Convexity in Contact Topology

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(Translated by Daniel Mathews)

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Contents

Translator's note

Hopefully most of this is accurate. Thanks to Patrick Massot for several corrections and suggestions. All translation errors and inaccuracies, however, should be ascribed to me alone. All footnotes are mine. Figures and references can be found in the original.

Introduction

This article tackles the study of convexity in contact geometry, as it has been defined in [EG]: a structure, symplectic or contact, is called convex if it is conformally invariant under the gradient flow of a proper Morse function. For symplectic manifolds, this property plays the same role as that of strict pseudoconvexity in complex analytic manifolds. It can only, for example, be shown on open manifolds having the homotopy type of polyhedra of half dimension and, in [EG], Ya. Eliashberg and M. Gromov show how it tempers the geometry and forbids certain exotic phenomena (see also [Gr] and [El1]). In contact geometry, the situation presents itself differently. First of all, the usual structures, on jet spaces of order 1, spheres and manifolds of contact elements, are all convex (see I.4.C). Then, the results obtained here show that in dimension 3, there exist numerous convex contact manifolds. In particular, certain exotic structures discovered by T. Erlandsson and D. Bennequin (see [Be]) are convex; in fact, we do not know any examples of non-convex structures.

The approach adopted is the following: given a proper Morse function f on a 3-dimensional manifold V, one tries to construct on V a contact structure ξ which is invariant under the flow of a gradient X of f. The study of contact fields (i.e. fields preserving the contact structure) shows that, if this structure ξ exists, the surface C of points of V where X is tangent to ξ must satisfy, vis-à-vis f, the following conditions (Proposition I.4.5):

(i) $f|_C$ is a proper Morse function;

(ii) the critical points of f are all on C and are exactly the critical points of $f|_C$;

(iii) f and $f|_C$ have the same local extrema.

A Morse function does not always admit surfaces satisfying these properties (see IV.1.B). Nevertheless, one can modify it, adding only several critical points of indices 1 or 2 in a position of elimination, so that such a surface C exists (Theorem IV.2.7). Also, being given C allows one effectively to construct the desired contact structure ξ (Theorem III.1.2). To obtain this, one puts on each handle an induced structure via immersion in a well-chosen model on $\mathbb{R}^3$. The difficulty is to adjust these immersions in order to be able to glue the pieces: this problem is localised along certain faces of the handles. Yet, in the neighbourhood of a surface, a contact structure is entirely described by the (singular) dimension-1 foliation that it traces on the surface. Moreover, each surface considered here, corresponding to a regular level set of f, is found, by construction, to be transverse in $\mathbb{R}^3$ to a vector field that preserves the model structure and holds the role of gradient of f. The crucial point then is to understand how, when one moves the surface by an isotopy keeping it transversal to this field, one modifies its foliation (Proposition II.3.6). At this point, a convex contact structure appears geometrically describable by a finite number of these foliations, carried by the different regular level sets of the function and determined only up to prior modifications.

Among these possible modifications of the foliation is the elimination of pairs of singularities (Lemma II.3.3). One can thus extend a result of Ya. Eliashberg that permits the elimination of certain complex points on a surface contained in the pseudo-convex boundary of a holomorphic domain (see [El1], Theorem 6.1 and [El2]). For that, in place of the theory of holomorphic curves in 4-dimensional symplectic manifolds, we use the following remarkable fact (Proposition II.2.6): in a contact 3-dimensional manifold, a surface generically possesses a transverse contact vector field. Thanks to this property of invariance, the problem of elimination reduces to the symplectic geometry of surfaces.

The problems studied in this article have been expounded to me by Yasha Eliashberg during some conversations which were marvellously enriching for me; I thank him heartily. I equally thank François Laudenbach and Jean-Claude

Sikorav for their numerous remarks and pertinent suggestions regarding this text.

I The notion of convexity

1 Preliminary definitions

A Symplectic and contact structures

A *symplectic structure on a vector space* V of dimension $2n$ is an exterior 2-form ω whose exterior n'th power is nonzero. The *orthogonal complement* of a subspace W of V is the subspace $\{v \in V \mid \forall w \in W,\ \omega(v,w) = 0\}$.

One says that W is *coisotropic* if it contains its orthogonal complement. Note that, if c is a nonzero real number, $c\omega$ is still a symplectic form and the orthogonal complement of W is the same for ω and $c\omega$.

A *symplectic structure on a vector bundle* of even rank is a field of symplectic forms on its fibres.

A *symplectic structure on a manifold* V of dimension $2n$ is a closed differential 2-form ω which induces on each tangent space a symplectic form.

A *contact structure* on a manifold V of dimension $2n+1$ is a completely non-integrable hyperplane field ξ, that is, defined locally by a 1-form α such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing. In other words, $d\alpha|_\xi$ is at every point a symplectic form. Multiplying α by a nowhere vanishing function f changes $d\alpha|_\xi$ to $f \cdot d\alpha|_\xi$, so that ξ is furnished with a *conformal symplectic structure*. We also remark that, if n is even, ξ is naturally oriented while, if n is odd, V is naturally oriented. In either case, any orientation transverse to ξ (which exists if and only if ξ admits a global equation $\alpha = 0$) orients at the same time ξ and V.

B Singular foliations of dimension 1

In this text, a *singular foliation* (of dimension 1) on a manifold M of dimension m is a foliation $\mathcal{F}$ defined by an atlas $\{U_i, X_i\}$ where: $\{U_i\}$ is a covering of M, X_i a vector field on U_i and, for every (i,j), there exists a nowhere vanishing function f_{ij} on $U_i \cap U_j$ such that $X_i = f_{ij} X_j$.

Remark 1.1 *If each U_i is furnished with a volume form θ_i, the data of X_i is equivalent to being that of the $(m-1)$-form $i(X_i)\theta_i$ (interior product of θ with X_i).*

We say that a vector field X on M *directs* $\mathcal{F}$ if, for all i, there exists a nowhere vanishing function f_i on U_i such that $X = f_i X_i$; we say that $\mathcal{F}$ is *orientable* if such a field exists.

C Characteristic foliation of a hypersurface

Let S be a hypersurface in a contact manifold (V, ξ) of dimension $2n+1$. The trace on ξ of the tangent bundle of S determines a distribution (of non-constant rank) of coisotropic subspaces in $\xi|_S$. The orthogonal distribution, via the conformal symplectic structure on $\xi|_S$, is of rank 0 on the singular locus Σ where ξ is tangent to S, and of rank 1 otherwise. It defines a singular foliation, in the sense of B, that we call the characteristic foliation of S. Locally, if θ is a volume form on S and β the 1-form induced by an equation for ξ, the characteristic foliation is defined by the vector field X such that $i(X)\theta = \beta \wedge (d\beta)^{n-1}$. One easily verifies that the characteristic foliation of S is orientable if and only if the normal bundle of S is isomorphic to the quotient bundle $(TV/\xi)|_S$.

Remark 1.2 *Outside the singular locus Σ, the characteristic foliation $\mathcal{F}$ of S has a transverse contact structure, $(\xi \cap TS)/\mathcal{F}$, invariant under holonomy (see [McD]). On Σ, $TS|_\Sigma = \xi|_\Sigma$ has a conformal symplectic structure, invariant under local vector fields which direct $\mathcal{F}$ (see 2.C).*

2 Characteristic hypersurface of a contact vector field

A Contact vector field

Let (V, ξ) be a contact manifold.

Definition 2.1 *A* contact vector field *on (V, ξ) is a vector field whose flow preserves ξ.*

It is well known (see [A]) that:

Proposition 2.2 *The contact vector fields on (V, ξ) are in bijective correspondence with the sections of the normal bundle to ξ, TV/ξ. In other words, any section of this quotient lifts to a unique contact vector field.*

Corollary 2.3 *Any contact vector field given locally can be extended globally.*

Remark. In the presence of an equation for ξ, i.e. of a trivialisation of TV/ξ, a section of TV/ξ is nothing other than a function, called the Hamiltonian of the corresponding contact vector field.

B Characteristic hypersurface

Let X be a contact vector field on (V, ξ).

Definition 2.4 *The* characteristic hypersurface *of X is the set $C = C(X)$ of points where X is tangent to ξ.*

On the space of vector fields (furnished with the C^∞ topology), the property of having a reduction modulo ξ transverse to the zero section of TV/ξ is generic. In this case, by abuse of language, we will say that the vector field is *generic*. Its characteristic hypersurface is then regular.

Proposition 2.5 *If X is generic, X is tangent to its characteristic hypersurface C and directs the characteristic foliation on it.*

PROOF The flow of X preserves X and ξ, therefore C, so that X is tangent to C.

Let now x be a point of C and α a local equation for ξ near x. The hypersurface C is defined locally by the equation $i(X)\alpha = 0$ (regular since X is generic). Also, as X is contact, the Lie derivative of α satisfies: $L_X\alpha = g\alpha$ for some function g. For $v \in T_xC \cap \xi_x$, we then have:

$$\begin{aligned} d\alpha(x) \cdot (X(x), v) &= (L_X\alpha)(x) \cdot v - (di(X)\alpha)(x) \cdot v \\ &= (g\alpha)(x) \cdot v - (di(X)\alpha)(x) \cdot v = 0 \end{aligned}$$

since the two terms are zero. Thus $X(x)$ is orthogonal to $T_xC \cap \xi_x$.

Moreover, if $X(x) = 0$, we have:

$$(L_X\alpha)(x) = (g\alpha)(x) = (di(X)\alpha)(x).$$

Therefore ξ is tangent to C at x. ■

Remark. If ξ is transversally orientable, there exist contact vector fields X with empty characteristic hypersurface; there are vector fields transverse to ξ, i.e. Reeb fields associated to various equations for ξ.

Example 2.6 *Any contact vector field X which is nonsingular or has nondegenerate singularities is generic.*

PROOF Let α be a local equation for ξ; we want to show that $d(i(X)\alpha)$ is nonzero at every point where $i(X)\alpha$ is zero. As X preserves ξ, $L_X\alpha = g\alpha$ for some function g. Then $di(X)\alpha = g\alpha - i(X)d\alpha$.

