects indicate that cell membranes are permeable to water, but impermeable to other molecules. Therefore, it is believed that water transport through cell membranes is facilitated by proteins acting as water channels, generally called 'aquaporins' ([**1**]). Although there is growing evidence that auqaporins are proteins embedded in the cell membrane ([**12**]), existence has been shown for only a few types of cells. The exact nature of aquaporins remains uncertain, although aqauporins are believed to be proteins ([**5**]). It is important to note that aquaporins merely facilitate water flux through the membrane, and do not drive it: aquaporins are channels, not pumps.

Quantitative modelling of water flux through membranes by aquaporins seems to have started with [**5**]. In this paper, the water flux was assumed to be proportional to the difference in molar free energy inside and outside the cell. However, the model did not predict the eventual slowing of cell swelling. Several other models using this assumption have been considered since, but no model provided a clear connection between the physical mechanism of the problem and the observed results. Furthermore, most models assumed spatially constant concentrations inside and outside the cell.

Recently, Pickard ([**8**]), made a first attempt to develop a specialized model for cell swelling, based on observations of Xenopus oocytes (frog egg cells) submitted to hypoosmotic shocks, that is, sudden exposure to low concentrations of solute. Again, the proposed model uses proportionality of free energy differences and water flux as a starting point. In this model, the surface tension of the cell membrane is not taken into account. This simplification is justified by an estimation argument. Numerical results are obtained after radial symmetry is imposed.

Unfortunately the justification of dropping surface tension does not seem to be valid if the cell is close to equilibrium. Although it should be noted that in most test tube situations the cell will burst long before it reaches equilibrium, the implications of assuming zero surface tension should be investigated. Furthermore, it should be checked if conclusions about the radial symmetric model can be generalized to non-symmetric situations as well. Finally, it is possible to include other phenomena in the model, such as elasticity of the membrane, water flow inside the cell, moving aquaporins etc.

In this thesis, the implications of taking surface tension into account will be investigated from a mathematical point of view. After introducing the model, existence, uniqueness and linear stability of equilibria will be investigated.

CHAPTER 1

Modelling osmosis

In this chapter, a mathematical model will be constructed for the cell, its membrane and the solute. The general domain Ω will be an open ball in three-dimensional space of radius R around the origin: $\Omega := \{x \in \mathbb{R}^3 : \|x\| < R\}$. It will be equipped with standard spherical coordinates

$$
\begin{aligned}
x &= r \sin\phi \cos\theta \\
y &= r \sin\phi \sin\theta \\
z &= r \cos\phi.
\end{aligned}
$$

The time index set will be $[0, \infty)$.

The cell is represented by a time-varying open subset C_t of Ω. For convenience, the following assumptions are made.

Assumption 1.1. At any time $t \geq 0$, the cell C_t is star-shaped with respect to the origin, that is

$$\forall x \in C_t \, \forall \alpha \in [0, 1] : \alpha x \in C_t.$$

By this assumption, ∂C_t can be parametrized by a function $p(\theta, \varphi, t)$. The initial shape of the cell is described by $p^0(\theta, \varphi)$.

1. Diffusion inside and outside the cell

Both inside and outside C_t, the solute can diffuse freely through the solvent. It is assumed that the solvent is stationary, in other words, transport of solute only takes place by means of diffusion.

Assumption 1.2. There is no flux of solvent.

Hence, the transport of solute is described by Fick's law of diffusion:

$$\mathbf{q} = -\kappa \nabla u \tag{1.1}$$

See [**7**] for a more detailed discussion of Fick's law. In this equation, $u : \Omega \times [0, \infty) \to \mathbb{R}$ is the local concentration of solute or *osmolality* at time t, measured in mole per cubic metre ($\mathrm{mol\,m^{-3}}$). The flux of solute, in mole per square metre second ($\mathrm{mol\,m^{-2}\,s^{-1}}$), is represented by a vector $\mathbf{q} : \Omega \times [0, \infty) \to \mathbb{R}^3$. $\kappa : \Omega \times [0, \infty) \to (0, \infty)$ is the diffusivity, which might vary over space and time.

Assumption 1.3. κ is constant both inside and outside C_t.

$$\kappa(x,t) = \begin{cases} \kappa_-, & \text{if } x \in C_t, \\ \kappa_+, & \text{if } x \in \Omega \setminus C_t, \end{cases}$$

with $\kappa_- < \kappa_+$.

The value of $\frac{\kappa_-}{\kappa_+}$ is typically ~ 0.25, see also [**13**]. The initial concentration of solute is denoted by $u^0 : \Omega \to \mathbb{R}$.

Naturally, it is assumed that mass is conserved. Moreover, it is assumed that there are no chemical reactions inside or outside the cell. Therefore, a conservation law can be derived by a simple limiting argument. Let $t \geq 0$ and $x \in \Omega \setminus \partial C_t$ be given. For a small ball of radius $\varepsilon > 0$ around x, the increase of solute after a small time $\Delta t > 0$ is

$$\int_{B_\varepsilon(x)} u(\xi, t + \Delta t) - u(\xi, t)\,\mathrm{d}\xi\,.$$

If u is sufficiently smooth,

$$\begin{aligned} u(\xi, t + \Delta t) - u(\xi, t) &= \Delta t\, u_t(\xi, t) + O(\Delta t^2) \\ &= \Delta t\, u_t(x, t) + O(\|\xi - x\| \Delta t) + O(\Delta t^2) \end{aligned}$$

It follows that the increase of solute is

$$\frac{4\pi}{3}\varepsilon^3 \Delta t\, u_t(x, t) + O(\varepsilon^4 \Delta t) + O(\varepsilon^3 \Delta t^2). \tag{1.2}$$

The net flow of solute into the ball is given by

$$-\int_t^{t+\Delta t} \int_{\partial B_\varepsilon(x)} \mathbf{q}(\xi, \tau) \cdot \mathbf{n}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}\tau = \int_t^{t+\Delta t} \int_{B_\varepsilon(x)} \operatorname{div} \mathbf{q}(\xi, \tau)\,\mathrm{d}\xi,$$

using the divergence theorem. Again assuming some regularity,

$$\operatorname{div} \mathbf{q}(\xi, \tau) = \operatorname{div} \mathbf{q}(x, \tau) + O(\|\xi - x\|) = \operatorname{div} \mathbf{q}(x, t) + O(\|\xi - x\|) + O(\tau - t).$$

Using this estimate,

$$\int_{\partial B_\varepsilon(x)} \mathbf{q}(\xi, \tau) \cdot \mathbf{n}\,\mathrm{d}\sigma(\xi) = \frac{4\pi}{3}\varepsilon^3 \operatorname{div} \mathbf{q}(x, t) + O(\varepsilon^4) + O(\varepsilon^3(\tau - t)).$$

Then the total flow solute into the ball during the time interval is

$$-\frac{4\pi}{3}\varepsilon^3 \Delta t\, \operatorname{div} \mathbf{q}(x, t) + O(\varepsilon^4 \Delta t) + O(\varepsilon^3 \Delta t^2). \tag{1.3}$$

Since the increase of solute should be equal to the net flow into the ball, (1.2) and (1.3) have to be equal:

$$-\operatorname{div} \mathbf{q}(x, t) + O(\varepsilon) + O(\Delta t) = u_t(x, t) + O(\varepsilon) + O(\Delta t).$$

Letting $\varepsilon, \Delta t \downarrow 0$,

$$-\operatorname{div} \mathbf{q} = u_t. \tag{1.4}$$

Combining (1.1) and (1.4) gives the diffusion equation

$$u_t = -\operatorname{div}(-\kappa \nabla u) = \kappa \Delta u \tag{1.5}$$

in C and $\Omega \times [0, \infty) \setminus C$.

Furthermore, it is assumed that the system is closed, that is, there is no flux accros $\partial\Omega$,

$$\mathbf{q}(R,\theta,\varphi,t)\cdot\mathbf{n} = -\kappa_+ \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{(R,\theta,\varphi,t)} = 0. \tag{1.6}$$

2. Impermeability to solute

The cell membrane, represented by ∂C_t is assumed to be permeable to the solvent, but impermeable to the solute. Using a limiting argument similar to the one above, boundary conditions can be derived.

Since there will be boundary conditions both on the inside and outside of ∂C_t, some additional notation is needed to describe the behaviour of functions on the boundary.

Definition 1.4. Let f be a function depending on x, and possibly other variables. Define

$$f\left(x^+,.\right) := f\left(r^+,\theta,\varphi,.\right) = \lim_{\rho\downarrow r} f(\rho,\theta,\varphi,.),$$
$$f\left(x^-,.\right) := f\left(r^-,\theta,\varphi,.\right) = \lim_{\rho\uparrow r} f(\rho,\theta,\varphi,.).$$

Denote the *jump* of f across x by

$$[f](x,.) := f\left(x^+,.\right) - f\left(x^-,.\right).$$

Let then $t \geq 0$ and $x \in C_t$ be given. Consider a cylinder of radius $\varepsilon > 0$, with the axis intersecting ∂C_t at x, normal to ∂C_t. Bound this cilinder by a fixed disk at distance ε from x and the portion of ∂C_t inside the membrane. This is a cylinder-like portion of space, with one face evolving through time. Denote this small portion of space at time τ by D_τ ($\tau \in (t, t+\Delta t)$). After a small time $\Delta t > 0$, the face containing x has moved a little due to boundary movement, while the other face hasn't moved. Furthermore, a small amount of solute has been transported into or out of the cylinder.

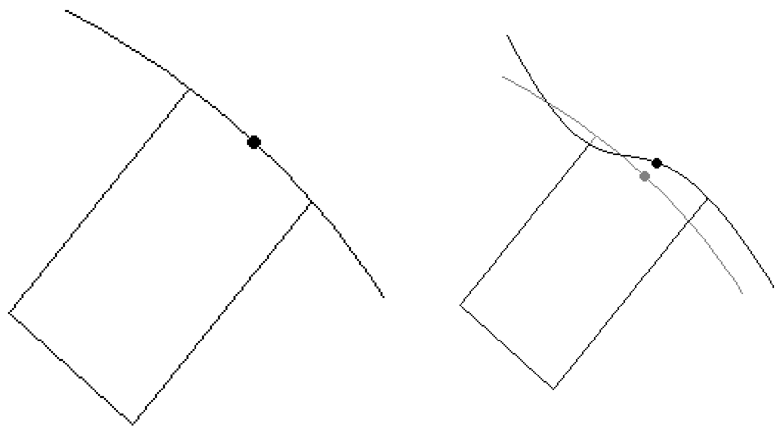

Figure 1. Impression of the cilinder before and after a small time interval

The flux into the cylinder is given by

$$\int_{\partial D_\tau} \mathbf{q}(\xi,\tau)\cdot\mathbf{n}\,\mathrm{d}\sigma(\xi)\,.$$

Since the membrane is impermeable to solute, and $\mathbf{q}(\xi,\tau) = \mathbf{q}(x^-,\tau) + O(\varepsilon) = \mathbf{q}(x^-,t) + O(\tau - t) + O(\|\xi - x\|)$, the contribution to this integral of the side of D_τ is of small order, leaving only the face away from ∂C_τ. It follows that this integral is

$$\pi\varepsilon^2\mathbf{q}(x^-,t)\cdot\mathbf{n} + O(\varepsilon^2(\tau - t)) + O(\varepsilon^3),$$

it follows that the amount of solute has increased with

$$\pi\varepsilon^2\Delta t\mathbf{q}(x^-,t)\cdot\mathbf{n} + O(\varepsilon^2\Delta t^2) + O(\varepsilon^3\Delta t).$$

On the other hand, the difference between the amount of solute before and after the time interval is also equal to

$$\int_{D_{t+\Delta t}} u(\xi,t+\Delta t)\,\mathrm{d}\xi - \int_{D_t} u(\xi,t)\,\mathrm{d}\xi\,.$$

As before,

$$u(\xi,\tau) = u(x^-,t) + O(\|\xi - x\|) + O(\tau - t),$$

which implies that the amount of solute inside D_τ is $u(x^-,t)$ times the volume of D_τ plus some higher order terms. Note that the volume of D_τ is given by

$$\pi\varepsilon^3 + \pi\varepsilon^2 v_{\mathbf{n}}(x,t)\Delta t + O(\varepsilon^2\Delta t^2),$$

where $v_{\mathbf{n}}(x,t) : \partial C \to \mathbb{R}$ is the normal velocity of ∂C_t. It follows that

$$\pi\varepsilon^2\Delta t\,\mathbf{q}(x^-,t)\cdot\mathbf{n} + O(\varepsilon^2\Delta t^2) + O(\varepsilon^3\Delta t) = u(x^-,t)\pi\varepsilon^2 v_{\mathbf{n}}(x,t)\Delta t + O(\varepsilon^2\Delta t^2).$$

Dividing by $\varepsilon^2\Delta t$ and letting $\varepsilon, \Delta t \downarrow 0$, this gives

$$u(x^-,t)v_{\mathbf{n}}(x,t) = \mathbf{q}(x^-,t)\cdot\mathbf{n}. \tag{1.7}$$

on ∂C. A similar argument for a cylinder on the outside gives

$$u(x^+,t)v_{\mathbf{n}}(x,t) = \mathbf{q}(x^+,t)\cdot\mathbf{n}. \tag{1.8}$$

3. Transport of solvent

Since the solvent is assumed to be stationary, transport of solvent through the membrane can be modeled by movement of the boundary. In order to derive a boundary condition, a few assumptions are needed.

Assumption 1.5. There is no tangential movement of the boundary, that is, the velocity field of the membrane is $v_{\mathbf{n}}\mathbf{n}$.

