Take-home exam for the master course "Differential Geometry" - January 4, 2014, to be handed in before or on February 3, 2014-

Exercise 1 (about the intrinsic torsion and foliations)

The goal of this exercise is to compute the intrinsic torsion of p-dimensional distributions viewed as G-structures. See sections 1.5 and 2.5 of the lecture notes for more details on distributions and their integrated counterparts foliations. It is strongly recommended to read and understand the examples of section 5.7 as a preparation for this exercise.

The linear picture: let V be an n-dimensional vector space and $W \subset V$ a p-dimensional vector subspace. Define the Lie group $GL(V, W) = \{g \in GL(V) \mid g(W) \subset W\}$. Its Lie algebra is given by:

$$gl(V, W) = \{A \in gl(V) \mid A(W) \subset W\}$$

A linear G-structure corresponding to a choice of a subspace W is called a p-direction. The standard model of p-directions is $(V, W) = (\mathbb{R}^n, \mathbb{R}^p)$ with symmetry group $GL(\mathbb{R}^n, \mathbb{R}^p)$. Thus, a p-direction is a linear $GL(\mathbb{R}^n, \mathbb{R}^p)$-structure.

The global picture: a $GL(\mathbb{R}^n, \mathbb{R}^p)$-structure $\mathcal{F}$ on an n-dimensional manifold M is called a p-dimensional distribution. Fix such a structure $\mathcal{F}$. Note that we will denote by $\mathcal{F}$ both the $GL(\mathbb{R}^n, \mathbb{R}^p)$-structure (choice of frames) and the distribution (choice of subspaces of TM).

1. Let $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a connection on M compatible with $\mathcal{F}$. Give a global description of the compatibility condition on ∇ in terms of $\Gamma(\mathcal{F})$ (i.e. ∇ is compatible with $\mathcal{F}$ if and only if ...). Recall that $\Gamma(\mathcal{F})$ is the set of vector fields tangent to the distribution $\mathcal{F}$.

2. Recall that we have the map $\partial : \mathrm{Hom}(V, gl(V, W)) \to \mathrm{Hom}(\Lambda^2 V, V)$ and the associated torsion space $\mathcal{T}(gl(V, W))$. Prove the following linear isomorphism:

$$\mathcal{T}(gl(V, W)) \cong \mathrm{Hom}(\Lambda^2 W, V/W)$$

3. Choose a connection ∇ compatible with $\mathcal{F}$ and compute the intrinsic torsion $\bar{T}$ of $\mathcal{F}$ (hint: use the equivariance of $\bar{T}$ together with your findings from question 2 to view $\bar{T}$ as a section of $\mathrm{Hom}(\Lambda^2 \mathcal{F}, TM/\mathcal{F})$, i.e. the vector bundle whose fiber at $x \in M$ is $\mathrm{Hom}(\Lambda^2 \mathcal{F}_x, T_x M/\mathcal{F}_x)$).

4. Prove that the intrinsic torsion vanishes if and only if $\mathcal{F}$ is involutive (hence, by the theorem of Frobenius, if and only if $\mathcal{F}$ is integrable).

Exercise 2 (about the Hopf fibration)

The next exercise discusses various descriptions and aspects of the Hopf fibration

$$\pi : S^3 \longrightarrow S^2, \quad \pi(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - tx), 2(ty + xz)),$$

which is a principal S^1-bundle, with the S^1-action on S^3:

$$(x, y, z, t) \cdot (a, b) = (ax - by, ay + bx, az - bt, at + bz).$$

In complex coordinates, writing

$$S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\},$$

the projection becomes

$$\pi(z_0, z_1) = (|z_0|^2 - |z_1|^2, i\bar{z}_0 z_1)$$

and the action is given by

$$(z_0, z_1) \cdot \lambda := (z_0\lambda, z_1\lambda). \tag{1}$$

We first fix some notations regarding the spaces we will be using.

- the space of quaternions:
$$\mathbb{H} = \{x + iy + jz + kt : x, y, z, t \in \mathbb{R}\}$$
where we recall that $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$ and, for
$$u = x + iy + jz + kt \in \mathbb{H}$$
one defines
$$u^* = x + iy + jz + kt, \ |u| = \sqrt{uu^*} = \sqrt{x^2 + y^2 + z^2 + t^2}.$$
Note that S^3 can be interpreted now as the space of quaternionic numbers of norm 1; as a consequence, using the multiplication of quaternions, S^3 becomes a group (incidentally, note that this is isomorphic to $SU(2)$- but this will not be used below).