If X is nonsingular at $x \in C$, $(i(X)d\alpha)(x)$ is not proportional to $\alpha(x)$ since $d\alpha(x)$ is nondegenerate on ξ_x. Thus, considering the expression for the Lie derivative, $di(X)\alpha$ is nonzero at x.

Now, if X has at x a nondegenerate singularity, its linearisation $A_x : T_xV \longrightarrow T_xV$ is invertible. Then the form $(di(X)\alpha)(x)$, which is equal to $\alpha(x) \circ A_x$, is nonsingular. ■

C Singularities of contact vector fields

Remarks.

(i) The singularities of a contact vector field lie on its characteristic hypersurface.

(ii) The divergence of a vector field at a singular point does not depend on the local volume form with which it is calculated: it is the trace of the linearisation.

Proposition 2.7 *Let (V, ξ) be a contact manifold of dimension $2n + 1$ and let X be a generic contact vector field. To each singularity x of X is associated a nonzero real number $c = c(x)$ (the coefficient of contraction) having the following properties:*

(i) $(n+1)c$ (resp. nc) is the divergence of X at x (resp. of $X|_C$ at x);

(ii) for any local equation α of ξ, which induces a form β on C, we have:

$$(L(X)\alpha)(x) = c\alpha(x) \quad \textit{and} \quad (L(X|_C)d\beta)(x) = c\, d\beta(x).$$

Proof As X is contact, $L(X)\alpha = g\alpha$ for some function g; thus, if it exists, the desired coefficient is $c = g(x)$. But, as X is generic, $g(x)$ is nonzero. Then:

$$L(X)d\alpha = dL(X)\alpha = dg \wedge \alpha + g\, d\alpha. \qquad \blacksquare$$

As $\beta(x) = 0$, we clearly have: $(L(X|_C)d\beta)(x) = c\, d\beta(x)$. Now, to see that c does not depend on the choice of α it suffices to show (i). But as $\alpha \wedge (d\alpha)^n$ and $(d\beta)^n$ are local volume forms on V and C respectively, (i) follows from the expressions above by derivation of a product. For example:

$$\begin{aligned} L(X)(\alpha \wedge (d\alpha)^n) &= (L(X)\alpha) \wedge (d\alpha)^n + \alpha \wedge L(X)(d\alpha)^n \\ &= g\alpha \wedge (d\alpha)^n + n\alpha \wedge L(X)d\alpha \wedge (d\alpha)^{n-1} = (n+1)g\alpha \wedge (d\alpha)^n. \end{aligned}$$

Corollary 2.8 *If x is a singularity of a generic contact vector field, the eigenvalue transverse to C (the tangent space to C is stable) is equal to c.*

Corollary 2.9 *Suppose that X is, for a certain metric, the gradient of a function f which has at x a Morse critical point of index i. If $c(x)$ is positive (respectively negative), then i is at most equal to n (resp. at least equal to $n+1$).*

Proof Let α be an equation of ξ near x and β the form induced on C.

The form $d\beta(x)$ is a symplectic form on T_xC. If $c(x)$ is positive, the tangent space at x to the stable manifold of $X|_C$ has dimension i, since the transverse eigenvalue is positive; moreover it is necessarily isotropic, that is, contained in its symplectic orthogonal complement (see remark 4.3). Then, i is at most equal to n. By similar reasoning on the unstable manifold, we see that, if $c(x) < 0$, then $i \geq n+1$. $\blacksquare$

3 Convex hypersurfaces

A Definition, example

Definition 3.1 *We say that a hypersurface S embedded in a contact manifold (V, ξ) is* convex *if there exists a contact vector field transverse to S.*

A convex hypersurface is therefore transversally orientable, that is, its tubular neighbourhoods are diffeomorphic to $S \times \mathbb{R}$. Also, any germ of a contact vector field along S, which is transverse to S, extends to a global contact vector field. The study of convex hypersurfaces is therefore closely linked to that of contact structures on $S \times \mathbb{R}$ invariant under the vertical vector field $\partial/\partial t$, where t denotes the coordinate on $\mathbb{R}$.

Example 3.2 *(Contactization of an exact symplectic manifold). We say that a symplectic manifold (W, ω) is* exact *if ω is the differential of a 1-form β called the Liouville form. By symplectic duality, it is equivalent to say that there exists on W a vector field X, called the Liouville vector field, whose flow dilates ω exponentially: $L(X)\omega = \omega$. If $(W, \omega = d\beta)$ is an exact symplectic manifold, the form $\beta + dt$ defines on $W \times \mathbb{R}$ a vertically invariant contact structure. Moreover, the Liouville field X, ω-dual to β, directs the characteristic foliation on the hypersurfaces $W \times \{t\}$, $t \in \mathbb{R}$.*

Remark 3.3 *(i) The contact structure thus obtained depends not only on the symplectic structure ω but also on the primitive β chosen. We observe however that, if we change β to $\beta + dh$, where h is a function on W, the diffeomorphism $\phi : W \times \mathbb{R} \longrightarrow W \times \mathbb{R}$, given by $\phi(x,t) = (x, t + h(x))$, satisfies $\phi^*(\beta + dt) = (\beta + dh) + dt$. This therefore establishes an isomorphism between the two contact structures.*

(ii) If $H \subset W$ is a hypersurface transverse to X, the form induced by β on H is contact. Indeed, $\beta \wedge (d\beta)^{n-1} = (1/n)\ i(X)\omega^n$ induces on H a volume form.

B Vertically invariant contact structures

Let S be a manifold of dimension $2n$. A contact structure ξ, transversally orientable and vertically invariant, on the cylinder $S \times \mathbb{R}$, can be defined by a global equation $\beta + u\ dt = 0$, where β and u are respectively a 1-form and a function on S such that:

$$\text{the form } \theta = (d\beta)^{n-1} \wedge (u\ d\beta + n\beta \wedge du) \text{ does not vanish on } S. \tag{1}$$

In fact $\theta \wedge dt = (\beta + u\ dt) \wedge (d(\beta + u\ dt))^n$.

We observe that:

(i) The set Σ where $u = 0$ is the trace on $S \times \{0\}$ of the characteristic hypersurface of the vector field $\partial/\partial t$; it's a regular hypersurface on which β induces a contact form since, along Σ, 1 is written $(d\beta)^{n-1} \wedge \beta \wedge du \neq 0$.

(ii) On the open set $\Omega = S \backslash \Sigma$, ξ is still defined by $\beta/u + dt = 0$ and we have: $\theta = u^{n+1}(d(\beta/u))^n$. Thus $(\Omega \times \mathbb{R}, \xi)$ is the contactization of the exact symplectic manifold $(\Omega, d(\beta/u))$.

Let Y be the vector field tangent to S defined by:

$$\beta \wedge (d\beta)^{n-1} = i(Y)\theta. \tag{2}$$

This vector field directs the characteristic foliation of $S \times \{0\}$ (see 1.C) and satisfies the relations below.

(iii) On Σ: $Y \cdot u = -1/n$. Indeed:

$$\beta \wedge (d\beta)^{n-1} = -n\ i(Y)[du \wedge \beta \wedge (d\beta)^{n-1}] = -n(Y \cdot u)\beta \wedge (d\beta)^{n-1},$$

since $i(Y)[\beta \wedge d\beta^{n-1}] = 0$.

(iv) On Ω, let X be the Liouville vector field of β/u defined by $\beta/u = i(X)d(\beta/u)$; we have:

$$X = nuY.$$

Indeed:

$$i(X)\theta = u^{n+1}i(X)\left(d\left(\frac{\beta}{u}\right)\right)^n = nu^{n+1}\frac{\beta}{u}\wedge\left(d\left(\frac{\beta}{u}\right)\right)^{n-1} = nu\beta\wedge(d\beta)^{n-1}.$$

Proposition 3.4 *Let S be a closed manifold of dimension $2n$ and let $\mathcal{F}$ be a singular foliation of dimension 1 on S (see 1.B). There exists on $S \times \mathbb{R}$ a vertically invariant contact structure which induces $\mathcal{F}$ as characteristic foliation on $S \times \{0\}$ if and only if there exists in S a hypersurface Σ transverse to $\mathcal{F}$ (in particular, avoiding the singularities of $\mathcal{F}$) such that:*

(i) the complement S' of an open tubular neighbourhood of Σ, whose fibres are in $\mathcal{F}$, is an exact symplectic manifold whose Liouville field directs $\mathcal{F}$ and exits transversally on the boundary;

(ii) the involution of the double cover $\partial S' \longrightarrow \Sigma$, obtained by following the leaves of $\mathcal{F}$ across the tube, preserves the contact structure induced on $\partial S'$ (see remark 3.3b) but reverses its transverse orientation.

Proof We suppose first of all that there exists on $S \times \mathbb{R}$ a vertically invariant contact structure ξ which induces $\mathcal{F}$ as characteristic foliation on $S \times \{0\}$. The intersection Σ of S with the characteristic hypersurface of the vector field $\partial/\partial t$ is a hypersurface of S transverse to $\mathcal{F}$ (see (1) and (3) above). On $\Omega = S\backslash\Sigma$, the vertical vector field is transverse to ξ, so ξ is transversally orientable and defined by a unique equation $\beta + dt = 0$, where β is necessarily a Liouville form on Ω. Using local equations near Σ and the relations (3) and (4) above, we see that the Liouville field X associated to β exits along $\partial S'$, if S' is chosen as in the statement. Finally, the contact structure ξ' defined by β on $\partial S'$ is the trace on $\partial S'$ of the contact structure transverse to $\mathcal{F}$ and invariant under the holonomy of $\mathcal{F}$. It follows that the involution of the double cover $\partial S' \longrightarrow \Sigma$ preserves ξ'; but, as X changes the direction of crossing Σ, the transverse orientation of ξ' is reversed.

Conversely, we now suppose that conditions (i) and (ii) are satisfied. We denote by $d\beta$ the exact symplectic structure on S' whose Liouville field X directs $\mathcal{F}$ and exits along $\partial S'$.