Assumption 1.6. The solvent is incompressible.

Both assumption seem to be reasonable. Since the bathing solution and the cytoplasm are water-like fluids, it is reasonable to assume incompressibility. Furthermore, it seems natural that the membrane will only move in the direction of its normal as a result of osmosis and surface tension.

At any time $t \geq 0$, consider a portion of the boundary S_t with θ, φ in a small domain

$$D_t := (\theta_0 - \Delta\theta, \theta_0 + \Delta\theta) \times (\varphi_0 - \Delta\varphi, \varphi_0 + \Delta\varphi)$$

for small $\Delta\theta, \Delta\varphi > 0$. Suppose that a given amount of mass crosses this portion of the membrane during a time interval of length Δt. By incompressibility, this is equivalent to a fixed volume V crossing this portion of the boundary

$$V = \iint_D \Delta t \, v_{\mathbf{n}}(\theta, \varphi, t) + O(\Delta t^2) \, \mathrm{d}\theta \, \mathrm{d}\varphi \, .$$

Assuming smoothness of the membrane, this can be approximated by

$$\Delta t \, v_{\mathbf{n}}(\theta_0, \varphi_0, t) + O(\Delta t |\theta - \theta_0|) + O(\Delta t \, |\varphi - \varphi_0|) + O(\Delta t^2).$$

It follows that

$$V = A(t)\Delta t \, v_{\mathbf{n}}(\theta_0, \varphi_0, t) + A(t)O(\Delta t \, \Delta\theta) + A(t)O(\Delta t \, \Delta\varphi) + A(t)O(\Delta t^2), \tag{1.9}$$

where $A(t)$ is the surface area of the small part of the membrane at time t, which is given by

$$A(t) = \iint_D \left\| \frac{\partial x}{\partial \theta} \times \frac{\partial x}{\partial \varphi} \right\| \mathrm{d}\theta \, \mathrm{d}\varphi \, .$$

The volume V of transported solvent is determined by the permeability of the membrane $\mathcal{P}(\theta, \varphi, t)$ ($\mathrm{m}^3 \, \mathrm{mol} \, \mathrm{s}^{-1} \, \mathrm{J}^{-1}$) and the jump in Helmholtz free energy $[\mu] : \partial C \to \mathbb{R}$ ($\mathrm{J} \, \mathrm{mol}^{-1}$) across the membrane. Integrating over S_t now gives

$$\begin{aligned} V &= \iint_{D_t} \mathcal{P}(\theta, \varphi, t)[\mu](\theta, \varphi, t) \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ &= A(t)\mathcal{P}(\theta_0, \varphi_0, t) + A(t)O(\Delta\theta) + A(t)O(\Delta\varphi) \end{aligned} \tag{1.10}$$

Setting (1.9) and (1.10) equal, dividing by $A(t)$ and letting $\Delta\theta, \Delta\varphi \downarrow 0$ gives

$$v_{\mathbf{n}} = [\mu]\mathcal{P} \tag{1.11}$$

The permeability of the membrane is determined by the density of aquaporins. The following assumptions are made about the aquaporins.

Assumption 1.7. The aquaporins are fixed in the membrane.

Assumption 1.8. An aquaporin is assumed to have a fixed permeance $\mathfrak{p}$.

These assumptions imply that the total permeability of S_t is constant in time. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{S_t} \mathcal{P} = \frac{\mathrm{d}}{\mathrm{d}t} \iint_{D_t} \mathcal{P}(\theta, \varphi, t) \left\| \frac{\partial x}{\partial \theta} \times \frac{\partial x}{\partial \varphi} \right\| \mathrm{d}\theta \, \mathrm{d}\varphi = 0.$$

By smoothness of the membrane and assuming some smoothness of $\mathcal{P}$, letting $\Delta\theta, \Delta\varphi \downarrow 0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{P}(\theta,\varphi,t)\left\|\frac{\partial x}{\partial\theta}\times\frac{\partial x}{\partial\varphi}\right\|\right) = 0.$$

Gauss' formula (see, for instance, [**3**]) can be used to describe the evolution of the cross product:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial x}{\partial\theta}\times\frac{\partial x}{\partial\varphi}\right\| = -2Hv_{\mathbf{n}}\left\|\frac{\partial x}{\partial\theta}\times\frac{\partial x}{\partial\varphi}\right\|$$

where H is the mean curvature of the membrane. Together, these equations give

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} = 2Hv_{\mathbf{n}}\mathcal{P}. \tag{1.12}$$

Unfortunately, this equation cannot be translated into a condition on $\mathcal{P}$ relative to the coordinates θ, φ in a straightforward way, since D_t might change over time.

For fixed θ, φ, the norm of the cross product can be computed:

$$\frac{\partial x}{\partial\theta} = \begin{pmatrix} p_\theta(\theta,\varphi,t)\sin\varphi\cos\theta - p(\theta,\varphi,t)\sin\varphi\sin\theta \\ p_\theta(\theta,\varphi,t)\sin\varphi\sin\theta + p(\theta,\varphi,t)\sin\varphi\cos\theta \\ p_\theta(\theta,\varphi,t)\cos\varphi \end{pmatrix},$$

$$\frac{\partial x}{\partial\varphi} = \begin{pmatrix} p_\varphi(\theta,\varphi,t)\sin\varphi\cos\theta + p(\theta,\varphi,t)\cos\varphi\cos\theta \\ p_\varphi(\theta,\varphi,t)\sin\varphi\sin\theta + p(\theta,\varphi,t)\cos\varphi\sin\theta \\ p_\varphi(\theta,\varphi,t)\cos\varphi - p(\theta,\varphi,t)\sin\varphi \end{pmatrix},$$

$$\left\|\frac{\partial x}{\partial\theta}\times\frac{\partial x}{\partial\varphi}\right\| = p(\theta,\varphi,t)^2\sqrt{\sin^2\varphi + \frac{p_\theta(\theta,\varphi,t)^2 + p_\varphi(\theta,\varphi,t)^2}{p(\theta,\varphi,t)^2}}.$$

4. Helmholtz free energy

For this model, the Helmholtz free energy consists of two relevant terms: the osmolality and the hydrostatic pressure. The jump $[\mu]$ is given by

$$[\mu](\theta,\varphi,t) = \mathcal{V}[P](\theta,\varphi,t) - \mathcal{V}\mathcal{R}\mathcal{T}[u](\theta,\varphi,t). \tag{1.13}$$

For an introductory text on chemical thermodynamics and Helmholtz free energy in particular, see [**11**]. In this formula, $\mathcal{V}$ is the molar volume of water, approximately $18\cdot 10^{-6}\mathrm{m}^3\,\mathrm{mol}^{-1}$. The hydrostatic pressure is denoted by P, $\mathcal{R}$ is the gas constant and $\mathcal{T}$ is the absolute temperature, which is assumed to be constant.

The jump $[P]$ of the hydrostatic pressure is caused by the surface tension of the membrane. Surface tension is not a material property, but a property of interfaces between materials. In order to enlarge a surface between two materials, work is required. Surface tension can be defined as the amount of work needed to enlarge a surface by one square metre. The behaviour films, like soap bubbles, can also be explained by surface tension, since they can be viewed as thin bodies of some material with *two* interfaces. The surface tension, usually denoted by γ ($\mathrm{N\,m^{-1}}$), is assumed to be constant:

ASSUMPTION 1.9. The surface tension does not depend on the surface strain.

Note that this assumption, together with 1.5 eliminates the surface strain from the *mechanical* behaviour of the membrane. Naturally, this assumption is fairly restrictive: a swollen cell will have a larger surface tension than a cell which is not swollen. However, the tendency of minimizing the membrance surface implied by surface tension is correct.

Using the Young-Laplace equation ([**10**], (13–5)), and the identity

$$H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where H is the mean curvature and R_1, R_2 are the principal radii, it follows that

$$[P] = -4\gamma H. \tag{1.14}$$

In case ∂C_t is a sphere, $H = \frac{1}{R}$. For a more explicit formula for H, see [**6**] p. 500–501. Note that this boundary condition will, in general, result in a nonconstant pressure difference, and hence in a nonconstant pressure on at least one side of the membrane. The solvent flow resulting from this nonconstant pressure is not taken into account by assumption 1.2.

5. The complete problem

The equations (1.5), (1.6), (1.7), (1.8), (1.11), (1.12), (1.13), (1.14) give the complete boundary value problem:

$$\begin{cases} u_t = \kappa \Delta u, & \text{in } (\Omega \times [0,\infty)) \setminus \partial C, \\ -\kappa \dfrac{\partial u}{\partial \mathbf{n}} = u v_{\mathbf{n}}, & \text{in- and outside } \partial C, \\ v_{\mathbf{n}} = -\dfrac{\mathcal{P}}{a}\left(\chi[u] + \psi H\right), & \text{on } \partial C, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} = 2H v_{\mathbf{n}} \mathcal{P}, & \text{on } \partial C, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial \Omega, \end{cases} \tag{1.15}$$

where $\chi := \mathcal{V}\mathcal{R}\mathcal{T} > 0$ and $\psi := 4\mathcal{V}\gamma > 0$.

Scaling time and space will result in a change in the coefficients in the first and third equation. The second and fourth equation will not change. Hence, it can be assumed, without loss of generality, that $\kappa_+ = 1$ and $R = 1$. This will result in a change of units, and different values for $\chi > 0$ and $\psi > 0$.

By construction, this problem has two conserved quantities: the total amount of solute inside and outside the cell, given by the integral of u over C_t and $\Omega \setminus C_t$, respectively. Denote these quantities by U_-, U_+, respectively, and set $U := U_- + U_+$.

In case of radial symmetry, the model can be simplified. Since the cell is spherical in this case, normal derivatives reduce to derivatives with respect to p and derivatives with respect to φ and θ disappear. Moreover, the equation for $\mathcal{P}$ can be eliminated:

$$\begin{aligned}\mathcal{P}(\theta,\varphi,t) &= \mathcal{P}(\theta,\varphi,0)\frac{\left\|\frac{\partial x}{\partial\theta}\Big|_t \times \frac{\partial x}{\partial\varphi}\Big|_t\right\|}{\left\|\frac{\partial x}{\partial\theta}\Big|_0 \times \frac{\partial x}{\partial\varphi}\Big|_0\right\|} \\ &= \mathcal{P}^0\left(\frac{p(t)}{p(0)}\right)^2\end{aligned}$$

where $\mathcal{P}^0$ is the inital permeability of the cell membrane. The mean curvature and normal velocity can also be expressed in terms of $p(t)$

$$\begin{aligned}&H = \frac{1}{p(t)},\\ &v_{\mathbf{n}}(t) = \dot{p}(t).\end{aligned}$$

Using these observations, the radial symmetric boundary value problem becomes

$$\begin{cases} u_t = \kappa\Delta u, & \text{in } (\Omega\setminus\partial C_t)\times[0,\infty),\\ -\kappa u_r = u\dot{p}, & \text{in- and outside } \partial C,\\ \dot{p} = -\dfrac{\chi}{p^2}[u] - \dfrac{\psi}{p^3}, & \\ u_r = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

where the constants, $\chi, \psi > 0$ are again scaled. Note that differentiability of u in $r = 0$ imposes $u_r = 0$. This additional condition will not be mentioned explicitly, since u is seen a function on Ω.

CHAPTER 2

Linearization

In order to study the stability of (1.16), it is first necessary to locate the equilibria and linearize the problem around these equilibria. The main difficulty of linearizing this problem is the free boundary. A coordinate transformation will be introduced to fix the free boundary. Finally, a concept of weak solution to the linearized problem will be introduced.

1. Equilibrium

The nature of the problem suggests a unique equilibrium. It turns out that this suggestion is correct, at least for non-degenerate parameter values.

THEOREM 2.1. *Given u^0 and p^0, such that $U_-, U_+ > 0$ with $u^0(x,t) \geq 0$ and $0 < p^0 < 1$, (1.16) has a unique equilibrium $(\overline{u}(x), \overline{r})$ with*

$$\overline{u}(x) = \begin{cases} \overline{u}_-, & \text{if } \|x\| < \overline{r}, \\ \overline{u}_+, & \text{if } \|x\| > \overline{r}, \end{cases}$$

where $0 < \overline{r} < 1$ and $\overline{u}_- > \overline{u}_+ > 0$ are constants.