- the groups
$$SO(n) = \{A \in GL_n(\mathbb{R} : A\,{}^tA = \mathrm{Id}, \ \det(A) = 1\}.$$
These groups will be used for $n \in \{2, 3\}$, when we use the identification and inclusion
$$S^1 \xrightarrow{\sim} SO(2) \hookrightarrow SO(3),$$
$$\cos(\alpha) + i\sin(\alpha) \xrightarrow{\sim} \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{2}$$

In particular, consider the right action of S^1 on $SO(3)$ coming from this inclusion and the multiplication of matrices:

$$A \cdot \lambda = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{for } \lambda = \cos(\alpha) + i\sin(\alpha) \in S^1 \tag{3}$$

For $A \in SO(3)$ we denote by $A_i \in \mathbb{R}^3$ the vector corresponding to the i^{th} row of A, $i \in \{1,2,3\}$. Note that the condition $A\,^tA = \mathrm{Id}$ can be written as

$$||A_1|| = ||A_2|| = ||A_3|| = 1, \langle A_1, A_2\rangle = \langle A_2, A_3\rangle = \langle A_1, A_3\rangle = 0$$

where $||\cdot||$ and $\langle \cdot,\cdot\rangle$ are the standard Euclidean norm and inner porduct on $\mathbb{R}^3$.

- the complex projective line $\mathbb{CP}^1$, consisting of all complex lines in $\mathbb{C}^2$ through the origin (i.e. 1-dimensional complex vector subspaces of $\mathbb{C}^2$). For $(z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$, we denote by

$$[z_0 : z_1] := \mathbb{C} \cdot (z_0, z_1) \in \mathbb{CP}^1$$

the induced line, so that we can write

$$\mathbb{CP}^1 = \{[z_0 : z_1] : (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}\},$$

$$[z_0 : z_1] = [z_0' : z_1'] \Longleftrightarrow \exists\, \lambda \in \mathbb{C} \text{ s.t. } z_0' = \lambda z_0, z_1' = \lambda z_1.$$

- Over $\mathbb{CP}^1$ we consider the tautological bundle E (of rank 2 as a real vector bundle) which is defined as the following sub-bundle of the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}^2$:

$$E := \{(l, v) : l \in \mathbb{CP}^1, v \in l\} \subset \mathbb{CP}^1 \times \mathbb{C}^2.$$

TO DO:

0. Check directly that $\pi : S^3 \longrightarrow S^2$ is a principal S^1-bundle.

1. Remark that, for any $u \in S^3$,

$$iu, ju, ku \in \mathbb{H} \cong \mathbb{R}^4$$

are orthogonal to u, i.e. they define vectors tangent to S^3 at u. Use this to prove that S^3 is parallelizable. More precisely, describe explicitly the resulting vector fields X_1 (using i), X_2 (using j) and X_3 (using k) which define a parallelism of S^3. For instance,

$$X_1(x, y, z, t) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} \in T_{x,y,z,t}S^3 \subset T_{x,y,z,t}\mathbb{R}^4.$$

2. Here is a slightly different argument for the parallelizability of S^3, using the group structure that comes from multiplication of quaternions (see above). Fix $u_0 = (1, 0, 0, 0) \in S^3$ and the tangent vectors

$$v_1 := \frac{\partial}{\partial y}(u_0), v_2 := \frac{\partial}{\partial z}(u_0), v_3 := \frac{\partial}{\partial t}(u_0) \in T_{u_0}S^3.$$

For $u = (x, y, z, t) \in S^3$, identified with the quaternion $x + iy + jz + kt$, consider the right multiplication by u:

$$R_u : S^3 \longrightarrow S^3, R_u(v) = vu,$$

its differential $(dR_u)_{u_0}$ at u_0 and

$$X_1(u) := (dR_u)_{u_0}(v_1) \in T_u S^3 \quad (uu_0 = u!)$$

and similarly X_2 and X_3. Show that these coincide with the vector fields you found above.

3. Show that the flow of X_1 induces the action (1) of S^1 on S^3.

4. Let $\theta^1, \theta^2, \theta^3$ the associated coframe (i.e. defined by $\theta^i(X_j) = \delta^i_j$). Compute its structure equations.