Lemma 3.5 *We can suppose that: (ii)' the involution of the double cover $\partial S' \longrightarrow \Sigma$ reverses the form induced by β on $\partial S'$.*

Proof Let $\bar{S}'$ be the manifold obtained as follows: we glue to S' the cylinder $\partial S' \times [0, \infty)$ along $\partial S' = \partial S' \times \{0\}$, connecting X with the vector field $\partial/\partial r$ where r is the coordinate on $[0, \infty)$; we still denote the extended vector field X. If η denotes the 1-form induced by β on $\partial S'$, we extend β to $\bar{S}'$ by setting

$\beta = e^r\eta$ on $\partial S' \times [0, \infty)$. Then $(\bar{S}', d\beta)$ is an exact symplectic manifold whose Liouville field is X.

Now let τ be the involution of the double cover $\partial S' \longrightarrow \Sigma$; by hypothesis, there exists a function, negative on $\partial S'$, denoted $-e^h$, satisfying $\tau^*\eta = -e^h\eta$; as τ^2 is the identity, we have: $\tau^* h = -h$. Let h_0 be a minimum of h on $\partial S'$ and let

$$S'_0 = S' \cap \{(y, r) \in \partial S' \times [0, \infty) \mid r \leq \frac{1}{2}[h(y) - h_0]\}.$$

Then the form induced by β on $\partial S'_0 \cong \partial S'$ is: $\eta_0 = e^{(h-h_0)/2}\eta$; by the following:

$$\tau^*\eta_0 = e^{\tau^*(h-h_0)/2}\tau^*\eta = -e^{-(h+h_0)/2}e^h\eta = -\eta_0.$$

Finally we have an isotopy which sends S'_0 to S' and respects the foliation by the orbits of X, which proves the lemma. ■

On $S' \times \mathbb{R}$, the equation $\beta + dt = 0$ defines a vertically invariant contact structure ξ which we seek to extend over $\Sigma \times \mathbb{R}$. For this, we suppose first of all that Σ is transversally orientable, and we take a split neighbourhood, $U \cong \Sigma \times (-1-\epsilon, 1+\epsilon)$, in which the leaves of $\mathcal{F}$ are the segments $\{pt\} \times (-1-\epsilon, 1+\epsilon)$. We choose the parametrisation so that:

- $\Sigma \cap U = \Sigma \times \{0\}$ and $\partial S' \cap U = \Sigma \times \{-1, 1\}$;
- on $S' \cap U$, X has the expression $-s(\partial/\partial s)$, where s is the coordinate on the interval $(-1-\epsilon, 1+\epsilon)$.

The relations $L(X)\beta = \beta$, $i(X)\beta = 0$ and property (ii)' show that $\beta|_{S' \cap U} = (1/s)\gamma$, where γ is a contact form on $\Sigma \times \{1\}$. Then the form $\gamma + s\, dt$ defines on U a contact structure which coincides with ξ on $(U \cap S') \times \mathbb{R}$.

Finally, if Σ is not transversally orientable, we pass to a cover of S in which it becomes so and we perform the preceding construction in an equivariant manner. ■

Remark 3.6 (F. Laudenbach) *If n is even and if S is orientable, the hypersurface Σ separates. Indeed, ξ is then orientable, thus transversely orientable, since $S \times \mathbb{R}$ is orientable. Then, the two sides of Σ are given by the sign of $\partial/\partial t$ relative to this transverse orientation.*

4 Convex contact structures

A Pseudo-gradients of a Morse function

Definition 4.1 *Let $f : M \longrightarrow \mathbb{R}$ be a Morse function, that is, a function all of whose critical points are nondegenerate. We say that a vector field X is a* pseudo-gradient *of f if there exists on M a Riemannian metric and a positive function ρ such that, everywhere in M, we have $X \cdot f \geq \rho||df||^2$. We then have a similar relation for any other Riemannian metric. For example: the gradient of f for a given metric verifies this inequality.*

We recall that a singularity x of a vector field X is *hyperbolic* if the linearisation A_x of X at x is hyperbolic, i.e., has no eigenvalue with zero real part. In this case, the theorem of the stable manifold asserts that the points having x for ω-limit (resp. α-limit) form an immersed submanifold called the stable manifold (resp. unstable); its tangent space at x is the stable (resp. unstable) manifold of the linearised field A_x. It is well known that:

Proposition 4.2 *Let f be a Morse function on a manifold M of dimension m and let X be a pseudo-gradient of f. Then:*

(i) the singularities of X are hyperbolic and are exactly the critical points of f;

(ii) at a critical point of f of index i, the stable (resp. unstable) manifold of X has dimension i (resp. $m - i$).

Remark 4.3 *Let A be a hyperbolic endomorphism of $\mathbb{R}^{2n}$ and let ω be a linear symplectic form on $\mathbb{R}^{2n}$. If $(e^{tA})^*\omega = e^{ct}\omega$ for c a positive constant and for all real t, then the stable manifold W^s of the linearised field A is isotropic (i.e. contained in its symplectic orthogonal complement). Indeed, for $v, w \in W^s$, $\omega(v, w) = e^{-ct}\,\omega(e^{tA}v, e^{tA}w)$ tends towards 0 when t tends towards $+\infty$, thus is zero. This allows us to extend corollary 2.9 to the case where X is a pseudo-gradient of a Morse function.*

B Notion and condition of convexity for a contact structure

In [EG], Ya. Eliashberg and M. Gromov propose the following definition.

Definition 4.4 *We say that a contact structure ξ on a manifold V is* convex *if there exists a proper Morse function $f : V \longrightarrow [0, \infty)$ having a complete pseudo-gradient which preserves ξ.*

The regular levels of f are then convex hypersurfaces. Moreover it follows from 2.C and 4.A that:

Proposition 4.5 *Let (V, ξ) be a contact manifold and $f : V \longrightarrow [0, \infty)$ a proper Morse function. If ξ is preserved by a pseudo-gradient of f, the characteristic hypersurface C of this vector field satisfies the following:*

(i) $f|_C$ is a proper Morse function;

(ii) the critical points of f are on C and are exactly the critical points of $f|_C$;

(iii) a critical point of index i for f gives, for $f|_C$, a critical point of index i if $i \leq n$ and of index $i - 1$ if $i \geq n + 1$.

In part III, we will show how to construct, conversely, convex contact structures on a 3-dimensional manifold V, starting from a Morse function and a surface in V satisfying the above conditions.

C Examples of convex contact structures

Example 4.6 *(Contactization of a Weinstein manifold)*

Definition 4.7 (Ya. Eliashberg and M. Gromov, EG) *We say that a symplectic manifold* (W, ω) *is* Weinstein *if there exists a proper Morse function* $f_0 : W \longrightarrow [0, \infty)$ *having a complete pseudo-gradient* X_0 *which dilates* ω *exponentially:* $L(X_0)\omega = \omega$. *Such a symplectic manifold is therefore exact since, as* ω *is closed, we have* $\omega = d\beta$ *where* $\beta = i(X_0)\omega$.

Under these conditions, the contact structure ξ *defined on* $W \times \mathbb{R}$ *by the equation* $\beta + dt = 0$ *is convex. Indeed, the field* $X = X_0 + t(\partial/\partial t)$ *preserves* ξ *since* $L(X)(\beta + dt) = \beta + dt$. *Moreover,* X *is a complete pseudo-gradient for the proper Morse function* $f : W \times \mathbb{R} \longrightarrow [0, \infty)$ *given by* $f(x, t) = f_0(x) + t^2$.

A typical example of a Weinstein manifold is the cotangent space (of any manifold whatsoever) furnished with its canonical symplectic structure ω. *In this case, we can choose* X_0 *so that* $\beta = i(X_0)\omega$ *differs from the canonical Liouville form by the differential of a function. The contactization of* β *is then isomorphic to the canonical contact structure on the space of 1-jets of functions (see remark 3.3a (3.3, (i))): this structure is consequently convex.*

Example 4.8 *The contact structure given on* S^{2n+1} *by the complex tangencies of the unit sphere in* $\mathbb{C}^{n+1}$ *is convex. Indeed, if* $z_j = x_j + iy_j$, $1 \leq j \leq n+1$, *are the coordinates, this structure has for example the form induced by* $-\sum y_j\, dx_j$; *we check then that the contact vector field associated to the hamiltonian* x_k *is a pseudo-gradient of the function* y_k. *The characteristic hypersurface of this vector field is the equatorial sphere with equation* $x_k = 0$.

Example 4.9 *(Canonical structure on the manifold of contact elements.) Let* $\pi : V \longrightarrow M$ *be the fibre bundle of contact elements on a manifold* M *of dimension* $n+1$. *Then the canonical contact structure on* V *(see [A]) is convex.*

Argument. *Given a proper Morse function* $f_0 : M \longrightarrow [0, \infty)$, *we choose a complete pseudo-gradient* X_0 *of* f_0 *having the following property: at each critical point of* f_0, *the eigenvalues of* X_0 *are real and distinct. Like all vector fields on* M, *the vector field* X_0 *lifts naturally to a contact vector field* X *on* V. *It turns out that* X *is a complete pseudo-gradient for some proper Morse function* $f : V \longrightarrow [0, \infty)$. *We obtain* f *by perturbing by perturbing the function* $f_0 \circ \pi$ *above a neighbourhood of the critical points of* f_0 *as follows: above such a point* x, *the vector field* X *is tangent to the fibre* $F = \pi^{-1}(x)$ *and is none other than the vector field induced naturally by the linearisation of* X_0 *on the projective cotangent space; as the eigenvalues of* X_0 *are real and distinct,* $X|_F$ *is the gradient of a Morse function* $g : F \longrightarrow [0, \infty)$ *having exactly* $n+1$ *critical points with distinct indices; it is this function* g, *properly weighted and extended, that we add to* $f_0 \circ \pi$.

Remark 4.10 *The characteristic hypersurface of the vector field* X *above is the conormal of the vector field* X_0.