PROOF. An equilibrium of (1.16) satisfies

$$u_t \equiv \dot{p} \equiv 0,$$

Naturally, this means that $p(t)$ is constant, which means that $C_t = C$ is time-independent. Since U_- and U_+ are conserved quantities, $\overline{r} \in (0,1)$. Furthermore, $u_r = 0$ both inside and outside ∂C. Then, using integration by parts,

$$0 = \int_C u\Delta u \,\mathrm{d}x = -\int_C \|\nabla u\|^2 \,\mathrm{d}x + \int_{\partial C} u\frac{\partial u}{\partial \mathbf{n}} \,\mathrm{d}\sigma = \int_C \|\nabla u\|^2,$$

using $u_t = 0$. Since also $\Delta u = 0$ inside C_t, $\|\nabla u\| = 0$ almost everywhere, and hence u is constant inside C_t. Similarly, u is also constant outside C_t. Since $U_-, U_+ > 0$, $\overline{u}_-$ and $\overline{u}_+$ are also both positive. Moreover,

$$\frac{\chi}{\overline{r}^2}[\overline{u}] + \frac{\psi}{\overline{r}^3} = 0,$$

$$[\overline{u}] = -\frac{\psi}{\chi\overline{r}} < 0 \tag{2.1}$$

which means in particular that $\overline{u}_- > \overline{u}_+$. Finally, the impermeability conditions give

$$\int_{C_t} u(x,t)\,\mathrm{d}x = \overline{u}_- \frac{4\pi}{3}\overline{r}^3 = U_-,$$

$$\int_{\Omega\setminus C_t} u(x,t)\,\mathrm{d}x = \overline{u}_+ \frac{4\pi}{3}(1-\overline{r}^3).$$

Substituting this into conservation of mass gives

$$U_- + U_+ = \frac{4\pi}{3}\left(\overline{u}_-\overline{r}^2 + \overline{u}_+(1-\overline{r}^3)\right) = \frac{4\pi\psi}{3\chi}\overline{r}^2 + \frac{U_+}{1-\overline{r}^3}$$

Clearly, the left-hand side of this expression does not depend on $\overline{u}_\pm$ and $\overline{r}$, and the right-hand side is increasing as a fuction of $\overline{r}$. Since the limit of the right-hand side for $\overline{r} \downarrow 0$ is smaller than the left-hand side and $+\infty$ for $\overline{r} \uparrow 1$, there is a unique solution. Given this unique solution $\overline{r}$, there is a unique solution for $\overline{u}_\pm$ as well. □

Note that leaving the curvature out of the equation, which is equivalent to setting $\psi = 0$, would result in a unique equilibrium satisfying $\overline{u}_+ = \overline{u}_-$: the equilibrium concentration would be constant, not just piecewise constant.

It turns out to be convenient to describe models by their equilibrium instead of their initial conditions. Given $\overline{u}$, χ and $\overline{r}$, setting

$$\psi = -[\overline{u}]\chi\overline{r}$$

will result in $([\overline{u}], \overline{r})$ being a stationary solution.

From now on, only perturbations of a given equilibrium will be considered, denoted by

$$v = u - \overline{u},$$
$$s = p - \overline{r}.$$

Note that $u_r = v_r$, $\Delta u = \Delta v$ if $r \neq p(t)$, $u_t = v_t$ and $\dot{p} = \dot{s}$.

2. Fixing the free boundary

The nonlinearity of (1.16) is due completely to the (nonlinear) movement of the boundary. A suitable change of variables can be used to fix the free boundary ∂C_t, although this will lead to nonlinearity in the other equations.

In order to study stability of the equilibrium, it is convenient to define new independent variables (ρ, τ) by

$$r = \rho + \Phi(\rho)s(\tau),$$
$$t = \tau,$$

where $\Phi : [0,1] \to [0,1]$ is a twice continuously differentiable function with $\operatorname{supp}\Phi \subset (0,1)$ and $\Phi \equiv 1$ in a neighborhood of $\overline{r}$. Note that this transformation maps points x in Ω to points ξ in $\mathbb{R}^3$.

The Jacobian of this transformation is

$$\begin{pmatrix} 1 + \Phi'(\rho)s(\tau) & \Phi(\rho)\dot{s}(\tau) \\ 0 & 1 \end{pmatrix},$$

which is invertible if $\Phi'(\rho)s(\tau) > -1$. Given Φ, this is certainly the case for sufficiently small s, that is, for $p(t)$ sufficiently close to $\overline{r}$. Since $\Phi(0) = \Phi(1) = 0$, the fixed boundaries $r = 0$ and $r = 1$ are preserved. Since $\Phi'(0) = 0$, the transformation is differentiable. Finally, since $\Phi(\overline{r}) = 1$, the free boundary $r = p(t)$ is mapped to $\rho = \overline{r}$. The requirement $\Phi \equiv 1$ for a neighborhood of $\overline{r}$ will result in more simple expressions for the boundary conditions.

Consider then the differential operators with respect to ρ and τ:

$$\begin{aligned}
\frac{\partial}{\partial \rho} &= \frac{\partial r}{\partial \rho}\frac{\partial}{\partial r} + \frac{\partial t}{\partial \rho}\frac{\partial}{\partial t} = (1 + \Phi'(\rho)s(\tau))\frac{\partial}{\partial r} \\
\frac{\partial}{\partial \tau} &= \frac{\partial r}{\partial \tau}\frac{\partial}{\partial r} + \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} = \Phi(\rho)\dot{s}(\tau)\frac{\partial}{\partial r} + \frac{\partial}{\partial t}
\end{aligned}$$

Hence, the differential operators with respect to r and t are

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{1}{1 + \Phi'(\rho)s(\tau)}\frac{\partial}{\partial \rho}, \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} - \frac{\Phi(\rho)\dot{s}(\tau)}{1 + \Phi'(\rho)s(\tau)}\frac{\partial}{\partial \rho}.
\end{aligned}$$

Applying these operators to v gives

$$\begin{aligned}
v_r &= \frac{1}{1 + \Phi'(\rho)s(\tau)}v_\rho \\
v_{rr} &= \frac{1}{1 + \Phi'(\rho)s(\tau)}\left(\frac{1}{1 + \Phi'(\rho)s(\tau)}v_{\rho\rho} - \frac{\Phi''(\rho)s(\tau)}{(1 + \Phi'(\rho)s(\tau)^2}v_\rho\right) \\
v_t &= v_\tau - \frac{\Phi(\rho)\dot{s}(\tau)}{1 + \Phi'(\rho)s(\tau)}v_\rho
\end{aligned}$$

Using the above identities, a new boundary problem for (v, s) with respect to coordinates (ρ, τ) can be derived.

$$\text{(2.2)} \qquad \begin{cases}
v_\tau = \dfrac{\kappa}{(1 + \Phi' s)^2}v_{\rho\rho} + F v_\rho, & \text{in } (\Omega \setminus \partial C') \times [0, \infty), \\
-\kappa v_\rho = (\overline{u} + v)\dot{s}, & \text{in- and outside } \partial C', \\
\dot{s} = -\dfrac{\chi}{(\overline{r} + s)^2}[\overline{u} + v] - \dfrac{\psi}{(\overline{r} + s)^3}, & \text{on } \partial C', \\
v_\rho = 0, & \text{on } \partial\Omega,
\end{cases}$$

where

$$F = \frac{\Phi\dot{s}}{1 + \Phi' s} - \frac{\kappa\Phi'' s}{(1 + \Phi' s)^3} + \frac{2\kappa}{(\rho + \Phi s))(1 + \Phi' s)},$$

and C' is the image of C_t. Note that the image of C_t is time-independent, and that F depends on s and $\dot{s}$.

3. Linearization

Since the boundary value problem (2.2) has no more free boundaries, it can be linearized. First, consider the coefficients of the transformed diffusion equation.

$$\frac{\kappa}{(1+\Phi' s)^2} = \kappa + \frac{\kappa\left((1+\Phi' s)^2 - 1\right)}{(1+\Phi' s)^2} = \kappa + \frac{2\kappa\Phi' s + \kappa\Phi'^2 s^2}{(1+\Phi' s)^2} = \kappa + O(s)$$

$$\frac{\Phi\dot{s}}{1+\Phi' s} = O(s)$$

$$\frac{\kappa\Phi'' s}{(1+\Phi' s)^3} = O(s)$$

$$\begin{aligned}\frac{2\kappa}{(\rho+\Phi s))(1+\Phi' s)} &= \frac{2\kappa}{\rho} + \frac{2\kappa\rho - 2\kappa(\rho+\Phi s))(1+\Phi' s)}{\rho(\rho+\Phi s))(1+\Phi' s)} \\ &= \frac{2\kappa}{\rho} - \frac{2\kappa\Phi s(1+\Phi' s) + 2\kappa\rho\Phi' s)}{\rho(\rho+\Phi s)(1+\Phi' s)} = \frac{2\kappa}{\rho} + O(s)\end{aligned}$$

It follows that the linearized diffusion equation is

$$v_\tau = \kappa v_{\rho\rho} + \frac{2\kappa}{\rho} v_\rho = \kappa\Delta v.$$

Consider then the impermeability condition on $\partial C'$.

$$\frac{\kappa}{1+\Phi' s} = \kappa + \frac{\kappa - \kappa(1+\Phi' s)}{1+\Phi' s} = \kappa - \frac{\kappa\Phi' s}{1+\Phi' s} = \kappa + O(s),$$

which means that the linearized impermeability condition on $\partial C'$ becomes

$$-\kappa v_\rho = \overline{u}\dot{s}.$$

Furthermore, in order to linearize the boundary movement equation, note that

$$\frac{\chi}{(\overline{r}+s)^2} = \frac{\chi}{\overline{r}^2} - \frac{2\chi}{\overline{r}^3}s + O(s^2), \frac{\psi}{(\overline{r}+s)^3} = \frac{\psi}{\overline{r}^3} - \frac{3\psi}{\overline{r}^4}s + O(s^2).$$

Substituting this into the equation gives

$$\begin{aligned}\dot{s} &= -\frac{\chi}{\overline{r}^2}[\overline{u}] - \frac{\chi}{\overline{r}^2}[v] + \frac{2\chi}{\overline{r}^3}s[\overline{u}] - \frac{\psi}{\overline{r}^3} + \frac{3\psi}{\overline{r}^4}s + o(s,v) \\ &= -\frac{\chi}{\overline{r}^2}[v] + \frac{\psi}{\overline{r}^4}s + O(s^2)\end{aligned}$$

by (2.1).

Finally, the equation for $\dot{s}$ can be plugged into the boundary condition for v on $\overline{r}$. This gives

$$\text{(2.3)} \quad \begin{cases} v_\tau = \kappa\Delta v, & \text{in } (\Omega\setminus\partial C')\times[0,\infty), \\ \kappa v_\rho = \dfrac{\overline{u}\chi}{\overline{r}^2}[v] - \dfrac{\overline{u}\psi}{\overline{r}^4}s, & \text{in- and outside } \partial C', \\ \dot{s} = -\dfrac{\chi}{\overline{r}^2}[v] + \dfrac{\psi}{\overline{r}^4}s, & \text{on } \partial C', \\ v_\rho = 0, & \text{on } \partial\Omega. \end{cases}$$

In accordance with initial conditions for u and r, initial perturbations will be denoted by v^0 and s^0.

4. Weak solutions

In order to study stability of the equilibrium, it is convenient to introduce a concept of weak solution. Let ϕ be a test function, that is, a pair ϕ_-, ϕ_+ of smooth functions on C' and $\Omega\backslash C'$, respectively. Multiplying the diffusion equation $v_\tau = \kappa\Delta v$ by such a test function ϕ and integrating by parts gives

$$\begin{aligned}
\int_\Omega v_\tau \phi \,\mathrm{d}\xi &= \int_\Omega \kappa\Delta v \phi \,\mathrm{d}\xi \\
&= -\int_\Omega \kappa\nabla v\cdot\nabla\phi \,\mathrm{d}\xi + \int_{\partial C'} -[\kappa\phi v_\rho]\,\mathrm{d}\sigma(\xi) \\
&= -\int_\Omega \kappa\nabla v\cdot\nabla\phi \,\mathrm{d}\xi + \frac{\chi}{\overline{r}^2}\int_{\partial C'} \left(\phi_-(\overline{r})\overline{u}_-[v] - \phi_+(\overline{r})\overline{u}_+[v]\right)\mathrm{d}\sigma(\xi) \\
&\qquad - \frac{\psi}{\overline{r}^4}\int_{\partial C'} \left(\phi_-(\overline{r})\overline{u}_- s - \phi_+(\overline{r})\overline{u}_+ s\right)\mathrm{d}\sigma(\xi) \\
&= -\int_\Omega \kappa\nabla v\cdot\nabla\phi \,\mathrm{d}\xi - \frac{\chi}{\overline{r}^2}\int_{\partial C'} \left([\phi\overline{u}][v] - \frac{\psi}{\chi\overline{r}^2}[\phi\overline{u}]s\right)\mathrm{d}\sigma(\xi) \\
&= -\int_\Omega \kappa\nabla v\cdot\nabla\phi \,\mathrm{d}\xi - \frac{\chi}{\overline{r}^2}\int_{\partial C'} \left([\phi\overline{u}][v] + \frac{[\overline{u}]}{\overline{r}}[\phi\overline{u}]s\right)\mathrm{d}\sigma(\xi)
\end{aligned}$$

Although this identity can be used to define a concept of weak solution, it is more convenient to weigh the inside and outside of the cell with a factor $\frac{1}{\overline{u}}$, since this will eliminate the complicated factor $[\phi\overline{u}]$ in the boundary integral:

$$\int_\Omega v_\tau \frac{\phi}{\overline{u}} \,\mathrm{d}\xi = -\int_\Omega \frac{\kappa}{\overline{u}}\nabla v\cdot\nabla\phi \,\mathrm{d}\xi - \frac{\chi}{\overline{r}^2}\int_{\partial C'} \left([\phi][v] + \frac{[\overline{u}]}{\overline{r}}[\phi]s\right)\mathrm{d}\sigma(\xi)\,. \tag{2.4}$$

In fact, any function satisfying this identity for all such function pairs ϕ in a suitable function space must be a solution of (2.3).

In view of (2.4), define the *bilinear* form T

$$T[(v,s),\phi] := \int_\Omega \frac{\kappa}{\overline{u}}\nabla v\cdot\nabla\phi \,\mathrm{d}\xi + \frac{\chi}{\overline{r}^2}\int_{\partial C'} [\phi][v] + \frac{[\overline{u}]}{\overline{r}}[\phi]s\,\mathrm{d}\sigma(\xi) \tag{2.5}$$

Note that this form is indeed linear in the two arguments (v, s) and ϕ.