5. Show that θ^1 can be interpreted as a connection on the Hopf fibration.

6. Show that the curvature of this connection, $K \in \Omega^2(S^2)$, is a non-zero constant (that you have to compute) times the area form σ of S^2,

$$\sigma = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2).$$

(be aware: the forms dx, etc make sense on the entire $\mathbb{R}^3$, but we only use their restrictions to S^2. And, while the forms dx, dy and dz are linearly independent on $\mathbb{R}^3$, their restrictions to S^2 do satisfy an extra-relation: $xdx + ydy + zdz = 0$ (why?)).

7. Show that

$$\pi_1 : S^3 \longrightarrow \mathbb{CP}^1, \ \pi_1(z_0, z_1) = [z_0 : z_1]$$

is a principal S^1-bundle, where the action of S^1 on S^3 is given by (1). What is the analogue of this in higher dimensions?

8. Show that there exists a diffeomorphism

$$\Psi : S^2 \xrightarrow{\sim} \mathbb{CP}^1 \tag{4}$$

uniquely determined by $\pi_1 = \Psi \circ \pi$ (i.e. identifying the Hopf fibration with the one from the previous item).

9. Show that one has a principal S^1-bundle

$$\pi_0 : SO(3) \longrightarrow S^2,$$

where the (right) action of S^1 on $SO(3)$ is the one from (3). What is the analogue of this in higher dimensions?

10. Show that the associated vector bundle $E[SO(3), \mathbb{R}^2, \rho]$, obtained from this principal S^1-bundle by attaching the fiber $\mathbb{R}^2$ via the representation of S^1

$$\rho : S^1 \xrightarrow{\sim} SO(2) \subset GL_2(\mathbb{R}) = GL(\mathbb{R}^2)$$

(see also (2)), is isomorphic to the tangent bundle of S^2.

11. Embedd $SO(3)$ inside $\mathrm{Fr}(S^2)$ and interpret it as an S^1-structure on S^2.

12. Identifying S^3 with the space of quaternionic numbers of norm 1,

$$S^3 \xrightarrow{\sim} \{u \in \mathbb{H} : |u| = 1\}, \ (x, y, z, t) \mapsto x + iy + jz + kt$$

and $\mathbb{R}^3$ with the space of pure quaternions

$$\mathbb{R}^3 \xrightarrow{\sim} \{v \in \mathbb{H} : v + v^* = 0\}, \ (a, b, c) \mapsto ai + bj + ck,$$

show that

$$A_u(v) := u^* v u$$

defines, for each $u \in S^3$, a linear map $A_u : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which, as matrix, gives

$$A_u \in SO(3).$$

13. Consider the induced map

$$\phi : S^3 \longrightarrow SO(3), \ u \mapsto A_u.$$

Show that the projection of the Hopf fibration (see the beginning) is precisely

$$\pi = \pi_0 \circ \phi.$$

(incidentally note that $\phi(u) = \phi(v)$ happens iff $v = u$ or $v = -u$- and this shows that $SO(3)$ can be identified with the real projective space $\mathbb{P}^3$.)

14. Show that

$$\phi(u \cdot \lambda) = \phi(u) \cdot \lambda^2 \ \ \forall\, u \in S^3, \ \lambda \in S^1,$$

where $u \cdot \lambda$ uses to the action of S^1 on S^3 given by (1) and $\phi(u) \cdot \lambda^2$ uses the action of S^1 on $SO(3)$ given by (3).

15. Consider the vector bundle $E_1 = E[S^3, \mathbb{R}^2, \rho]$ associated to the Hopf fibration by attaching the fiber $\mathbb{R}^2$ via the representation ρ of S^1 (see item 10 above) and, similarly, $E_2 = E[S^3, \mathbb{R}^2, \rho^2]$ defined using the square ρ^2 of the representation ρ:

$$\rho^2 : S^1 \longrightarrow GL_2(\mathbb{R}), \rho^2(\lambda) = \rho(\lambda)^2 = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{pmatrix}$$

for $\lambda = \cos(\alpha) + i\sin(\alpha) \in S^1$. Show that E_2 is isomorphic to the tangent bundle of S^2 and E_1 is isomorphic to $\Psi^*(E)$ (where E is the tautological line bundle- see the beginning, and for Ψ see (4)).