II On the characteristic foliation of surfaces in dimension 3

1 Properties of characteristic foliations

We are interested here in singular foliations on a surface S which can be realised as characteristic foliations by embedding S in a 3-dimensional contact manifold. In such a manifold, naturally oriented, the normal bundle of S is isomorphic to the bundle $\wedge^2 TS$; this allows us to speak of germs of contact structures along S without specifying the ambient manifold.

A General form of characteristic foliations

Definition 1.1 *We say that a singularity x of a vector field Y is* isochore[1] *if the divergence of Y at x is zero. An isochore singularity of Y is then an isochore singularity of $f \cdot Y$ for any function f; this notion is therefore well defined for singular foliations in the sense of 1.1.B.*

Proposition 1.2 *Let $\mathcal{F}$ be a singular foliation on a surface S. We fix an orientation on the manifold $\wedge^2 TS$ and we are interested only in germs of contact structures along S which give this orientation.*

(i) $\mathcal{F}$ is the characteristic foliation induced on S by a germ of contact structures if and only if $\mathcal{F}$ is without isochore singularities.

(ii) If S is closed, two germs of contact structures which induce the same characteristic foliation $\mathcal{F}$ are isomorphic: they are conjugate by a germ of a diffeomorphism which is isotopic to the identity through diffeomorphisms preserving $\mathcal{F}$.

Proof (i) The absence of isochore singularities is necessary; indeed, if α is a contact form which induces on S a form β which is null at x, the form $d\beta(x)$, which is none other than $d\alpha(x)|_{\mathrm{Ker}\ \alpha(x)}$, is nondegenerate; in other words, the vector field Y given near x by $\beta = i(Y)d\beta$ has nonzero divergence at x.

The converse and (ii) rest on the following fact: let S_0 be an orientable surface; a 1-form $\alpha = \beta_t + u_t\, dt$ on $S_0 \times \mathbb{R}$ is contact if and only if:

$$u_t\, d\beta_t + \beta_t \wedge \left(du_t - \frac{\partial \beta_t}{\partial t} \right) \quad \text{is nowhere zero.} \tag{3}$$

In particular, β_0 being given, the pairs $(u_0, (\partial\beta_t/\partial t)|_{y=0})$ which satisfy this inequality for $t = 0$, with a fixed sign, form a convex set; yet these pairs are those which determine a contact structure. We now suppose that S is orientable and we take on S an area form ω such that $\omega \wedge dt$ gives the chosen orientation on $\wedge^2 TS \cong S \times \mathbb{R}$. We suppose additionally that

[1] Straight from the French!

$\mathcal{F}$ is transversally orientable, that is, given by an equation $\beta = 0$, where β is a 1-form on S. We denote by u the function defined on S by $d\beta = u\omega$, and we take a 1-form γ on S such that the 2-form $\beta \wedge \gamma$ is positive or zero with respect to ω, and strictly positive outside the singular locus of β. We then set $\beta_t = \beta + t(du - \gamma$. The condition 3 shows immediately that the 1-form $\beta_t + u\ dt$ defines a contact structure near $S \times \{0\}$ in $S \times \mathbb{R}$; indeed:

$$u\ d\beta + \beta \wedge \left(du - \frac{\partial \beta_t}{\partial t}|_{t=0} \right) = u^2\omega + \beta \wedge \gamma.$$

Yet, as $\mathcal{F}$ has no isochore singularities, u is nonzero at each point of the singular locus of β.

Finally, if S is not orientable or if $\mathcal{F}$ is not transversally orientable, we remedy this problem by passing to a cover of order 2 or 4 on which we carry out the preceding construction in an invariant manner.

(ii) Passing if necessary to a double cover of S on which $\mathcal{F}$ becomes transversally orientable, we reduce to the case where the two germs of contact structures are transversally orientable. They then admit equations α_0 and α_1 which induce on S the same form. The formula 3 shows that the kernel ξ_s of $\alpha_s = (1-s)\alpha_0 + s\alpha_1$ is, near S, a contact structure for all $s \in [0,1]$.

We now seek, by J. Moser's method, an isotopy φ_s, $s \in [0,1]$, which takes ξ_0 to ξ_s, i.e. satisfies: $\alpha_0 \wedge \varphi_s^*\alpha_s = 0$. This condition shows that the path $s \mapsto \varphi_s^*\alpha_s$ remains on the ray $\{r\alpha_0, r > 0\}$ in the space of 1-forms; in other words:

$$\varphi_s^*\alpha_s \wedge \frac{\partial}{\partial s}(\varphi_s^*\alpha_s) = 0 \quad \text{for all } s.$$

Denoting by X_s the infinitesimal generator of (φ_s), this relation can be written:

$$(L(X_s)\alpha_s)|_{\xi_s} = -\frac{\partial \alpha_s}{\partial s}|_{\xi_s}.$$

We take for X_s the vector field satisfying at the same time

$$i(X_s)\alpha_s = 0 \quad \text{and} \quad (i(X_s)d\alpha_x)|_{\xi_S} = -\frac{\partial \alpha_s}{\partial s}|_{\xi_s}.$$

This system has a unique solution by definition of a contact structure. Moreover, if v is a vector of $\xi_x \cap TS = \xi_0 \cap TS$, we have $(\partial\alpha_s/\partial s)(v) = 0$, then $d\alpha_s(X_s, v) = 0$. This shows that X_s is tangent to $\mathcal{F}$ along S. We finally use that S is closed to integrate X to an isotopy. ∎

B Generic properties of characteristic foliations

The space of singular foliations on a surface S (in the sense of I.1.B) has a natural topology as the quotient of the space of plane fields by the null section in $\wedge^2 TS$. If now S is embedded in an oriented 3-dimensional manifold V, the function which to a plane field on V associates the induced foliation on S is open. As the set of contact structures forms an open set, its image is an open set in the space of singular foliations of S. Also, contact structures being locally stable by a theorem of J. Gray [G], we see:

Lemma 1.3 *Let $\mathcal{P}$ be a property of singular foliations which is C^∞-generic and let S be a surface embedded in a contact manifold (V, ξ). We can move S by a C^∞-small isotopy so that its characteristic foliation satisfies $\mathcal{P}$.*

Example 1.4 *Recall that a vector field on a closed surface is said to be Morse-Smale if it satisfies the three following properties:*

(i) the singularities and the periodic orbits of X are hyperbolic;

(ii) the α-limit set (resp. ω-limit) of every point is a singularity or a limit cycle;

(iii) there are no connections between saddles.

After a theorem of M. Peixoto, a vector field on a closed orientable surface is C^∞-generically Morse-Smale.

Let then S be a closed orientable surface in a contact manifold (V, ξ). If ξ is transversally orientable, the characteristic foliation of S is directed by a vector field that we can make Morse-Smale by a C^∞-small isotopy of S in V.

2 Convex surfaces

A Dividing set of a convex surface

Recall that a surface S, embedded in a 3-dimensional contact manifold (V, ξ), is called convex if there exists a contact vector field transverse to S. Such a surface is transversally orientable, therefore orientable. It follows immediately from propositions I.3.4 and II.1.2(b) that:

Proposition 2.1 *Let (V, ξ) be a 3-dimensional contact manifold, S a closed orientable surface embedded in V and $\mathcal{F}$ its characteristic foliation. Then the surface S is convex if and only if there exists on S a curve Γ transverse to $\mathcal{F}$, in general disconnected, which decomposes S into subsurfaces where $\mathcal{F}$ can be directed by a dilating vector field, for a certain area form, and exiting through the boundary.*

*Remark.*In particular, if S is convex, all leaves of $\mathcal{F}$ cut Γ at most once.

In the following, we will say that Γ is the *dividing set* of S (see remark 2.3 (2.3)). The data of a contact vector field X transverse to S realises this dividing set as the curve of points of S where X is tangent to ξ.

Proposition 2.2 *(i) Let S be a closed surface. Two vertically invariant contact structures on $S \times \mathbb{R}$ which define the same orientation and induce the same characteristic foliation $\mathcal{F}$ on $S \times \{0\}$ are isotopic: they are conjugate by a product diffeomorphism $\varphi \times Id$, where φ is isotopic to the identity through diffeomorphisms which preserve $\mathcal{F}$. Moreover, if the dividing set of S associated to the vertical vector field is the same for the two structures, it is preserved all along the isotopy.*

(ii) For $i = 0, 1$ let S_i be a convex surface in a contact manifold (V_i, ξ_i); let $\mathcal{F}_i$ be the characteristic foliation and X_i a contact vector field transverse to S_i (the data of X_i orients S_i). If S_0 and S_1 are closed (compact without boundary) and there exists a diffeomorphism from S_0 to S_1 which respects orientations and sends $\mathcal{F}_0$ to $\mathcal{F}_1$, then there exists a germ of a contact diffeomorphism, from (V_0, S_0) to (V_1, S_1), which sends X_0 to X_1.