The next step is to construct a suitable function space for v and s. For the above expression to be meaningful, $v(.,\tau)$ should have at least a weak gradient inside and outside C' and a trace on $\partial C'$, both inside and outside C'. A sensible choice is $H^1(C') \times H^1(\Omega \setminus C')$, intersected with the space of radially symmetric functions. Denote the resulting function space by $\hat{H}^1(\Omega; C')$, where a pair (v_-, v_+) is seen as a function on Ω (see [**2**], Chapter 5 for an introduction to Sobolev spaces). Also, slightly abusing notation, the pair $(\nabla v_-, \nabla v_+)$ of gradients is denoted by ∇v.

The weight factors used to simplify (2.4) can be put into the space by defining an alternative inner product for the underlying L^2-space:

$$(\phi_1, \phi_2)_{L^2(\Omega)} := \int_\Omega \frac{\phi_1 \phi_2}{\overline{u}} \,\mathrm{d}\xi, \tag{2.6}$$

which is equivalent to the usual L^2-inner product and will be denoted by $(.,.)$. The associated norm will be denoted by $\|.\|$.

Since v is also time-dependent, it can be viewed as a function from some time-index set to $H^1(C') \times H^1(\Omega \setminus C')$. A suitable choice is

$$L^2\left(0, T; \hat{H}^1(\Omega; C')\right),$$

and using the weak time-derivative. See [**2**], §5.9.2 for an introduction to Sobolev spaces involving time. Following [**2**], §5.9.1, the left-hand side of the above identity can be generalised for v_τ in the dual space of $\hat{H}^1(\Omega; C')$, denoted by $\hat{H}^{-1}(\Omega; C')$. Using similar proofs, results analogous to the results for H^{-1} can be derived. The pairing of $\hat{H}^{-1}(\Omega; C')$ and $\hat{H}^1(\Omega; C')$ will be denoted by $\langle .,. \rangle$. Note that $L^2(\Omega)$ can be embedded in $\hat{H}^{-1}(\Omega; C')$ continuously by

$$f^0 \mapsto \left(f : \hat{H}^1(\Omega; C') \to \mathbb{R}; \phi \mapsto (f^0, \phi)\right).$$

Note that this embedding satisfies $\langle f, \phi \rangle = (f^0, \phi)$.

For s, the function space $C([0,T]; (-\overline{r}, 1-\overline{r}))$ is used. Again, the time-derivative is replaced by a weak derivative. Since any continuous function is integrable on compact time interval, the concept of weak solution is well-defined ([**2**], §5.9.2). Note that this does not mean that every continuous function has a weak derivative.

DEFINITION 2.2. A pair (v, s) with

$$v \in L^2(0,T; \hat{H}^1(\Omega; C')), \qquad v_\tau \in L^2(0,T; \hat{H}^{-1}(\Omega; C')),$$
$$s \in C([0,T]; (-\overline{r}, 1-\overline{r}))$$

is a weak solution of (2.3) on the time interval $[0,T]$ if

$$\langle v_\tau, \phi \rangle + T[(v,s), \phi] = 0,$$
$$\dot{s} = -\frac{\chi}{\overline{r}^2}[v] + \frac{\psi}{\overline{r}^4} s$$

for any $\phi \in \hat{H}^1(\Omega; C')$ and almost every time $t \in [0,T]$. A pair (v, s) with

$$v \in L^2(0,\infty; \hat{H}^1(\Omega; C')), \qquad v_\tau \in L^2(0,\infty; \hat{H}^{-1}(\Omega; C')),$$
$$s \in C([0,\infty); (-\overline{r}, 1-\overline{r}))$$

is called a weak solution of (2.3) if it is a weak solution on any time interval $[0,T]$.

Note that, by the analogon of Theorem 3(i) from [**2**], §5.9.1, initial values for v and s can be prescribed.

CHAPTER 3

Linear Stability

In order to obtain results about the stability of the system, it is useful to study some basic properties of the linearised system first.

1. Conserved quantities

The nonlinear problem (1.15) has two conserved quantities: the amount of solute inside and outside the cell. Due to the linearization, these two quantities are no longer conserved. However, the linearized system (2.3) has two conserved quantities that are closely related to the mass inside and outside the cell.

THEOREM 3.1. *The quantities*

$$\begin{aligned}&\int_{C'} v \,\mathrm{d}\xi + \overline{u}_- \int_{\partial C'} s \,\mathrm{d}\sigma(\xi),\\ &\int_{\Omega\setminus C'} v \,\mathrm{d}\xi - \overline{u}_+ \int_{\partial C'} s \,\mathrm{d}\sigma(\xi)\end{aligned}$$

are conserved by (2.3).

PROOF. This can be shown by integrating over C' and $\Omega \setminus C'$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{C'} v \,\mathrm{d}\xi &= \langle v_\tau, \overline{u}_- 1_{C'}\rangle\\ &= -(\kappa\overline{u}_-\nabla v, \nabla 1_{C'}) - \frac{\chi\overline{u}_-}{\overline{r}^2}\int_{\partial C'}\left([1_{C'}][v] + \frac{[\overline{u}]}{\overline{r}}[1_{C'}]s\right)\mathrm{d}\sigma(\xi)\\ &= -\overline{u}_-\int_{\partial C'}\left(-\frac{\chi}{\overline{r}^2}[v] + \frac{\psi}{\overline{r}^4}s\right)\mathrm{d}\sigma(\xi)\\ &= -\overline{u}_-\int_{\partial C'}\dot{s}\,\mathrm{d}\sigma(\xi) = -\overline{u}_-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\partial C'} s\,\mathrm{d}\sigma(\xi),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega\setminus C'} v\,\mathrm{d}\xi &= -(\kappa\overline{u}_+\nabla v, \nabla 1_{\Omega\setminus C'})\\ &\qquad - \frac{\chi\overline{u}_+}{\overline{r}^2}\int_{\partial C'}\left([1_{\Omega\setminus C'}][v] + \frac{[\overline{u}]}{\overline{r}}[1_{\Omega\setminus C'}]s\right)\mathrm{d}\sigma(\xi)\\ &= -\overline{u}_+\int_{\partial C'}\left(\frac{\chi}{\overline{r}^2}[v] - \frac{\psi}{\overline{r}^4}s\right)\mathrm{d}\sigma(\xi)\\ &= \overline{u}_+\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\partial C'} s\,\mathrm{d}\sigma(\xi)\,.\end{aligned}$$

Note that the conservation laws indeed resemble the original laws of conservation of mass. □

REMARK. Note that the sum of the two conserved quantities gives that

$$\int_{\Omega} v \,\mathrm{d}\xi - [\overline{u}] \int_{\partial C'} s \,\mathrm{d}\sigma(\xi)$$

is a conserved quantity as well; the linearized version of global conservation of mass. If $\psi = 0$, $[\overline{u}] = 0$ as well, and and the total amount of mass is conserved by the linear system.

Perturbations which have both conserved quantities equal to zero are of special interest. Therefore, the concept of a *admissible* perturbation is introduced.

DEFINITION 3.2. A perturbation (v, s) is called admissible if

$$\begin{aligned}
\int_{C'} v \,\mathrm{d}\xi + \overline{u}_- \int_{\partial C'} s \,\mathrm{d}\sigma(\xi) &= 0, \\
\int_{\Omega \setminus C'} v \,\mathrm{d}\xi - \overline{u}_+ \int_{\partial C'} s \,\mathrm{d}\sigma(\xi) &= 0.
\end{aligned}$$

A perturbation v is called admissible if there exists a (unique) s such that (v, s) is admissible.

The relation between s, and the integrals inside and outside C' can be explored a little further. Define

$$\begin{aligned}
w_0 &= \begin{cases} w_-, & \text{if } \rho < \overline{r}, \\ w_+, & \text{if } \rho > \overline{r} \end{cases} \\
&= \frac{1}{\sqrt{\frac{\overline{u}_-}{|C'|} + \frac{\overline{u}_+}{|\Omega \setminus C'|}}} \begin{cases} -\frac{\overline{u}_-}{|C'|}, & \text{if } \rho < \overline{r}, \\ \frac{\overline{u}_+}{|\Omega \setminus C'|}, & \text{if } \rho > \overline{r}, \end{cases}
\end{aligned}$$

such that $\|w_0\|_{L^2(\Omega)} = 1$, $w_+ > 0$ and

$$\begin{aligned}
&\frac{1}{\overline{u}_-} \int_{C'} w_0 \,\mathrm{d}\xi + \frac{1}{\overline{u}_+} \int_{\Omega \setminus C'} w_0 \,\mathrm{d}\xi = (w_0, 1) = 0, \\
&[w_0] = \sqrt{\frac{\overline{u}_-}{|C'|} + \frac{\overline{u}_+}{|\Omega \setminus C'|}}.
\end{aligned}$$

Since by the conservation laws $(v, 1) = 0$ for any admissible v,

$$v = \tilde{v} + (v, w_0) w_0,$$

where $\tilde{v}$ has mean zero both inside and outside C', which means in particular that $(\tilde{v}, w_0) = 0$.

For admissible v, it is possible to construct the s such that (v, s) is admissible. This can be done by conputing the s_0 corresponding to w_0 first, using the conservation laws

$$|C'|w_- + \overline{u}_-|\partial C'|s_0 = 0,$$
$$|\Omega \setminus C'|w_+ - \overline{u}_+|\partial C'|s_0 = 0;$$
$$[w_0] = \left(\frac{\overline{u}_+|\partial C'|}{|\Omega \setminus C'|} + \frac{\overline{u}_-|\partial C'|}{|C'|}\right) s_0$$

It follows that

$$s_0 = \frac{|\Omega \setminus C'||C'|}{\overline{u}_+|\partial C'||C'| + \overline{u}_-|\partial C'||\Omega \setminus C'|}[w_0].$$

Note that in particular,

$$\frac{-[\overline{u}]}{\overline{r}} s_0 = \frac{-[\overline{u}]|\Omega \setminus C'||C'|}{\overline{r}\left(\overline{u}_+|\partial C'||C'| + \overline{u}_-|\partial C'||\Omega \setminus C'|\right)}[w_0] =: \theta[w_0]$$

where $\theta > 0$ and

$$\begin{aligned}
\theta &= \frac{-[\overline{u}]|\Omega \setminus C'||C'|}{\overline{r}\left(\overline{u}_+|\partial C'||C'| + \overline{u}_-|\partial C'||\Omega \setminus C'|\right)} \\
&= \frac{-[\overline{u}]\left(\frac{4\pi}{3}\right)^2 \overline{r}^3(1-\overline{r}^3)}{\overline{r}\overline{u}_+ 4\pi\overline{r}^2\frac{4\pi}{3}\overline{r}^3 + \overline{r}(\overline{u}_+ - [\overline{u}])4\pi\overline{r}^2\frac{4\pi}{3}(1-\overline{r}^3)} \\
&= \frac{-[\overline{u}]\frac{1}{3}(1-\overline{r}^3)}{\overline{u}_+\overline{r}^3 + (\overline{u}_+ - [\overline{u}])(1-\overline{r}^3)} \\
&< \frac{-[\overline{u}]\frac{1}{3}(1-\overline{r}^3)}{-[\overline{u}](1-\overline{r}^3)} = \frac{1}{3}.
\end{aligned} \tag{3.1}$$

Moreover, $\theta \uparrow \frac{1}{3}$ as $[\overline{u}] \to -\infty$.

The conservation laws also imply that the s corresponding to $\tilde{v}$ is always zero. Then, for any admissible v, setting

$$s = \frac{\overline{r}\theta}{-[\overline{u}]}(v, w_0)[w_0] \tag{3.2}$$

will result in an admissible perturbation (v, s).

Moreover, by [**2**], §5.8, theorem 1 and continuity of the trace operator, it follows that for some constant $\gamma > 0$,

$$|[v]| = |[\tilde{v}] + (v, w_0)[w_0]| \leq \gamma\|\nabla v\|_{L^2(\Omega)} + |(v, w_0)|[w_0],$$

using $[w_0] = w_+ - w_- > 0$. However, for $[\tilde{v}]$, a stronger result can be obtained. For $\overline{u}_- = \overline{u}_+ = 1$, there exist constants $A_-, A_+ > 0$, only depending on $\overline{r}$, such that

$$|\tilde{v}(\overline{r}^-)| \leq \sqrt{A_- \int_{C'} |\nabla\tilde{v}|^2 \, \mathrm{d}\xi},$$

$$|\tilde{v}(\overline{r}^+)| \leq \sqrt{A_+ \int_{\Omega\setminus C'} |\nabla\tilde{v}|^2 \, \mathrm{d}\xi},$$

for any $\tilde v$ with mean zero inside and outside C'. It follows that

$$|\tilde v(\overline{r}^-)| \leq \sqrt{\overline{u}_- A_- \int_{C'} \frac{|\nabla \tilde v|^2}{\overline{u}_-}\,\mathrm{d}\xi},$$

$$|\tilde v(\overline{r}^+)| \leq \sqrt{\overline{u}_+ A_+ \int_{\Omega\setminus C'} \frac{|\nabla \tilde v|^2}{\overline{u}_+}\,\mathrm{d}\xi}$$

for *any* $\overline{u}$. Combining these two estimates gives

$$\begin{aligned}
[\tilde v]^2 &\leq 2|\tilde v(\overline{r}^-)|^2 + 2|\tilde v(\overline{r}^+)|^2 \\
&\leq 2A_-(\overline{u}_+ - [\overline{u}]) \int_{C'} \frac{|\nabla \tilde v|^2}{\overline{u}_-}\,\mathrm{d}\xi \; + 2A_+\overline{u}_+ \int_{\Omega\setminus C'} \frac{|\nabla \tilde v|^2}{\overline{u}_+}\,\mathrm{d}\xi \qquad (3.3)\\
&\leq \frac{A}{2}\overline{u}_+ \int_\Omega \frac{|\nabla \tilde v|}{\overline{u}}\,\mathrm{d}\xi \; - \frac{A}{2}[\overline{u}] \int_{C'} \frac{|\nabla \tilde v|^2}{\overline{u}_+}\,\mathrm{d}\xi = A(\overline{u}_+ - [\overline{u}])\|\nabla \tilde v\|^2_{L^2(\Omega)}
\end{aligned}$$

where $A = 4\max\{A_-, A_+\}$, which only depends on $\overline{r}$.

Finally, using Poincaré's inequality ([**2**], §5.6.1, Theorem 3),

$$\|v\|^2_{L^2(\Omega)} = \|\tilde v\|^2_{L^2(\Omega)} + (v, w_0)^2 \leq \alpha \|\nabla \tilde v\|^2_{L^2(\Omega)} + (v, w_0)^2. \qquad (3.4)$$

It should be noted that because of radial symmetry, the unknown s can be eliminated out of the perturbation problem completely for admissible perturbations. This can be done by substituting the conserved quantities for s in the equation for $\dot s$ and plugging this into the boundary conditions on $\partial C'$. However, since this is only possible for radial symmetric perturbations, this will not be done explicitly.

2. Eigenvalues for $\psi = 0$

In case $\psi = 0$, the system is partially decoupled: the term containing s drops out of T since $[\overline{u}] = 0$, which makes it possible to define an ordinary bilinear form B on $\hat H^1(\Omega; C')$.

$$B[\phi^1, \phi^2] = (\kappa \nabla \phi^1, \nabla \phi^2) + \frac{\chi}{\overline{r}^2} \int_{\partial C'} [\phi^1][\phi^2]\,\mathrm{d}\sigma(\xi)\,.$$

This observation gives rise to a linear operator in the following way.

DEFINITION 3.3. Let $L : \hat H^1(\Omega; C') \to \hat H^{-1}(\Omega; C')$ be such that $Lv = f$ if

$$\langle f, \phi\rangle = -B[v, \phi]$$

for any test function $\phi \in \hat H^1(\Omega; C')$.

Moreover, B is an inner product on a suitable subspace of $\hat H^1(\Omega; C')$:

$$\overline{H}^1(\Omega; C') := \left\{ \phi \in \hat H^1(\Omega; C') : \int_\Omega \phi\,\mathrm{d}\xi = 0 \right\}.$$

Denote the dual of $\overline{H}^1(\Omega; C')$ by $\overline{H}^{-1}(\Omega; C')$.

LEMMA 3.4. *The space $\overline{H}^1(\Omega; C')$, equipped with $B[.,.]$, is a Hilbert space.*

PROOF. Symmetry and nonnegativity are satisfied by definition. Moreover, $B[0,0] = 0$ trivially. Let then $B[\phi, \phi] = 0$. Then $\nabla\phi = 0$ almost everywhere inside and outside C', which implies that ϕ is constant almost everywhere inside and outside C'. It also follows that $[\phi] = 0$ almost everywhere in the trace sense, which means that the constants inside and outside C' are equal. Since $\phi \in \overline{H}^1(\Omega; C')$, it must be the zero function.

Let then ϕ_n be a Cauchy sequence. This sequence has a limit ϕ in $H^1(C') \times H^1(\Omega \setminus C')$, equipped with its natural inner product. Then

$$\int_\Omega |\nabla\phi - \nabla\phi_n|^2 \, \mathrm{d}\xi \to 0.$$

Moreover, since the trace operator on H^1 is continuous,

$$\int_{\partial C'} |[\phi - \phi_n]|^2 \, \mathrm{d}\sigma(\xi) \to 0,$$

which implies that $\phi_n \to \phi$ according in $\overline{H}^1(\Omega; C')$. □

The main point of interest is the existence of eigenfunctions of L. These can be studied more easily by studying the eigenfunctions of the inverse of L, which can be proven to exist using Riesz' Representation Theorem ([**2**], §D.3).

LEMMA 3.5. *$L : \overline{H}^1(\Omega; C') \to \overline{H}^{-1}(\Omega; C')$ has a bounded inverse $S : \overline{H}^{-1}(\Omega) \to \overline{H}^1(\Omega; C')$. Seen as a mapping from $\overline{H}^1(\Omega; C')$ to itself or $L^2(\Omega)$ to itself, it is also symmetric, negative and compact.*

PROOF. For any $f \in \overline{H}^{-1}(\Omega)$, consider the functional $\phi \mapsto -\langle f, \phi\rangle$ on $\overline{H}^1(\Omega; C')$. By definition of the dual space, this functional is bounded with respect to $B[.,.]$. Then, by Riesz' Representation Theorem, there exists a unique $v \in \overline{H}^1(\Omega; C')$ such that $B[v, \phi] = -\langle f, \phi\rangle$ for every $\phi \in \overline{H}^1(\Omega; C')$. Then L^{-1} exists, say $L^{-1} = S$.

Boundedness of S follows from a straightforward computation:

$$\begin{aligned} \|Sf\|_{H^1(\Omega;C')} &= B[Sf, Sf] = -\langle f, Sf\rangle \\ &\leq \sup_{\|\phi\|_{\overline{H}^1(\Omega;C')}=1} \{|\langle f, \phi\rangle|\} = \|f\|_{\overline{H}^{-1}(\Omega;C')}. \end{aligned}$$

Consider then $S : \overline{H}^1(\Omega; C') \to \overline{H}^1(\Omega; C')$:

$$\begin{aligned} &B[Sf, \phi] = -\langle f, \phi\rangle = -\langle \phi, f\rangle = B[S\phi, f] = B[f, S\phi], \\ &B[Sf, f] = -\langle f, f\rangle \leq 0. \end{aligned}$$

Similarly, for $S : L^2(\Omega) \to L^2(\Omega)$,

$$\begin{aligned} (Sf, \phi) &= (\phi, Sf) = \langle \phi, Sf\rangle = -B[S\phi, Sf] \\ &= -B[Sf, S\phi] = \langle f, S\phi\rangle = (f, S\phi) \\ (Sf, f) &= (f, Sf) = \langle f, Sf\rangle = -B[Sf, Sf] \leq 0. \end{aligned}$$

By continuity of the trace operator and the Rellich-Kondrachov Compactness theorem ([**2**], §5.7), $\overline{H}^1(\Omega; C')$ is compactly embedded in $L^2(\Omega)$, which is itself continuously embedded in $H^{-1}(\Omega; C')$. Then S from $L^2(\Omega)$ to itself and S from $H^1(\Omega; C')$ to itself are both compact. $\square$

Using this result, existence of eigenfunctions of S can be shown.

THEOREM 3.6. *There exists a countable orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of L. This basis is also an orthogonal basis of $\overline{H}^1(\Omega; C')$.*

PROOF. Since $S : L^2(\Omega) \to L^2(\Omega)$ is symmetric and compact, there exists a countable orthonormal basis $\{w_k : k \in \mathbb{N}\}$ of $L^2(\Omega)$ consisting of eigenfunctions of S by the Hilbert-Schmidt Theorem ([**9**], §3.6). For any w_k, $(w_k, \phi) = -\mu_k B[w_k, \phi]$ for all $\phi \in \overline{H}^1(\Omega; C')$. Then

$$-\left(\mu_k^{-1} w_k, \phi\right) = B[w_k, \phi],$$

which implies that $Lw_k = \lambda_k w_k$ for $\lambda_k = \mu_k^{-1}$. Hence, $\{w_k : k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of L. Note that all eigenvalues λ_k are negative, since

$$-\lambda_k = -\lambda_k(w_k, w_k) = -(Lw_k, w_k) = B[w_k, w_k] > 0$$

Note that the eigenfunctions w_k are in $\overline{H}^1(\Omega; C')$. Since also $\overline{H}^1(\Omega; C')$ is a subspace of $L^2(\Omega)$, the eigenfunctions w_k must span $\overline{H}^1(\Omega; C')$ as well. $\square$

REMARK. By negativity of S, all eigenvalues of L are negative. Form the Hilbert-Schmidt theorem it also follows that the eigenvalues λ_k of L form a decreasing sequence

$$\lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

The eigenfunctions w_k can be described in more detail by applying separation of variables.

THEOREM 3.7. *The eigenfunctions w_k of L are given by*

$$w_k(\rho) = \begin{cases} \frac{A}{\rho} \sin\left(\sqrt{\frac{|\lambda_k|}{\kappa_-}}\rho\right), & \text{if } \rho < \overline{r}, \\ \frac{B}{\rho} \cos\left(\sqrt{|\lambda_k|}\rho\right) + \frac{C}{\sqrt{|\lambda_k|}\rho} \sin\left(\sqrt{|\lambda_k|}\rho\right), & \text{if } \rho > \overline{r}, \end{cases}$$

where $A, B \in \mathbb{R}$ are constants. Moreover, all eigenvalues are simple.

PROOF. The eigenfunctions w_k satisfy

$$\Delta w = w_{\rho\rho} + \frac{2}{\rho} w_\rho = \lambda w,$$

which is equivalent to

$$\kappa(\rho w)'' = \lambda \rho w.$$

For $\lambda < 0$, the general solution of this ordinary differential equation in w is given by

$$w = \begin{cases} \frac{A}{\rho} \sin\left(\sqrt{\frac{|\lambda|}{\kappa_-}}\rho\right) \frac{C}{\rho} \cos\left(\sqrt{\frac{|\lambda|}{\kappa_-}}\rho\right), & \text{if } \rho < \overline{r}, \\ \frac{B}{\rho} \cos\left(\sqrt{|\lambda|}\rho\right) + \frac{D}{\rho} \sin\left(\sqrt{|\lambda|}\rho\right), & \text{if } \rho > \overline{r}, \end{cases}$$

where A, B, C, D are constants, using the assumption $\kappa_+ = 1$. In order for w to be well-defined in $\rho = 0$, at least the limit of the numerator for $\rho \downarrow 0$ should be zero. This is the case if $C = 0$. In this case, not only the limit of the numerator is zero, but it also converges fast enough to ensure existence of w at $\rho = 0$. Note that $w' = \frac{(\rho w)' - w}{\rho}$. Using this identity, the boundary condition $w_\rho = 0$ on $\rho = 1$ gives

$$\left(\sqrt{|\lambda|}\cos(\sqrt{|\lambda|}) - \sin(\sqrt{|\lambda|})\right) D = \left(\sqrt{|\lambda|}\sin(\sqrt{|\lambda|}) - \cos(\sqrt{|\lambda|})\right) B.$$

The coefficients in the above equality cannot both be zero. Note first that if there is any λ for which both are zero, both $\sin(\sqrt{|\lambda|})$ and $\cos(\sqrt{|\lambda|})$ are nonzero, since the sine and cosine do not share any zeros. Suppose then that the left coefficient is zero. Then

$$\sqrt{|\lambda|} = \frac{\sin(\sqrt{|\lambda|})}{\cos(\sqrt{|\lambda|})}.$$

Substituting this into the right coefficient and multiplying by $\cos(\sqrt{|\lambda|})$ gives

$$\sin^2(\sqrt{|\lambda|}) - \cos^2(\sqrt{|\lambda|}) = 0$$

which means that $\sqrt{|\lambda|} = \frac{\pi}{4} + m\frac{\pi}{2}$ for $m \in \mathbb{N}$. Substituting this back into the first equation immediately gives that at least one coefficient is nonzero, since there is no m such that $\frac{\pi}{4} + m\frac{\pi}{2} = \pm 1$. It follows that the solution space for (B, D) is one-dimensional, which implies that the multiplicity of λ is at most 2.

Consider then $A = 1$, $B = D = 0$, which means that $w(\rho) = 0$ for $\rho > \overline{r}$. Then also $w_\rho(\overline{r}^+) = 0$, which means that the jump at $\overline{r}$ should be zero as well. Using again the jump condition, $w_\rho(\overline{r}^-)$ should also be zero:

$$\frac{\sin\left(\sqrt{\frac{|\lambda|}{\kappa_-}}\overline{r}\right)}{\overline{r}} = 0$$

$$\frac{\sqrt{\frac{|\lambda|}{\kappa_-}}\cos\left(\sqrt{\frac{|\lambda|}{\kappa_-}}\overline{r}\right)}{\overline{r}} - \frac{\sin\left(\sqrt{\frac{|\lambda|}{\kappa_-}}\overline{r}\right)}{\overline{r}^2} = 0$$

which is impossible, since the sine and cosine share no zeros. It follows that the solution space for (A, B, D) cannot be two-dimensional, which means that λ must be a simple eigenvalue. $\square$

Using the spectrum of L, the eigenfunctions associated to the complete linear problem (2.3) with $\psi = 0$ can be derived.