Proof (ii) follows immediately from (i) which is proved exactly like (ii) in proposition 1.2. The isotopy consists of sliding along the leaves of $\mathcal{F}$ to make one of the contact structures turn into the other. This is possible in general only if S is closed. ∎

Remark 2.3 *Proposition 2.2 shows that the characteristic foliation $\mathcal{F}$ of a convex surface S totally determines, up to isotopy through curves transverse to $\mathcal{F}$, the dividing set Γ, that is the trace on S of the characteristic surface of a transverse contact field. In paragraph 3, we will see to what extent this curve reveals the geometry of the characteristic foliation of S. But first we give geometric criteria for convexity and non-convexity and we show in particular that an orientable surface is generically convex. This genericity, exceptional, is related to the fact that every open connected set in $\mathbb{R}$ (respectively, in $\mathbb{C}$) is convex (respectively pseudo-convex): in dimension 3, the minimal dimension of contact manifolds, convexity is a degenerate property.*

B Examples of non-convex surfaces

A contact structure on $S \times \mathbb{R}$ invariant under $\partial/\partial t$ is (locally) defined by equations of the type $\beta + u\, dt = 0$ where β and u are respectively a 1-form and a function on (a neighbourhood of) S such that:

$$u\, d\beta + \beta \wedge du \quad \text{is nowhere zero.} \tag{4}$$

The characteristic foliation $\mathcal{F}$ of S is then defined by $\beta = 0$. If ω is an area form on S and if Y is the vector field which directs $\mathcal{F}$ defined by $i(Y)\omega = \beta$, then condition 4 says

$$u \operatorname{div}_\omega(Y) - Y \cdot u \neq 0. \tag{5}$$

This shows immediatlely that the characteristic foliation of a closed convex surface S cannot be defined by a closed (nonsingular) form. For example, the invariant tori of the Hopf fibration in S^3 are not convex for the standard structure. We see similarly that, if S is convex, its characteristic foliation $\mathcal{F}$ possesses no closed leaf having a first return map tangent to the identity. Indeed, in a neighbourhood of one such leaf F, the foliation $\mathcal{F}$ admits an equation $\beta = 0$ where $d\beta|_F$ is identically zero; it is then impossible to find a function u such that $u\ d\beta + \beta \wedge du$ is nonzero on F since $u|_F$ necessarily has critical points. Finally, convexity forbids certain connections of saddles; to be precise, we say:

Definition 2.4 *Let x be a non-isochore singularity of a singular foliation $\mathcal{F}$. We say that we* positively orient *$\mathcal{F}$ at x when we choose, to direct $\mathcal{F}$ near x, a vector field for which the divergence at x is positive.*

If S is convex, no leaf of its characteristic foliation joins two saddles while being a stable separatrix for both when they are positively oriented. This results for example from 5: if, near such a leaf F, we orient the foliation by a vector field Y directed from the saddle x_0 towards the saddle x_1, we must have $u(x_0)$ negative and $u(x_1)$ positive. Yet, by 5 u can be zero only when decreasing in the direction of Y.

C Examples of convex surfaces

Definition 2.5 *We say that a singular foliation $\mathcal{F}$ on a closed surface S is* Morse-Smale *if it satisfies the following conditions:*

(i) the singularities and the closed leaves of $\mathcal{F}$ are hyperbolic;

(ii) the limit set of each half-leaf is a singularity or a closed leaf;

(iii) $\mathcal{F}$ has no connections between saddles.

We say that $\mathcal{F}$ is almost Morse-Smale *if it satisfies (i), (ii) and:*
(iii)' when we orient $\mathcal{F}$ positively near saddles, the associated stable manifolds do not intersect.

Proposition 2.6 *Let S be an orientable closed surface embedded in a contact manifold (V, ξ). If the characteristic foliation $\mathcal{F}$ of S is almost Morse-Smale, then S is convex.*

Proof By (b) of proposition 1.2 it suffices to construct on $S \times \mathbb{R}$ a contact structure invariant under $\partial/\partial t$ which makes $\mathcal{F}$ the characteristic foliation on $S \times \{0\}$. Around each closed leaf (resp. each focus), we take an annulus (resp. a disk) with boundary transverse to $\mathcal{F}$. Near saddles, we orient $\mathcal{F}$ positively. Using (ii) of definition 2.5, we place bands around their stable manifolds so that the union of these annuli, discs and bands forms a surface S_0 with boundary transverse to $\mathcal{F}$ (see Figure 1). By construction, using (iii)' of definition 2.5, on a neighbourhood U of S_0, $\mathcal{F}$ is directed by a vector field Y exiting along ∂S_0 and for which singularities have positive divergence. There then exists an area form ω on S such that $\mathrm{div}_\omega(Y) > 0$ on U. We set $u = 1$, then $u\mathrm{div}_\omega(Y) - Y \cdot u > 0$ on U.

On the surface with boundary $S' = \mathrm{Cl}(S \backslash S_0)$, $\mathcal{F}$ is a nonsingular foliation transverse to the boundary and without closed leaves.[2] By (ii), as S is orientable, S' is a union of annuli foliated by segments going from one boundary to another. We can then conclude by using proposition 2.1 or the following elementary reasoning. We choose on S' a nonsingular vector field Y' directing $\mathcal{F}$ and coinciding with $\pm Y$ on a collar neighbourhood U' of $\partial S'$ in $U \cap S'$. We set $u' = \pm 1$ on U' accordingly as $Y' = \pm Y$; we then seek to extend to S' the germ of u' on the boundary, so as to have: $u' \mathrm{div}_\omega(Y') - Y' \cdot u' > 0$ on S'. This extension follows immediately from the following remark.

Remark 2.7 *Let $h : [0,1] \longrightarrow \mathbb{R}$ be a function positive at 0 and negative at 1. There exists a function $v : [0,1] \longrightarrow \mathbb{R}$ equal to 1 near 0 and equal to -1 near 1 such that $vh - dv/d\theta$ is positive; we take $v(\theta) = w(\theta) \exp(\int_0^\theta h(\sigma) d\sigma)$ where $w : [0,1] \longrightarrow \mathbb{R}$ is a suitable decreasing function.*

3 Deformations of characteristic foliations

A A reduced form for characteristic foliations

Let S be a closed orientable surface embedded in a 3-dimensional contact manifold (V, ξ) with a Morse-Smale characteristic foliation $\mathcal{F}$. By proposition 2.6, there exists a germ of a contact vector field transverse to S. Given any neighbourhood U of S, it's easy to extend this germ to a contact vector field for which the flow defines an embedding $S \times \mathbb{R} \longrightarrow V$ with image $V_0 \subset U$. On $V_0 \cong S \times \mathbb{R}$, $\xi_0 = \xi|_{V_0}$ is a contact structure invariant under $\partial/\partial t$ and the characteristic surface of this contact vector field is a cylinder $\Gamma \times \mathbb{R}$ where Γ decomposes $S = S \times \{0\}$ as indicated in 2.1. Then any function $h : S \longrightarrow \mathbb{R}$ has for its graph a convex surface S_h contained in V_0 having the "same dividing set Γ" as S.

Proposition 3.1 *There exists a function $h : S \longrightarrow \mathbb{R}$ such that the characteristic foliation $\mathcal{F}_h$ of S_h is Morse-Smale and gives, on each component S' of the surface obtained by decomposing S_h along Γ, the following:*

(i) if S' is a disk, $\mathcal{F}_{h|_{S'}}$ has a unique singularity which is a focus and has no closed leaves: topologically it is a radial foliation;

(ii) if S' is not a disk, $\mathcal{F}_{h|_{S'}}$ has exactly one closed leaf and only has saddles for singularities.

Moreover we can choose h non-positive.

We will show this proposition in C; it also follows from proposition 3.6.

[2] Cl here denotes closure.

B Elimination of singularities

Definition 3.2 *Given a singular foliation without isochore singularities on a surface, we say that a focus x_0 and a saddle x_1 are in* simple elimination position *(resp. in* cyclic elimination position*) if when we positively orient the foliation near x_1, one and only one stable separatrix comes from x_0 (resp. two stable separatrices come from x_0).*

Lemma 3.3 (Elimination lemma) *(see [El1] theorem 6.1 and [El2]). With the notation and the hypotheses of 3.A, let x_0 and x_1 be a focus and a saddle of $\mathcal{F}$ in simple or cyclic elimination position.*

(i) There exists in S an annulus A disjoint from Γ and satisfying:

- *the only singularities of $\mathcal{F}$ on A are x_0 and x_1;*
- *$\mathcal{F}|_A$ has no closed leaf;*
- *$\mathcal{F}$ is transverse to the boundary of A.*

The two configurations are shown in figures 2 and 3.

(ii) There exists a function $k : A \longrightarrow (-\infty, 0]$ with support in the interior of A and such that the characteristic foliation on the graph of k has no singularity.

Proof (i) Let S' be the connected component of x_1 in the surface obtained by decomposing S along Γ. There exists on S' a vector field which directs $\mathcal{F}$, exiting along $\partial S'$ and dilating some area form on S'. In particular this vector field positively orients $\mathcal{F}$ near x_1 and the stable manifold $W^s(x_1)$ lies in S'.

If x_0 and x_1 are in cyclic elimination position, we choose for A an annular neighbourhood of the union $\{x_0\} \cup W^s$.

If x_0 and x_1 are in simple elimination position, choose one of the two: either the other branch of W^s comes from a focus x_2, or it comes from a closed leaf F necessarily disjoint from Γ. In the first case, we take for A a disk neighbourhood of the union $\{x_0, x_2\} \cup W^s$, minus a disk around x_2. In the second case, we first take an annulus A' around F with boundary transverse to $\mathcal{F}$; the branch of W^s which comes from F then cuts $\partial A'$ in a point x; we take for A a neighbourhood of the union of the arc which joins x to x_0 in W^s and of the component of x in $\partial A'$.

(ii) As A is disjoint from Γ, the contact structure on $A \times \mathbb{R}$ is the contactisation of a Liouville form β on A; in other words, it has an equation of the form $\beta + dt = 0$. Thus, for any function $k : A \longrightarrow \mathbb{R}$ the characteristic foliation on the graph A_k of k is defined by $\beta + dk = 0$.

Let then ω be any area form on A and Y the vector field given by $i(Y)\omega = \beta$. We seek to add to Y the ω-Hamiltonian Y_k of a function k with support in the interior of A such that $Y + Y_k$ is nonsingular (here Y_k is defined by

$i(Y_k)\omega = dk$. For this we take, on A, a foliation by circles parallel to the boundary which we denote $\mathcal{G}$. On a neighbourhood B of ∂A in A, $\mathcal{F}$ and $\mathcal{G}$ are transverse. On $A \backslash B$, the vector field Y is bounded. We therefore choose a function $k : A \longrightarrow (-\infty, 0]$, with support in the interior of A, constant on the leaves of $\mathcal{G}$, and for which the ω-Hamiltonian Y_k is very large on $A \backslash B$. Then $Y + Y_k$ is nonzero on $A \backslash B$. On B, Y is nonsingular and is transverse to Y_k there or Y_k is nonzero. Then $Y + Y_k$ is everywhere nonzero. ∎

Remark 3.4 *In the case where x_0 and x_1 are in cyclic elimination position, we thus create a closed leaf.*

In the case where x_0 and x_1 are in simple elimination position all leaves go from one boundary to the other of the annulus.

We can easily check that this construction preserves the Morse-Smale character of the foliation.

C End of the proof of proposition 3.1

Let S' be a component of the surface obtained by decomposing S along Γ. On S', we choose a vector field Y which directs $\mathcal{F}$ and which dilates a given area form; the foci and the closed orbits of Y are then repulsive.

(i) We suppose that S' is a disk. Then S' does not contain any closed orbits since Y is dilating. If $x_1 \in S'$ is a saddle of Y, its stable manifold lies in S', therefore x_1 is in elimination position with a focus. When we have eliminated all the saddles, there remains only one focus.

(ii) We now suppose that S' is not a disk. The orbits of Y which leave from a focus $x_0 \in S'$ cannot go across a closed orbit $F \subset S'$. Nor can they all exit since S' is not a disk. It follows that at least one goes towards a saddle $x_1 \in S'$ such that we can eliminate all the foci of S'. Now, as the α-limit set of every point in S' is in S', S' contains at least one closed orbit. If it contains only one, we stop. If it contains two, F and F', then S' is not an annulus and there exists at least one saddle x in S' for which one and only one separatrix comes from F'. Indeed, if not, let $y_1, \dots, y_p$ be the saddles for which one separatrix (and in fact the whole stable manifold) comes from F'; the set of points of S' which have for α-limit one of the y_i, or F', is a connected component of S', but S' is connected. By the inverse procedure to cyclic elimination, we replace F' by a focus x_0 and a saddle x_1 in cyclic elimination position. The separatrix of x which came from F' now comes from x_0 so that x_0 and x are in simple elimination position.

D Foliations adapted to a given dividing set

Let S be a convex closed surface in a 3-dimensional contact manifold (V, ξ), and let X be a contact vector field transverse to S whose flow defines an embedding

$S \times \mathbb{R} \longrightarrow V$. We denote by Γ the dividing set of S associated to X, the curve consisting of points of S where X is tangent to ξ, and we denote by S_Γ the compact surface with boundary obtained by decomposing S along Γ.

Definition 3.5 *(i) An* admissible isotopy *of S in V is an isotopy of S through surfaces transverse to X, which in particular avoid singularities of X.*

(ii) A singular foliation on S is adapted *to Γ if the foliation induced on S_Γ is directed by a vector field which dilates an area form and which exits transversely through the boundary ∂S_Γ.*

Proposition 3.6 *Let $\mathcal{F}$ be a foliation on S adapted to Γ. Then there exists an admissible isotopy $\delta_s : S \longrightarrow V$, $s \in [0,1]$, such that the characteristic foliation on $\delta_1 S$ is $\delta_1 \mathcal{F}$. Moreover, for all $s \in [0,1]$, the dividing set of $\delta_s S$ associated to X is $\delta_s \Gamma$.*

Proof We denote by $\mathcal{F}_0$ the characteristic foliation of S and by ξ_0 the vertically invariant contact structure induced on $S \times \mathbb{R}$ by the flow of X, $\psi : S \times \mathbb{R} \longrightarrow V$. We take an area form ω on S such that $\omega \wedge dt$ orients $S \times \mathbb{R}$ like ξ_0; finally we take a closed tubular neighbourhood A of Γ, small enough so that $\mathcal{F}$ and $\mathcal{F}_0$ foliate it by segments from one boundary to the other.

On $(S \setminus \mathrm{int} A) \times \mathbb{R}$, ξ_0 admits a unique equation of the type $i(Y_0)\omega + dt = 0$, where Y_0 is a vector field on $S \setminus \mathrm{int} A$ which directs $\mathcal{F}_0$ and which dilates ω. Also, as $\mathcal{F}$ is adapted to Γ there exists on $S \setminus \mathrm{int} A$ a vector field Y which directs $\mathcal{F}$ and which dilates a certain area form; observing that $\mathrm{div}_{\pm e^g \omega}(Y) = e^{-g} \mathrm{div}_\omega (e^g Y)$, we replace Y with a vector field Y_1 which dilates ω. For $s \in [0,1]$, we set $Y_s = (1-s)Y_0 + sY_1$. Then, the equation $i(Y_s)\omega + dt = 0$ defines, for each s in $[0,1]$, a vertically invariant contact structure ξ_s on $(S \setminus \mathrm{int} A) \times \mathbb{R}$. Now, on a small neighbourhood U of A in S, we take vector fields Y_0' and Y_1' which respectively direct $\mathcal{F}_0$ and $\mathcal{F}$ and which coincide with $\pm Y_0$ and $\pm Y_1$ on $U \cap (S \setminus \mathrm{int} A)$; for $s \in [0,1]$, we again set $Y_s' = (1-s)Y_0' + sY_1'$. On $U \times \mathbb{R}$, the contact structure ξ_0 is defined by a unique equation of the type $i(Y_0')\omega + u_0 dt$; the function u_0 is zero on Γ, is equal to ± 1 wherever $Y_0' = \pm Y_0$ and satisfies on U: $u_0 \mathrm{div}_\omega (Y_0') - Y_0' \cdot u_0 > 0$.

Then, using remark 2.7, we form a family u_s of functions on U such that, for all $s \in [0,1]$, we have $u_s \mathrm{div}_\omega (Y_s') - (Y_s' \cdot u_s) > 0$, with $u_s = \pm 1$ wherever $Y_s' = \pm Y_s$. We thus obtain on $S \times \mathbb{R}$ a family still denoted ξ_s, $s \in [0,1]$, of vertically invariant contact structures; by construction, the characteristic surface of the vertical vector field is $\Gamma \times \mathbb{R}$ for each structure ξ_s, and the characteristic foliation induced by ξ_1 on $S \times \{0\}$ is none other than $\mathcal{F}$.

J. Moser's method (see the proof of proposition 1.2) then provides a family of vertically invariant vector fields on $S \times \mathbb{R}$ which, since S is closed, integrates to an isotopy φ_s satisfying $\varphi_s^* \xi_s = \xi_0$; moreover the diffeomorphisms $\varphi_s : S \times \mathbb{R} \longrightarrow S \times \mathbb{R}$ preserve $\partial / \partial t$ therefore $\Gamma \times \mathbb{R}$; it follows that $\varphi_s^{-1}(S \times \{0\})$ is always transverse to $\partial / \partial t$ and is decomposed by its intersection with $\Gamma \times \mathbb{R}$. Composing with a vertical translation, we can arrange that $\varphi_s^{-1}(X \times \{0\})$ extends to $S \times (-\infty, 0]$. We then set $\delta_s = \psi \circ \varphi_s^{-1}|_{S \times \{0\}}$. ∎

Remark. The previous proposition allows us, with lemma 3.3, to eliminate singularities and prove proposition 3.1. Equally it gives other reduced forms for the characteristic foliation of convex surfaces; for example:

Example 3.7 *(Foliation associated to a handle decomposition). Let* (S, X, Γ, S_Γ) *be as above. By* handle decomposition *of* S_Γ*, we mean a finite collection of arcs* $\gamma_1, \dots, \gamma_r$ *, disjoint in* S_Γ*, going from boundary to boundary, and such that the complement in* S_Γ *of a regular neighbourhood* Ω *of* $\partial S_\Gamma \cup \gamma_1 \cup \dots \cup \gamma_r$ *is a disjoint union of disks* $\Delta_1, \dots, \Delta_q$*.*

To any handle decomposition of S_Γ*, we associate a singular foliation of* S_Γ*, unique up to homeomorphism, in the following manner: on each disk* Δ_i*, we put a radial foliation and, on* Ω*, we take the foliation described in Figure 4; this foliation is directed by a vector field exiting through* ∂S_Γ*, entering through* $\partial\Omega \backslash \partial S_\Gamma$*, which has no closed orbits and which has for singularities precisely* r *saddles with positive divergence for which the unstable manifolds are the* γ_j*; note that the stable manifolds of these saddles come from the centres of the disks* Δ_i*. By gluing, we construct on* S *foliations adapted to* Γ*, without closed leaves.*

III Construction of convex contact structures in dimension 3

1 Convex contact structures and essential surfaces

A Existence results

Definition 1.1 *(i) Let* V *be a 3-dimensional manifold and* $f : V \longrightarrow [0, \infty)$ *a proper Morse function. We say that a surface* C *embedded in* V*, not necessarily connected, is* f-essential *if it satisfies the following three properties:*

(a) $f|_C$ *is a proper Morse function;*

(b) all critical points of f *are on* C *and are exactly the critical points of* $f|_C$*;*

(c) a critical point of index 1 or 2 for f *is of index 1 for* $f|_C$*; equivalently* f *and* $f|_C$ *have the same local extrema.*

(ii) We say that a contact structure on an oriented 3-dimensional manifold is positive if it induces the given orientation.

Theorem 1.2 (Existence theorem) *Let* V *be a 3-dimensional oriented manifold and* $f : V \longrightarrow [0, \infty)$ *a proper Morse function. There exists on* V *a contact structure which is preserved under a complete pseudo-gradient of* f *if and only if there exists in* V *an* f*-essential surface* C*.*

Remark. In I.4, we saw that the existence of an f-essential surface is necessary; we will prove that this is also sufficient. The problem of existence of essential surfaces for a given function will be discussed in part IV; from this discussion will follow a version of the theorem of R. Lutz and J. Martinet (see [Ma]) for convex contact structures, namely:

Theorem 1.3 *Any oriented 3-dimensional manifold carries a positive convex contact structure.*

Definition 1.4 *(Ya. Eliashberg [El3]). We say that a contact structure on a 3-dimensional manifold V is* overtwisted *if there exists a 2-dimensional disk embedded in V, for which the characteristic foliation has a limit cycle (with exactly one singularity in the interior according to [El3], but the arguments of II.3 show that this condition adds nothing).*

R. Lutz has described a procedure for constructing on any 3-dimensional manifold an overtwisted contact structure [Lu]; we will give a "convex" version showing that:

Corollary 1.5 *Any oriented 3-dimensional manifold carries a positive convex overtwisted contact structure.*

By a theorem of M. Gromov and Ya. Eliashberg (see [Gr] and [El1]), overtwisted contact structures are not symplectically fillable (see [El1] and [EG] for the definition). It follows that there exist contact structures which are convex but not symplectically fillable, answering a question of [EG].