THEOREM 3.8. *The eigenfunctions associated to* (2.3) *with* $\psi = 0$ *are given by*

$$\left(w_k, -\frac{\chi}{\overline{r}^2\lambda_k}[w_k]\right);$$

$$\left(w_0 \equiv \frac{1}{|\Omega|}, 0\right);$$

$$(0, 1);$$

where w_k *are the normalized eigenfunctions of* L *with associated eigenvalues* λ_k*.*

PROOF. By definition of L, setting $v = e^{\lambda_k \tau} w_k$ for an eigenfunction w_k of L will result in $v_\tau = \lambda_k v$ and $\dot{s} = \lambda_k s$, which means that the pair (v, s) is an eigenfunction of the linear system. Note that the system of eigenfunctions for $\overline{H}^1(\Omega; C')$ can be extended to a basis for $\hat{H}^1(\Omega; C')$ by adding $w_0 \equiv |\Omega|^{-\frac{1}{2}}$, which is mapped to the zero function by L. Since its jump at $\rho = \overline{r}$ is zero, the pair $w_0 \equiv 1, s = 0$ is an eigenfunction with eigenvalue 0. Finally, if $v \equiv 0$, $s_0 = 1$ results in an eigenfunction with eigenvalue 0. Note that the extended set $\{w_k : k \geq 0\}$ is still orthonormal with respect to $(.,.)$ and orthogonal $B[.,.]$. $\square$

REMARK. By orthonormality of the $\{w_k : k \geq 0\}$, any function $\phi \in L^2(\Omega)$ can be written as

$$\phi = \sum_{k=0}^{\infty} (\phi, w_k) w_k = \frac{1}{|\Omega|} \int_\Omega \phi + \sum_{k=1}^{\infty} (\phi, w_k) w_k$$

In case $\psi = 0$, the problem seems to be stable. There are two neutral eigenvalues, but these can be associated to conservation laws. Together with the negativity of the other eigenvalues, this is a strong indication that any perturbation satisfying the conservation laws will vanish as $\tau \to \infty$. When proving well-posedness of the linearized problem for $\psi = 0$, it also becomes clear that admissible perturbations converge to zero at exponential speed.

3. Weak solutions for $\psi = 0$

In case $\psi = 0$, a more explicit formula for solutions of (2.3) with $\psi = 0$ can be found. This formula can be derived by computing Galerkin approximations. See [**2**], §7.1.2 for an example.

THEOREM 3.9. *The unique weak solution of* (2.3) *with* $\psi = 0$ *is given by*

$$v(\rho, \tau) = \frac{1}{|\Omega|} \int_\Omega v^0 + \sum_{k=1}^{\infty} (v^0, w_k) e^{\lambda_k \tau} w_k(\rho)$$

$$s(\tau) = s^0 - \sum_{k=1}^{\infty} (v^0, w_k) \frac{\chi[w_k]}{\overline{r}^2 \lambda_k} (1 - e^{\lambda_k \tau})$$

for $v^0 \in L^2(\Omega)$ *and* $s^0 \in (-\overline{r}, 1 - \overline{r})$.

PROOF. For $N \in \mathbb{N}$, consider the function

$$v_N(\rho, \tau) = d_N^0 + \sum_{k=1}^{N} d_N^k(\tau) w_k$$

where the w_k are the eigenfunctions from the previous section. By definition of Galerkin approximation, the functions v_N should satisfy the equation when only v_k $(k \leq N)$ are used as test functions. An approximation s_N for s is obtained by replacing v by v_N:

$$\begin{aligned}
&\langle (v_N)_\tau, w_k\rangle + B[v_N, w_k] = 0,\\
&d_N^k(0) = (v^0, w_k),\\
&\dot{s}_N = -\frac{\chi}{\overline{r}^2}[v_N],
\end{aligned}$$

for any $k \le N$. Since $(v_N)_\tau$ is a finite combination of w_k, the pairing of $(v_N)_\tau$ and w_k is actually an $L^2(\Omega)$ inner product. By construction of w_k,

$$\dot{d}_N^k(\tau) + \lambda_k d_N^k(\tau) = 0,$$

which, together with the initial conditions for d_N^k, means that $d_N^k(\tau) = (v^0, w_k)e^{\lambda_k\tau}$. The solution for s_N can be found by substituting $[v_N]$ and integrating from 0 to τ. Hence, an approximate solution is given by

$$\begin{aligned}
v_N(\rho,\tau) &= \frac{1}{|\Omega|}\int_\Omega v^0 + \sum_{k=1}^N (v^0, w_k)e^{\lambda_k\tau}w_k(\rho),\\
s_N(\tau) &= s^0 - \sum_{k=1}^N (v^0, w_k)\frac{\chi[w_k]}{\overline{r}^2\lambda_k}(1 - e^{\lambda_k\tau}).
\end{aligned}$$

Next, fix $T \ge 0$, and note that by orthogonality of $\{w_k, k \ge 0\}$

$$\begin{aligned}
\|v_N(.,\tau)\|_{L^2(\Omega)}^2 &= \sum_{k=0}^N |(v^0, w_k)|^2 e^{2\lambda_k\tau} \le \sum_{k=0}^N |(v^0, w_k)|^2 \le \|v^0\|_{L^2(\Omega)}^2,\\
B[v_N(.,\tau), v_N(.,\tau)] &= \sum_{k=0}^N |(v^0, w_k)|^2 e^{2\lambda_k\tau} B[w_k, w_k]\\
&\le \sum_{k=0}^N |(v^0, w_k)|^2 B[w_k, w_k] = B[v^0, v^0].
\end{aligned}$$

Hence, the norm of v_N in $L^2(0,T;\hat{H}^1(\Omega;C'))$ is also bounded. The first computation also shows that $v_N(.,\tau)$ converges in $L^2(\Omega)$, uniformly in τ, to some limit v. Let then $\phi \in \hat{H}^1(\Omega;C')$) with $B[\phi,\phi] \le 1$ and $\phi = \phi^1 + \phi^2$ such that ϕ^1 is in the span of $\{w_0, \dots, w_N\}$ and ϕ^2 is orthogonal to $w_0, \dots, w_N$. Then, by construction of v_N,

$$\begin{aligned}
|\langle (v_N)_\tau, \phi\rangle| &= |((v_N)_\tau, \phi)| = |((v_N)_\tau, \phi^1)| = |B[v_N, \phi^1]|\\
&\le B[v_N, v_N]B[\phi^1, \phi^1] \le B[v_N, v_N] \le B[v^0, v^0],
\end{aligned}$$

which means that the norm of $(v_N)_\tau$ in $L^2(0,T;\hat{H}^{-1}(\Omega;C'))$ is bounded as well. Moreover, note that

$$\begin{aligned}
|s_N(\tau)| &\le |s^0| + \sum_{k=1}^N \left|(v^0, w_k)\frac{\chi[w_k]}{\overline{r}^2\lambda_k}\right|^2 \le C\left(|s^0| + \sum_{k=1}^N |(v^0, w_k)|^2\right)\\
&\le C(|s^0| + \|v^0\|)
\end{aligned}$$

by continuity of the trace operator. It follows that the norm of s_N in $L^2(0,T;(-\overline{r},1-\overline{r}))$ is bounded. Again, this computation also shows uniform convergence, this time of $s_N(\tau)$ in $C(0,T,(-\overline{r},1-\overline{r}))$.

By these estimates and weak compactness ([**2**], §D.4, Th. 3), $(v_N)_\tau$ converges weakly along some subsequence, say $(v_{N_l})_\tau \rightharpoonup \hat{v}$. Using Bochner's Theorem ([**2**], §E.5, Th. 8), it can be checked that $\hat{v} = v_\tau$:

$$\begin{aligned}
\int_0^T \langle v_{N_l}, \dot{\phi} w\rangle \, \mathrm{d}\tau &= \int_0^T \langle \dot{\phi} v_{N_l}, w\rangle \, \mathrm{d}\tau = \left\langle \int_0^T \dot{\phi} v_{N_l} \, \mathrm{d}\tau, w \right\rangle \\
&= \left\langle -\int_0^T \phi (v_{N_l})_\tau \, \mathrm{d}\tau, w \right\rangle = -\int_0^T \langle \phi (v_{N_l})_\tau, w\rangle \, \mathrm{d}\tau \\
&= -\int_0^T \langle (v_{N_l})_\tau, \phi w\rangle \, \mathrm{d}\tau
\end{aligned}$$

for any $w \in \hat{H}^1(\Omega; C'))$, $\phi \in C_c^\infty(0,T)$. Taking limits for $l \to \infty$ gives

$$\int_0^T \langle v, \dot{\phi} w\rangle \, \mathrm{d}\tau = -\int_0^T \langle \hat{v}, \phi w\rangle \, \mathrm{d}\tau \,.$$

As a consequence,

$$\left\langle \int_0^T \dot{\phi} v \, \mathrm{d}\tau, w \right\rangle = \int_0^T \langle v, \dot{\phi} w\rangle \, \mathrm{d}\tau = -\int_0^T \langle \hat{v}, \phi w\rangle \, \mathrm{d}\tau = \left\langle -\int_0^T \phi \hat{v} \, \mathrm{d}\tau, w \right\rangle$$

It follows that

$$\int_0^T \dot{\phi} v \, \mathrm{d}\tau = -\int_0^T \phi \hat{v} \, \mathrm{d}\tau,$$

which means by definition that $v_\tau = \hat{v}$.

Consider then

$$\phi(\xi,\tau) := \sum_{k=0}^{m} \phi^k(\tau) w_k(\xi)$$

for smooth functions ϕ^k. By substituting this into the equations for v_N and summing up to $N_l \geq m$, it follows that

$$\int_0^T \langle (v_N)_\tau, \phi\rangle + B[v_N, \phi] \, \mathrm{d}\tau = 0.$$

Taking weak limits for $l \to \infty$ now gives

$$\int_0^T \langle v_\tau, \phi\rangle + B[v, \phi] \, \mathrm{d}\tau = 0.$$

By density of functions of the type ϕ in $L^2(0,T;\overline{H}^1(\Omega;C')$, it follows that

$$\langle v_\tau, \phi\rangle + B[v,\phi] = 0$$

for any $\phi \in \overline{H}^1(\Omega;C')$ and almost every $t \in [0,T]$. Furthermore, by construction,

$$\int_0^T \dot{\phi} s_{N_l} \, \mathrm{d}\tau = \frac{\chi}{\overline{r}^2} \int_0^T \phi[v_{N_l}] \, \mathrm{d}\tau \,.$$

Passing to weak limits gives that

$$\int_0^T \dot{\phi} s \, \mathrm{d}\tau = \frac{\chi}{\bar{r}^2} \int_0^T \phi[v] \, \mathrm{d}\tau \, .$$

Moreover, by uniformity of convergence of v_N and s_N in the appropriate spaces, and the observation that the weak limits above should be equal to these uniform limits, v and s satisfy the initial conditions.

Finally, for uniqueness, note that it is sufficient to show that the only weak solution of (2.3) with $\psi = 0$ and $v^0 \equiv 0$, $s^0 = 0$ is $v \equiv 0$, $s \equiv 0$. Using the equation for v,

$$\langle v_\tau, v \rangle + B[v, v] = \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{1}{2} \|v\|_{L^2(\Omega)} \right) + B[v, v] = 0.$$

Since $B[v, v] \geq 0$ and $v^0 \equiv 0$, $\|v(., \tau)\|_{L^2(\Omega)} = 0$ for all $\tau \geq 0$. Similarly,

$$s(\tau)^2 = \int_0^\tau 2\dot{s}s = \int_0^\tau s \frac{\chi}{\bar{r}^2}[0] = 0$$

for any $\tau \geq 0$. □

It appears that the linearized system converges back to some other equilibrium after a perturbation at an exponential rate. The new equilibrium is not necesserily the same as the original equilibrium, but the conservation laws suggest that this is due to linearization and not to instability of the original system. It is expected that the nonlinear system is stable as well.

4. Eigenvalues for $\psi > 0$

In case $\psi > 0$, the symmetric structure of (2.3) is lost. Therefore it can not be expected that there exists an orthogonal complete system of eigenfunctions. However, to get an impression of stability, it is informative to examine the eigenvalues and eigenfunctions associated to (2.3). The eigenfunctions are of the same form as the w_k in theorem 3.7, but the boundary conditions now give a different system of equation, which also involves s. The resulting system depends on λ, and the zeros of its determinant are the eigenvalues of the system. The nontrivial homogeneous solution(s) of this system represent the eigenfunctions.

Unfortunately, the resulting equation is transcendental. Although this means that there is no closed expression for the solutions, it is still possible to derive a result. The determinant $D(\lambda, \psi)$ depends on λ through functions of the form

$$\cosh(\omega\lambda), \frac{\sinh(\omega\lambda)}{\sqrt{\lambda}},$$

where ω depends on the parameters, it is analytic as a function of λ. Moreover, ψ appears as a multiplicative constant in some, but not all terms of D. Moreover, the terms containing ψ are of one order lower in λ; that is,

$$D(\lambda, \psi) = D(\lambda, 0) + \psi \frac{f(\lambda)}{\lambda}$$

for some function $f(\lambda)$, $\lambda \neq 0$. In case $\psi = 0$, it follows from theorem 3.7 that the equation has a sequence of roots of multiplicity one on the negative real axis. The Implicit Function Theorem now states that any eigenvalue will be simple for sufficiently small ψ, but this is not a priori uniform in ψ. It *can* be shown that the root $\lambda = 0$ has multiplicity 2 for any $\psi \geq 0$, both geometric and algebraic.

The disadvantage is that there is no estimate of how large ψ can become before eigenvalues can collide and become complex. It is even possible that there are complex eigenvalues for any positive ψ. Even worse, it is also not known if there can be roots on the imaginary axis. The conclusion that all eigenvalue have negative real part can therefore not be drawn.

Although the analysis fails to provide a clear picture of the spectrum, intuition indicates that all eigenvalues should be stable: the problem without surface tension is stable, and surface tension generally has a stabilizing effect. Moreover, no counterexample has been found so far. Numerical experiments suggest that the eigenvalues remain real and negative. A typical example is shown in figure 1. It can be seen that the first two negative eigenvalues move closer together, but this movement seems to stagnate for large values of ψ.

Contrary to the nonzero eigenvalues, it is possible to keep track of the double eigenvalue zero. This eigenvalue keeps geometric mulitplicity two for $\psi \geq 0$. As before, the pair $v \equiv 1$, $s = 0$ has eigenvalue zero. The other eigenfunction is given by

$$v(\rho) = \begin{cases} -\frac{\psi}{\chi \overline{r}^2}, & \text{if } \rho < \overline{r}, \\ 0, & \text{if } \rho > \overline{r}, \end{cases}$$
$$s = 1$$

No linear combination of these two eigenfunctions is admissible: although it is possible to reconstruct w_0 from these eigenfunctions, this will result in an s which is not admissible. Note also that taking limits for $\psi \downarrow 0$ gives the original stationary eigenfunctions.

5. Weak solutions for $\psi \geq 0$

Using another Galerkin approximation, existence and uniqueness of weak solution can be derived for the general case $\psi \geq 0$. Since an orthonormal system of eigenfunctions is not available, another system has to be constructed. Unfortunately, the eigenfunctions associated to $\psi = 0$ will not work, since then the finite-dimensional problem does not satisfy both conservation laws.

A system that *does* result in a finite-dimensional approximation with the same conservation laws can be constructed by considering Δ inside and outside C' with Neumann conditions on $\partial\Omega$ and $\partial C'$. This is equivalent to setting $\chi = \psi = 0$. Using arguments similar to the proofs of Lemma 3.5 and Theorem 3.6, it can be shown that there exists a system of eigenfunctions of this operator which is orthogonal with respect to the H^1-norm on $\hat{H}^1(\Omega; C')$ and orthonormal with respect to the (scaled) L^2-norm. Note that the the eigenfunctions have support in either C' or

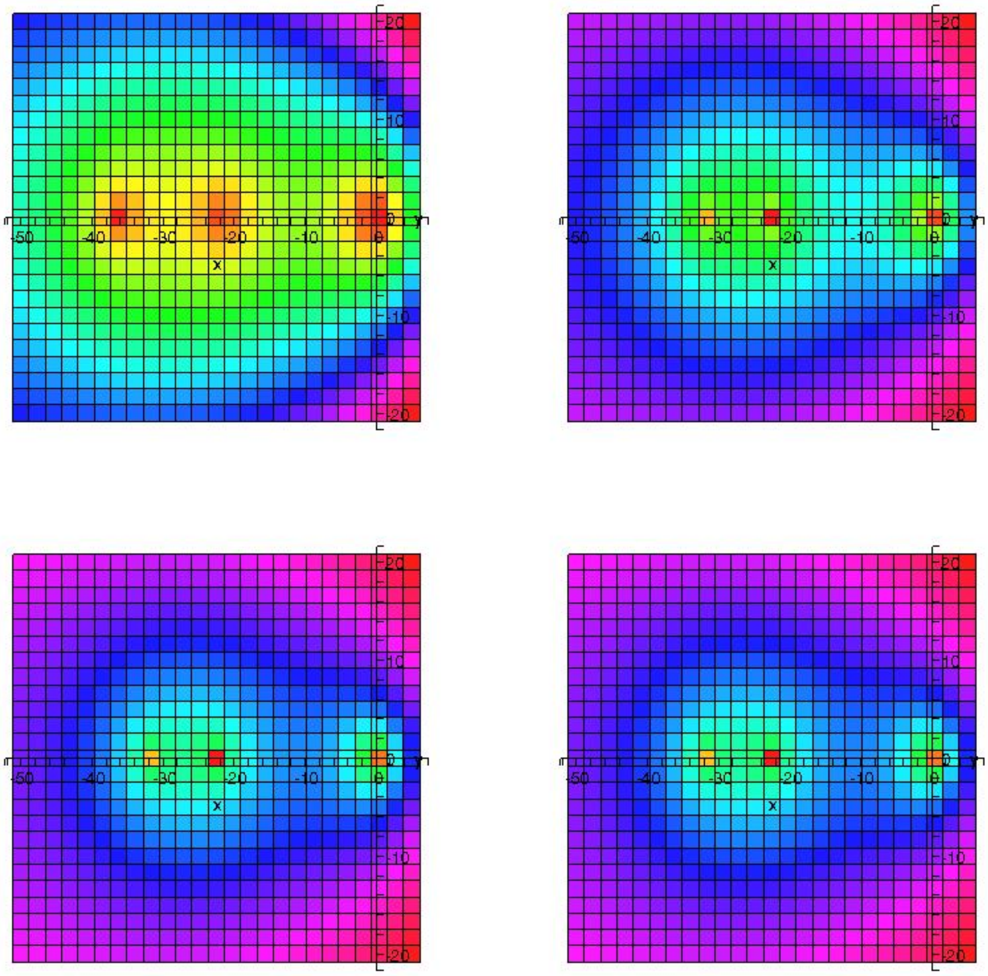

Figure 1. Density plot of $\log(1 + |D(\lambda, \psi)|)$ for $\kappa_- = 0.4$, $\overline{u}_- = 1$, $\chi = 1$ and $\overline{r} = 0.4$, with $\psi = 0$ (top-left), $\psi = 10$ (top-right), $\psi = 500$ (bottom-left) and $\psi = 100000$ (bottom-right)

$\Omega \setminus C'$, since the system associated to this operator is decoupled. Moreover, the constant functions inside and outside C', are eigenfunctions.

Existence and uniqueness of weak solutions is shown first for admissible initial conditions. All eigenfunctions, except for the two piecewise constant functions, are admissible when paired with $s = 0$. Replacing the constant eigenfunctions with w_0 from section 1, which is admissible when paired with s_0, gives a system of functions $\{w_k : k \geq 0\}$ which can always be made admissible. This will turn out to be convenient when constructing weak solutions. Note also that since w_0 is piecewise constant, the modified system is still orthonormal with respect to the scaled L^2-norm.

Lemma 3.10. *There exist positive constants β, γ such that*

$$T[(v,s),v] \geq \beta \|\nabla v\|^2_{L^2(\Omega)} - \gamma \|v\|^2_{L^2(\Omega)}$$

for all admissible perturbations $v \in \hat{H}^1(\Omega; C')$, $s \in \mathbb{R}$.

Proof. First, note that

$$\begin{aligned} [v]^2 + \frac{[\overline{u}]}{\overline{r}}[v]s &= [\tilde{v}]^2 + 2[\tilde{v}](v,w_0)[w_0] + (v,w_0)^2[w_0]^2 \\ &\qquad - \theta[w_0](v,w_0)[\tilde{v}] - \theta(v,w_0)^2[w_0]^2 \\ &= [\tilde{v}]^2 + (2-\theta)[\tilde{v}](v,w_0)[w_0] + (1-\theta)(v,w_0)^2[w_0]^2. \end{aligned}$$

Using Cauchy's inequality ([**2**], §B.2.b) with ε and (3.1), it follows that

$$\begin{aligned} [v]^2 + \frac{[\overline{u}]}{\overline{r}}[v]s &\geq [\tilde{v}]^2 - [\tilde{v}]^2 - \frac{(2-\theta)^2}{4}(v,w_0)^2[w_0]^2 + (1-\theta)(v,w_0)^2[w_0]^2 \\ &= -\frac{\theta^2}{4}(v,w_0)^2[w_0]^2 \geq -\frac{(v,w_0)^2[w_0]^2}{36}. \end{aligned}$$

Hence,

$$\begin{aligned} T[(v,s),v] &= \int_\Omega \frac{\kappa}{\overline{u}}|\nabla v|^2 \,\mathrm{d}\xi \; + \frac{\chi}{\overline{r}^2}\int_{\partial C'} \left([v]^2 - \frac{[\overline{u}]}{\overline{r}}[v]s\right)\mathrm{d}\sigma(\xi) \\ &\geq \kappa_- \int_\Omega \frac{1}{\overline{u}}|\nabla v|^2 \,\mathrm{d}\xi \; - (v,w_0)^2\frac{\chi}{\overline{r}^2}\int_{\partial C'} \frac{[w_0]^2}{36}\,\mathrm{d}\sigma(\xi) \\ &\geq \beta\|\nabla v\|^2_{L^2(\Omega)} - \gamma\|v\|^2_{L^2(\Omega)} \end{aligned}$$

using (3.4). □

Using this estimate, existence and uniqueness of weak solutions can be shown.

Theorem 3.11. *The linear problem* (2.3) *with initial values $v^0 \in L^2(\Omega)$, $s^0 \in (-\overline{r}, 1-\overline{r})$ has a unique weak solution if (v^0, s^0) is admissible.*

Proof. As before, define

$$v_N(\rho,\tau) = \sum_{k=0}^{N} d_N^k(\tau) w_k(\rho).$$

By the argument from section 1, setting

$$s_N(\tau) := d_N^0(\tau)s_0 = (v_N(.,\tau), w_0)s_0$$

guarantees that $(v_N(.,\tau), s_N(\tau))$ is admissible for any time $\tau \geq 0$.

The coefficients $d_N^k(\tau)$ are chosen such that

$$\langle (v_N)_\tau, w_l\rangle + T[(v_N, s_N), w_l] = 0$$

for any $l \geq 0$. Using linearity, it follows that these coefficients should satisfy

$$\begin{aligned}
\dot{d}_N^l(\tau) &= -d_N^0(\tau)T[(w_0, s_0), w_l] - \sum_{k=1}^{N} d_N^k(\tau)T[(w_k, 0), w_l] \\
&= -d_N^0(\tau)\int_{\partial C'} \frac{\chi}{\overline{r}^2}[w_l][w_0] - \frac{\psi}{\overline{r}^4}[w_l]s_0 \,\mathrm{d}\sigma(\xi) \;+ \sum_{k=1}^{N} d_N^k(\tau)B[w_k, w_l]
\end{aligned} \tag{3.5}$$

where $B[.,.]$ is as in the case $\psi = 0$. Initial conditions for d_N^k can be obtained by projection of v^0 onto the finite dimensional space for v_N.

$$d_N^k(0) = (w_k, v^0)$$

Using standard existence theory for ODE's, there exists a unique solution of this system. That is, absolutely continuous functions $d_N^l(\tau), \cdots, d_N^N(\tau)$ on any time interval $[0, T]$ which satisfy the inital conditions and the ODE for almost every $\tau \in [0, T]$.

Multiplying (3.5) by $d_N^l(\tau)$ and summing for $l = 1 \dots N$ gives

$$\langle (v_N)_\tau, v_N\rangle + T[(v_N, s_N), v_N] = 0.$$

Using $\frac{\mathrm{d}}{\mathrm{d}\tau}\|v_N(., \tau)\|_{L^2(\Omega)} = \langle (v_N)_\tau, v_N\rangle$ and Lemma 3.10 gives that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\|v_N(., \tau)\|^2_{L^2(\Omega)}\right) + 2\beta\|\nabla v_N(., \tau)\|^2_{L^2(\Omega)} \leq 2\gamma\|v_N(., \tau)\|^2_{L^2(\Omega)}, \tag{3.6}$$

for almost every $0 \leq \tau \leq T$. Since $\|\nabla v_N\|^2_{L^2(\Omega)}$ is nonnegative, the differential form of Gronwall's inequality ([**2**], §B.2) implies that

$$\|v_N(., \tau)\|_{L^2(\Omega)} \leq e^{\gamma\tau}\|v_N(0)\|_{L^2(\Omega)} \leq e^{\gamma\tau}\|v^0\|_{L^2(\Omega)}$$

for $0 \leq \tau \leq T$. Integrating 3.6, it follows that

$$\|v_N(., \tau)\|_{L^2(0,T;\hat{H}^1(\Omega;C'))} \leq \frac{e^{\gamma T} - 1}{\gamma}\|v^0\|_{L^2(\Omega)}. \tag{3.7}$$

Next, let $\phi \in \hat{H}^1(\Omega; C'))$ with $\|\nabla\phi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)} \leq 1$, and write $\phi = \phi^1 + \phi^2$, where ϕ^1 is in the space spanned by $\{w_k : k = 0, 1, \dots, N\}$ and ϕ^2 in it's orthogonal complement. By (3.5), and construction of v_N,

$$|\langle (v_N)_\tau, \phi\rangle| = |\langle (v_N)_\tau, \phi^1\rangle| = |T[(v_N, s_N), \phi^1]|.$$

Using the Cauchy-Schwarz inequality, continuity of the trace operator, the definition of s_N and 3.7,

$$\begin{aligned}
|T[(v_N, s_N), \phi^1]| &\leq \tilde{\gamma}(\|\nabla v_N\|_{L^2(\Omega)} + \|v_N\|_{L^2(\Omega)})(\|\nabla\phi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)}) \\
&\leq \hat{\gamma}\|v_N\|_{L^2(\Omega)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\|(v_N)_\tau\|_{H^{-1}(\Omega;C')} \leq \hat{\gamma}\|v^0\|_{L^2(\Omega)} \\
&\|(v_N)_\tau\|_{L^2(0,T;\hat{H}^{-1}(\Omega;C'))} \leq \overline{\gamma}\|v^0\|_{L^2(\Omega)}
\end{aligned}$$

The above estimates mean that v_N is bounded in $L^2(0,T;\hat{H}^1(\Omega;C'))$ and $(v_N)_\tau$ in $L^2(0,T;\hat{H}^{-1}(\Omega;C'))$. Then, by weak compactness, there is some subsequence $\{v_{N_l}\}$ of $\{v_N\}$ such that

$$v_{N_l} \rightharpoonup v,$$
$$(v_{N_l})_\tau \rightharpoonup v_\tau$$

weakly. Moreover,

$$s_{N_l} \rightharpoonup (v,w_0)s_n =: s$$

weakly in $L^2([0,T])$.