B Scheme of the proof of theorem 1.2

Let $a_0 < a_1 < \cdots$ be the critical values of f, which we suppose are distinct (only to simplify the exposition), and let $b_0 < b_1 < \cdots$ be intermediate regular values, i.e. so that $a_0 < b_0 < a_1 < b_1 < \cdots$. We set $V_k = \{x \in V \mid f(x) \leq b_k\}$ and $C_k = C \cap V_k$.

Then V_{k+1} is obtained from V_k by attaching a single handle of index equal to the index of f at the critical point x_{k+1} of value a_{k+1}. As C is f-essential, C_{k+1} is obtained simultaneously from C_k by attaching a handle of index equal to the index of $f|_C$ at x_{k+1}. Precisely, let $H_i = D^i \times D^{3-i}$ be a handle of index $i = 0, 1, 2, 3$; the attachment of H_i to V_k is given by an embedding $\varphi : \partial D^i \times D^{3-i} \longrightarrow \partial V_k$; the pair (V_{k+1}, V_k) only depends on the isotopy class of φ. For $j \leq i$, let D^j be the sub-dis $D^j \times \{0\}$ contained in D^i; then for an appropriate choice of φ, the handle that we attach to C_k is $D^j \times D^{2-j}$ with $j = 0, 1, 1, 2$ when $i = 0, 1, 2, 3$; we glue it along the restriction of φ to $\partial D^j \times D^{2-j} \subset \partial D^i \times D^{3-i}$.

By induction on k, we will construct on V_k a positive contact structure ξ_k, as well as a pseudo-gradient X_k of $f_k = f|_{V_k}$, which preserves ξ_k and for which the characteristic surface is C_k. For this we distinguish four cases corresponding to the different possible indices. It is not necessary to worry about the problem of completeness since we can always handle it afterwards; indeed:

Remark 1.6 *Let c be a given positive number, S a closed surface and ξ a vertically invariant contact structure on $S \times [0, 1]$. Then there exists on $S \times [0, 1]$ a contact structure ξ' having the following properties:*

(i) ξ' coincides with ξ near the boundary;

(ii) ξ' is preserved by a vector field X' which is equal to $\partial/\partial t$ near the boundary, and whose orbits are the segments $\{\cdot\} \times [0,1]$ covered in time c.

PROOF We extend ξ to a vertically invariant contact structure on $S \times \mathbb{R}$ and we choose a diffeomorphism $\rho : [0,c] \longrightarrow [0,1]$ which coincides with the identity near 0 and with a translation near c; we then take for ξ' and X' the images under $Id \times \rho$ of ξ and $\partial/\partial t$. ∎

2 Attachment of handles of index 0 and 3

A The model

On $\mathbb{R}^3$ oriented by $dx \wedge dy \wedge dz$, the plane field with equation $dz + uy\, dx + vx\, dy = 0$, $u, v \in \mathbb{R}$, is a positive contact structure if and only if $v - u > 0$. This plane field is preserved by all vector fields of the form

$$ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cz\frac{\partial}{\partial z}, \quad a, b, c \in \mathbb{R}, \quad \text{with } c = a + b;$$

indeed, their flow at time t is given by $(x, y, z) \mapsto (e^{at}x, e^{bt}y, e^{ct}z)$. Finally, for $v - u > 0$ and $c = a + b$, the characteristic surface of the contact vector field so defined has equation: $cz + (au + bv)xy = 0$.

Let ζ_0 be the contact structure with equation $dz - y\, dx + x\, dy = 0$. The contact vector fields

$$Z_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} \quad \text{and } Z_3 = -Z_0$$

both have for characteristic surface the plane $\{z = 0\}$ are pseudo-gradients respectively of $g_0 = x^2 + y^2 + z^2$ and $g_3 = -g_0$.

We denote by H_3 the handle of index 3: $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ which we orient by ζ_0; we denote by F_3 the boundary of H_3 furnished with the orientation induced by the entering vector field Z_3: this orientation is opposite to the usual orientation of the unit sphere in $\mathbb{R}^3$ as the boundary of the ball.

B Handles of index 0

As a_0 is the minimum of f, there exists a diffeomorphism of V_0 onto the closed ball $B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ which respects orientations, which sends C_0 to $B^3 \cap \{z = 0\}$ and which, up to an affine transformation of $\mathbb{R}$, conjugates $f_0 = f|_{V_0}$ to $x^2 + y^2 + z^2$. Then the inverse of this diffeomorphism transforms ζ_0 into a contact structure ξ_0 on V_0 and sends the vector field Z_0 to a pseudo-gradient X_0 of f_0; by construction, this pseudo-gradient preserves ξ_0 and has C_0 as its characteristic surface.

Any "attachment" of a handle of index 0 can be dealt with in the same manner.

C Handles of index 3

On V_k we have, by the inductive hypothesis, a contact structure ξ_k, as well as a pseudo-gradient X_k of $f_k = f|_{V_k}$ which preserves ξ_k and has characteristic surface C_k.

Definition 2.1 *Let $S \subset \partial V_k$ be a surface. We will say that an isotopy δ_s of embeddings of S in V_k is* admisible *if, for all s, $\delta_s S$ is transverse to X_k in V_k and cuts C_k along $\delta_s(S \cap C_k)$.*

It is clear that such an isotopy extends to an isotopy of embeddings $\bar{\delta}_s : V_k \longrightarrow V_k$ which are *admissible* in the following sense:

- for all s, $\bar{\delta}_s$ sends C_k to C_k;
- for all s, $\bar{\delta}_s^*$ is still a pseudo-gradient of f_k and evidently preserves the positive contact structure $\bar{\delta}_s^* \xi_k$.

We now suppose that V_{k+1} is obtained from V_k by attaching a handle of index 3, that is, by gluing a ball onto a spherical component S of ∂V_k. Simultaneously C_{k+1} is obtained by attaching to C_k a disk along $S \cap \partial C_k$; this intersection is tthus a connected curve Γ. We denote by $\phi : F_3 \longrightarrow S$ a gluing diffeomorphism which respects orientations and sends $F_3 \cap \{z = 0\}$ onto Γ.

Lemma 2.2 *We can find an admissible isotopy $\delta_s : S \longrightarrow V_k$, $s \in [0, 1]$, such that there exists a germ of a diffeomorphism $\psi : (H_3, F_3) \longrightarrow (V_k, \delta_1 S)$ having the following properties:*

(i) $\psi|_{F_3}$ is isotopic to $\delta_1\phi$ through diffeomorphisms of F_3 in $\delta_1 S$ which send $F_3 \cap \{z = 0\}$ to $\delta_1\Gamma$;

(ii) ψ takes ζ_0 to ξ_k and Z_3 to X_k.

Proof By proposition II.2.2, it suffices to find an admissible isotopy δ_s for which the diffeomorphism $\delta_1\phi : F_3 \longrightarrow \delta_1 S$ respects orientations and sends the characteristic foliation induced by ζ_0 to that induced by ξ_k. This isotopy is immediately given by proposition II.3.6 since the foliation obtained on S by transporting via ϕ the characteristic foliation on F_3 is adapted to Γ. ■

Now let $\bar{\delta}_s$ be an admissible isotopy of embeddings $V_k \longrightarrow V_k$ which extends the isotopy δ_s of the above lemma (see 2.1). We can attach H_3 to V_k by gluing on the one hand $\bar{\delta}_1^*(\xi_k)$ to ζ_0, and on the other $\bar{\delta}_1^*(X_k)$ to Z_3. We then extend f_k to this manifold via a function on H_3 which admits Z_3 as a pseudo-gradient and equals $(a_{k+1} - x^2 - y^2 - z^2)$ near the origin. On the other components of ∂V_k, we add an exterior collar up to the level b_{k+1}; there, we extend X_k trivially then ξ_k in an invariant manner.

3 Attachment of handles of index 1 and 2

A The model

On $\mathbb{R}^3$ oriented by $dx \wedge dy \wedge dz$, let ζ_1 be the positive contact structure with equation $dz + y\,dx + 2x\,dy = 0$. The contact vector fields

$$Z_1 = 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \quad \text{and} \quad Z_2 = -Z_1$$

have as characteristic surface the plane $\{z = 0\}$ and are pseudo-gradients respectively of $g_1 = x^2 - y^2 + z^2$ and $g_2 = -g_1$.

Given $\epsilon > 0$, we denote by $H_1 = H_1(\epsilon)$ the handle of index 1 $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 \leq \epsilon^2,\ y^2 \leq 1\}$ and we denote by $F_1 = F_1(\epsilon)$ the surface $H_1 \cap \{y = \pm 1\}$. The data of ζ_1 and Z_1 orient H_1 and F_1. Similarly, we denote by H_2 the handle of index 2 $\{(x, y, z) \in \mathbb{R}^3 \mid y^2 \leq \epsilon^2,\ x^2 + z^2 \leq 1\}$ and we denote by F_2 the surface $H_2 \cap \{x^2 + z^2 = 1\}$; H_2 and F_2 are oriented by the data of ζ_1 and Z_2. If we parametrise F_2 by $(\theta, y) \mapsto (x = \sin\theta,\ y,\ z = \cos\theta)$, $\theta \in [0, 2\pi]$, the orientation described previously is given by $d\theta \wedge dy$. In addition the characteristic foliation induced by ζ_1 has equation: $(y\cos\theta - \sin\theta)\ d\theta + 2\ \sin\theta\ dy = 0$; this appears therefore as in figure 5.