Let then

$$f(\xi,\tau) = \sum_{k=0}^{m} f^k(\tau)w_k(\xi)$$

for smooth functions f^k. For $N_l \geq m$,

$$\int_0^T \langle (v_{N_l})_\tau, f\rangle + T[(v_{N_l}, s_{N_l}), f]\,\mathrm{d}\tau = 0. \tag{3.8}$$

Taking weak limits now gives

$$\int_0^T \langle v_\tau, f\rangle + T[(v,s), f]\,\mathrm{d}\tau = 0, \tag{3.9}$$

which, by density of functions like f in $L^2(0,T;\hat{H}^1(\Omega;C'))$ means that

$$\langle v,\phi\rangle + T[(v,s),\phi] = 0$$

for every $\phi \in \hat{H}^1(\Omega;C')$ and almost every $0 \leq \tau \leq T$. Furthermore, by the analogon [**2**], §5.9.2, Theorem 3, $v \in C([0,T];\hat{H}^1(\Omega;C'))$, which also implies that s is continuous.

Clearly, $(v(.,\tau),s)$ is admissible at every time $0 \leq \tau \leq T$. From the calculations in the proof of Theorem 3.1, it follows that

$$\dot{s} = -\frac{\chi}{\bar{r}^2}[v] + \frac{\psi}{\bar{r}^4}s$$

at almost every $0 \leq \tau \leq T$.

Next, let $f \in C^1([0,T];\hat{H}^1(\Omega;C'))$ with $v(\xi,T) \equiv 0$. From (3.8), (3.9) it follows that

$$\int_0^T -\langle f_\tau, v\rangle + T[(v,s),f]\,\mathrm{d}\tau = (v(.,0), f(.,0)),$$
$$\int_0^T -\langle f_\tau, v_{N_l}\rangle + T[(v_{N_l},s_{N_l}),f]\,\mathrm{d}\tau = (v_{N_l}(.,0), f(.,0)).$$

Passing to weak limits gives that

$$(v(.,0), f(.,0)) = \int_0^T -\langle f_\tau, v\rangle + T[(v,s),f]\,\mathrm{d}\tau = (v^0, f(.,0)).$$

Since $f(.,0)$ can be any function in $\hat{H}^1(\Omega;C'))$, it follows that $v(.,0) = v^0$.

Finally, for uniqueness, it i[(v, s), v]s shown that if $v^0 \equiv 0$, $s^0 = 0$, the only solution is $v \equiv 0$, $s \equiv 0$. Note that

$$\langle v_\tau, v\rangle + T[(v,s),v] = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\|v\|^2_{L^2(\Omega)}\right) + T[(v,s),v] = 0$$

Using the estimate from Lemma 3.10,

$$T[(v,s),v] \geq -\gamma\|v\|^2_{L^2(\Omega)}.$$

Then Gronwall's inequality implies $v \equiv 0$. Since (v^0, s^0) is admissible, it follows that $s \equiv 0$ as well. □

REMARK. The requirement for initial values to be admissible is not restrictive. If the initial values (v^0, s^0) are not admissible, a (unique) linear combination of the eigenfunctions with eigenvalue zero can be subtracted such that the result *is* admissible. By linearity, the unique solution is then sum of the solution of the problem with the modified initial values, and the linear combination of the stationary eigenfunctions.

For small values of ψ, a stronger version of Lemma 3.10 can be proven.

LEMMA 3.12. *There exists a $\psi_0 = \psi_0(\kappa, \chi, \overline{r}, \overline{u}_+) > 0$ such that if $\psi < \psi_0$,*

$$T[(v,s),v] \geq \beta\|\nabla v\|_{L^2(\Omega)^2} + \gamma\|v\|_{L^2(\Omega)^2}$$

for all admissible perturbations $v \in \hat{H}^1(\Omega; C')$, $s \in \mathbb{R}$.

PROOF. First, note that by (3.2) and Cauchy's inequality

$$\begin{aligned}
[v]^2 + \frac{[\overline{u}]}{\overline{r}}[v]s &= [v]^2 - \theta[w_0](v, w_0)[v] = [v]^2 - \theta[w_0]\left(\frac{[v]-[\tilde{v}]}{[w_0]}\right)[v] \\
&= (1-\theta)[v]^2 + \theta[\tilde{v}][v] \geq \left(\frac{1}{2}-\theta\right)[v]^2 - \frac{\theta^2}{2}[\tilde{v}]^2
\end{aligned}$$

where $\frac{1}{2} - \theta > \frac{1}{6}$ using (3.1). It follows that

$$\begin{aligned}
T[(v,s),v] &= \int_\Omega \frac{\kappa}{\overline{u}}|\nabla v|^2\,\mathrm{d}\xi + \frac{\chi}{\overline{r}^2}\int_{\partial C'}\left([v]^2 + \frac{[\overline{u}]}{\overline{r}}[v]s\right)\mathrm{d}\sigma(\xi) \\
&\geq (\kappa\nabla\tilde{v}, \nabla\tilde{v}) + \frac{\chi|\partial C'|}{\overline{r}^2}\left(\left(\frac{1}{2}-\theta\right)[v]^2 - \frac{\theta^2}{2}[\tilde{v}]^2\right) \\
&\geq \frac{\kappa_-}{2}\|\nabla\tilde{v}\|^2_{L^2(\Omega)} + \left(\frac{\kappa_-}{2A(\overline{u}_+ - [\overline{u}])} - \frac{\chi|\partial C'|\theta^2}{2\overline{r}^2}\right)[\tilde{v}]^2 \\
&\quad + \frac{\chi|\partial C'|}{\overline{r}^2}\left(\frac{1}{2}-\theta\right)[v]^2
\end{aligned}$$

using (3.3). Note that the coefficient

$$\nu := \frac{\kappa_-}{2A(\overline{u}_+ - [\overline{u}])} - \frac{\chi|\partial C'|\theta^2}{2\overline{r}^2}$$

is continuous as a function of $[\overline{u}] < 0$. Moreover, it is positive if $[\overline{u}] = 0$ since also $\theta = 0$ in this case, and the limit for $[\overline{u}] \to -\infty$ is negative. Since $\psi = -\chi[\overline{u}]\overline{r}$ there exists a ψ_0 such that $\nu > 0$ if $\psi < \psi_0$.

If this is the case, Cauchy's inequality gives

$$\begin{aligned} T[(v,s),v] &\geq \frac{\kappa_-}{2}\|\nabla\tilde{v}\|^2_{L^2(\Omega)} + \alpha\left(\frac{[v]-[\tilde{v}]}{[w_0]}\right)^2 \\ &\geq \frac{\kappa_-}{2}\|\nabla\tilde{v}\|^2_{L^2(\Omega)} + \alpha(v,w_0)^2 \end{aligned}$$

Applying (3.4) proves the lemma. □

The inequalities used in the above proof also imply stability of the linear problem.

Theorem 3.13. *Let (v,s) be the weak solution of* (2.3) *with admissible initial values $v^0 \in L^2(\Omega)$, $s^0 \in (-\overline{r}, 1-\overline{r})$. Then, if $\psi < \psi_0$, v converges to zero in both $L^2(\Omega)$ and $\hat{H}^1(\Omega; C')$ and s converges to zero in $\mathbb{R}$ as $\tau \to \infty$.*

Proof. By Lemma 3.10,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\|v(.,\tau)\|^2_{L^2(\Omega)}\right) &+ 2\beta\|\nabla v(.,\tau)\|^2_{L^2(\Omega)} \\ &\leq 2\langle v_\tau, v\rangle + 2T[(v,s),v] - 2\gamma\|v(.,\tau)\|^2_{L^2(\Omega)} = -2\gamma\|v(.,\tau)\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.10}$$

which implies

$$\|v(.,\tau)\|_{L^2(\Omega)} \leq e^{-\gamma\tau}\|v^0\|_{L^2(\Omega)}$$

by Gronwall's inequality. Integrating from 0 to T,

$$\begin{aligned} \|v(.,T)\|^2_{L^2(\Omega)} - \|v^0\|^2_{L^2(\Omega)} + 2\beta\int_0^T \|\nabla v(.,\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\ \leq -2\gamma\int_0^T \|v(.,\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\,. \end{aligned} \tag{3.11}$$

Since the integral on the right-hand side converges as $T \to \infty$, the integral on the left-hand side must converge as well. It follows that

$$\|\nabla v(.,\tau)\|^2_{L^2(\Omega)} \to 0$$

as $T \to \infty$. Since $s(\tau) = (v(.,\tau), w_0)$, s converges to zero at least as fast as $\|v(.,\tau)\|_{L^2(\Omega)}$. □

Conclusion

Including surface tension in the model changes the equilibrium substantially: in the model without surface tension, equilibrium concentrations are constant globally, whereas concentrations are only piecewise constant in the model with surface tension. This is an indication that surface tension should be included. Intuitively, solute concentrations should in general by higher inside cells. Again, it should be noted that in some real-life experiments, the cell membrane will fail long before the system can reach equilibrium.

For the linearized system, including surface tension really changes the mathematical properties of the problem. The model is still stable when adding small enough surface tension, but in this case is different from the problem without surface tension. The symmetry of the differential operator is lost, which means that it can no longer be expected that there exists a complete orthonormal system of eigenfunctions. Including the surface tension spoils this nice structure, but leaves stability intact if the surface tension is not too large.

If the surface tension becomes large, the problem remains well-posed, but stability cannot be shown using coercivity anymore. There is a range of possibilities for the asymptotic behaviour in this case: for example, there might be perturbation which grow at first, but converge to zero later. It might also be possible that there are perturbation that really blow up.

First of all, nonlinear stability should be investigated. Since the linear problem is stable in all relevant directions, it is expected that for sufficiently small surface tension the original model is stable as well.

Further research should also consider more general models, most important the non-symmetric case. It is expected that non-symmetric situations have a strong tendency to become symmetric because of surface tension. Furthermore, varying surface tension might be a good candidate to include in the model.

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Nomenclature

$B_\varepsilon(x)$	ε-ball around x	
C	Cell ($\subset \Omega \times [0, \infty)$)	
C_t	Cell at time t ($\subset \Omega$)	
γ	Surface tension of the membrane	$\mathrm{N\,m^{-1}}$
κ	Diffusivity of solute	$\mathrm{m^2\,s^{-1}}$
μ	Helmholtz free energy	$\mathrm{J\,mol^{-1}}$
$\mathbf{n}$	Outward normal vector field	
Ω	Domain ($\subset \mathbb{R}^3$)	
$p(\theta, \varphi, t)$	Parametrization of ∂C_t	
$p^0(\theta, \varphi)$	Parametrization of ∂C_0	
$\mathfrak{p}$	Permeance of an aquaporin	$\mathrm{m^3\,mol\,s^{-1}\,J^{-1}}$
$\mathcal{P}$	Permeability of the membrane	$\mathrm{m\,mol\,s^{-1}\,J^{-1}}$
P	Hydrostatic pressure	$\mathrm{N\,m^{-2}}$
$\mathbf{q}$	Molar flux of solute	$\mathrm{mol\,m^{-2}\,s^{-1}}$
(r, θ, φ)	Spherical coordinates for Ω	m, rad, rad
ρ	Transformed radial variable	
R	Radius of Ω	m
$\mathcal{R}$	Gas constant	$\mathrm{J\,K^{-1}\,mol^{-1}}$
s	Perturbation of ∂C_t	
t	Time variable	s
τ	Transformed time variable	
$\mathcal{T}$	Absolute temperature	K
u	Osmolality	$\mathrm{mol\,m^{-3}}$
u^0	Initial osmolality	$\mathrm{mol\,m^{-3}}$
U	Total amount of solute C_t	mol
v	Osmolality perturbation	
v^0	Initial osmolality perturbation	
$v_{\mathbf{n}}$	Normal velocity of the membrane	$\mathrm{m\,s^{-1}}$
$\mathcal{V}$	Molar volume of water, $\sim 18 \cdot 10^{-6}$	$\mathrm{m^3\,mol^{-1}}$
x	Spatial variable	m

MAPLE commands

The following commands have been used to produce Figure 1.

```
> restart:
> with(plots):
Parameter values
> kappa := 0.4;
> r := 0.4;
> chi := 1;
> u := 1;
Auxiliary functions
> f1 := lambda -> r*sqrt(kappa)*cosh(sqrt(lambda/kappa)*r)
>       - kappa*sinh(sqrt(lambda/kappa)*r)/sqrt(lambda);
> f2 := lambda -> cosh(sqrt(lambda)*(1-r))
>       - sinh(sqrt(lambda)*(1-r))/sqrt(lambda);
> f3 := lambda -> (1-r)*cosh(sqrt(lambda)*(1-r))
>       + (r*lambda-1)*sinh(sqrt(lambda)*(1-r))/sqrt(lambda);
> f4 := lambda -> sinh(sqrt(lambda/kappa)*r)/sqrt(lambda);
Determinant
> F := lambda -> -f1(lambda)*( -u*lambda*r^2*f2(lambda)
>      + (psi/chi/r+lambda*r^3/chi)*f3(lambda) )
>      + (u + psi/chi/r)*lambda*r*f4(lambda)*f3(lambda);
> psi := 0;
> densityplot(log(1+abs(F(x+I*y))), x=-50..5,
>             y=-20..20, colorstyle=HUE);
> psi := 10;
> densityplot(log(1+abs(F(x+I*y))), x=-50..5,
>             y=-20..20, colorstyle=HUE);
> psi := 50;
> densityplot(log(1+abs(F(x+I*y))), x=-50..5,
>             y=-20..20, colorstyle=HUE);
> psi := 100000;
> densityplot(log(1+abs(F(x+I*y))), x=-50..5,
>             y=-20..20, colorstyle=HUE);
```