We can easily show that:

Lemma 3.1 *For $i = 1, 2$ and $\epsilon > 0$ given, let h_i be a non-singular germ of a function along F_i, equal to a negative constant on F_i. Then h_i extends to a function on H_i which coincides with g_i near the origin, for which Z_i is a pseudo-gradient.*

B Handles of index 2

We suppose that V_{k+1} (resp. C_{k+1}) is obained by attaching to V_k (resp. to C_k) a handle of index 2 (resp. of index 1). This attachment is given by an embedding $\phi : F_2 \longrightarrow S = \partial V_k$ which respects orientations and which meets $\Gamma = \partial C_k$ exactly along $F_2 \cap \{z = 0\}$. The attaching curve Θ, the image under ϕ of $F_2 \cap \{y = 0\}$, therefore cuts Γ in two points and is thus divided into two arcs denoted Θ_+ and Θ_-. Finally, we denote by S_Γ the surface obtained by decomposing S along Γ. To construct the contact structure ξ_{k+1} and the vector field X_{k+1} on V_{k+1} it suffices, by lemma 3.1, to show that:

Lemma 3.2 *We can find an admissible isotopy $\delta_s : S \longrightarrow V_k$, $s \in [0, 1]$, such that there exists an annulus A around Θ, and a germ of a diffeomorphism $\psi : (H_2, F_2) \longrightarrow (V_k, \delta_1 A)$ having the following properties:*

- *(i) $\psi|_{F_2}$ is isotopic to $\delta_1\phi$ through embeddings of F_2 in $\delta_1 A$ which meet $\delta_1\Gamma$ exactly along $F_2 \cap \{z = 0\}$;*
- *(ii) ψ takes ζ_1 to ξ_k and Z_2 to X_k.*

Proof We begin by building an admissible isotopy $\delta'_s : S \longrightarrow V_k$, $s \in [0,1]$, such that there exists an annulus A around Θ, and a diffeomorphism $\psi' : F_2 \longrightarrow \delta'_1 A$ which respects orientations, which meets $\delta'_1 \Gamma$ exactly along $F_2 \cap \{z = 0\}$ and which conjugates the characteristic foliations induced respectively by ζ_1 and ξ_k. For this, we take two arcs γ_+ and γ_- having their endpoints on Γ and satisfying the following conditions (see figure 6):

- γ_+ and γ_- are contained in a tubular neighbourhood Ω of Θ in S and are respectively isotopic to Θ_+ and Θ_- in Ω; moreover they do not cut Γ in their interiors;
- $\gamma_\pm$ crosses $\Theta_\pm$ at a single point $m_\pm$;
- in Ω, Θ cuts Γ between γ_+ and γ_-.

We then extend the data of γ_+ and γ_- to a handle decomposition of S_Γ (see example II.3.7). The associated foliation induces on S a foliation $\mathcal{F}$ adapted to Γ (Definition II.3.5) which, on an annulus A around Θ, is conjugate to the germ of the characteristic foliation of F_2 along the circle $\{y = 0, x^2 + z^2 = 1\}$. Proposition II.3.6 provides an admissible isotopy $\delta'_s : S \longrightarrow V_k$ such that $\delta'_1 A$ has characteristic foliation $\delta'_1(\mathcal{F})$. We thus obtain the desired diffeomorphism $\psi' : F_2 \longrightarrow \delta'_1 A$.

Now, we extend ψ' to a germ of a diffeomorphism, still denoted ψ', $(H_2, F_2) \longrightarrow (V_k, \delta'_1 A)$, which sends Z_2 to X_k. Thus, ξ_k (resp. $\psi'_* \zeta_1$) induces on $S \times \mathbb{R}$, via δ'_1 and the flow of X_k, a vertically invariant contact structure η_0 (resp. η). It then suffices to establish the following fact:

Lemma 3.3 (Sub-lemma) *We can extend η to $S \times \mathbb{R}$ as a vertically invariant contact structure η giving on $S \times \{0\}$ the same characteristic foliation and the same dividing set as η_0.*

Proof that 3.3 implies 3.2. As S is closed, we can now argue from the uniqueness of vertically invariant contact structure which induce a given characteristic foliation on $S \times \{0\}$ (proposition II.2.2): there exists an isotopy $\varphi_s : S \times \mathbb{R} \longrightarrow S \times \mathbb{R}$, which preserves at once $\partial/\partial S$ and the levels $S \times \{t\}$, such that φ_1 straightens η_1 to η_0. We then obtain an admissible isotopy δ_s and the desired diffeomorphism ψ by correcting by φ_1 the isotopy δ'_s and the diffeomorphism ψ'. ∎

Proof (of 3.3) As ψ' meets $\delta'_1 \Gamma$ exactly along $F_2 \cap \{z = 0\}$, there exists a function $h : A \longrightarrow \mathbb{R}$ such that, if η_0 is defined near a point of $A \times \mathbb{R}$ by an equation $\beta + u\, dt = 0$, then η_1 is defined near this point by $\beta + e^h u\, dt = 0$. We extend h arbitrarily in a neighbourhood of Γ.

On $S_\Gamma \times \mathbb{R}$, η_0 induces a vertically invariant contact structure, globally defined by an equation of the form $i(Y)\omega + u\, dt = 0$ where:

- ω is an area form on S_Γ;

- Y is a vector field which exits along ∂S_Γ and which directs the foliation on S_Γ induced by $\mathcal{F}$, that is, the foliation associated to the handle decomposition chosen on S_Γ;
- u is a positive function on the interior of S_Γ, zero at the boundary and satisfying $u \operatorname{div}_\omega(Y) - Y \cdot u > 0$.

Let A_Γ be the part of S_Γ corresponding to A; on $A_\Gamma \times \mathbb{R}$, η induces a contact structure with equation $i(Y)\omega + e^h u\, dt = 0$. From which: $u(\operatorname{div}_\omega(Y) - Y \cdot h) - Y \cdot u > 0$, in other words:

$$u\,(Y \cdot h) < u \operatorname{div}_\omega(Y) - Y \cdot u. \tag{6}$$

We must therefore extend h to S_Γ preserving this inequality. We observe that, on a sufficiently small neighbourhood of ∂S_Γ, the function h given arbitrarily satisfies 6 since u is zero on ∂S_Γ. The fact the we can then extend h results from the two following remarks:

- On the interior of S_Γ, where $u > 0$ 6 says $Y \cdot h < \operatorname{div}_\omega(Y) - Y \cdot \log u$. Yet each orbit of Y which exits A_Γ goes in finite time to ∂S_Γ without cutting A_Γ again. On such a segment of the orbit, h is given near its endpoints, but the variation of $-\log u$ is infinite and $\operatorname{div}_\omega Y$ is bounded; we can therefore extend h over the segment.
- An orbit of Y which enters into A_Γ comes to a focus without cutting A_Γ first. On a time interval of the type $(-\infty, \tau_0]$, h is only given near τ_0. The condition 6, which bounds its derivative from above by a quantity which is strictly positive and bounded from below, does not prevent us from extending h to a function with compact support. ■

C Handles of index 1

We suppose that V_{k+1} (resp. C_{k+1}) is obtained from V_k (resp. C_k) by attaching a handle of index 1 to two points p and q of $\Gamma = \partial C_k \subset S = \partial V_k$. We denote by p_0 and q_0 the points with coordinates $(0, 1, 0)$ and $(0, -1, 0)$ in $\mathbb{R}^3$. By lemma 3.1, it suffices to establish the following fact:

Lemma 3.4 *There exists a germ of a diffeomorphism $(V_k, p, q) \longrightarrow (H_1, p_0, q_0)$ which sends ξ_k to ζ_1 and X_k to Z_1.*

It is in this lemma, whose proof is easy, that the orientability of V is needed.

IV Construction of essential surfaces

In this part, we give methods for constructing, on 3-dimensional manifolds, Morse funcitons having essential surfaces (see definition III.1.1). I have had the pleasure of discussing this question with several people, in particular Slava Kharlamov, François Laudenbach, Christine Lescop and Alexis Marin; I thank them for their suggestions and remarks.

1 Some examples

A Examples of essential surfaces

Example 1.1 *(F. Laudenbach). Let V_0 be a compact 3-dimensional manifold with connected boundary $C = \partial V_0$, and let $f_0 : V_0 \longrightarrow \mathbb{R}$ be a function having the following properties:*

(i) f_0 is nonsingular and its restriction to C is a Morse function;

(ii) any local minimum (resp. local maximum) of $f_0|_C$ is a local minimum (resp. local maximum) of f_0 on V_0.

Then there exists on the double $V = V_0 \cup_C V_0$ of V_0 a Morse function f for which C is an essential surface.

Remark 1.2 *We will see later (lemma 2.2) that, if V_0 possesses a function f_0 satisfying (i) and (ii) then V_0 is a handlebody.*

Proof A simple way to construct the double V of V_0 is the following: we take a Morse function $g_0 : (V_0, C) \longrightarrow ([0,1], 1)$ without singularities near the boundary. We take on $V_0 \times [-1,1]$ the function $g(x,s) = g_0(x) + s^2$ and we set $V = \{g = 1\} \subset V_0 \times [-1,1]$. We have a smooth manifold which is identified with the double of V_0 via the two functions $V_0 \longrightarrow V$, $x \mapsto (x, \pm(1 - g_0(x))^{1/2})$, which send C to $C \times \{0\} \subset V$.

Now let π be the projection $V_0 \times [-1,1] \longrightarrow V_0$ and let f be the restriction to V of $f_0 \circ \pi$. As the kernel of $d(f_0 \circ \pi)$ contains at each point $\partial/\partial s$ at as the tangent space to V is defined by $d(g_0 \circ \pi) + 2s\, ds = 0$, we see that the critical points of f all lie on $C \times \{0\} = V \cap (V_0 \times \{0\})$ and correspond exactly to the critical points of $f_0|_C$. Moreover condition (ii) follows since each local minimum (resp. maximum) of $f|_C$ is a minimum (resp. maximum) of f. ■

Example 1.3 *(V.M. Kharlamov). Let Γ be a link in S^3 and $\pi : V \longrightarrow S^3$ a branched double cover over Γ. We suppose there exists a Seifert surface C_0, bounded by Γ, and a Morse function f_0 on S^3 satisfying the following conditions:*

(i) the critical points of f_0 lie on $C_0 \backslash \Gamma$ and are exactly the critical points of $f_0|_{C_0}$;

(ii) $f_0|_{C_0}$ has no local minimum nor local maximum on Γ.

Then $C = \pi^{-1}(C_0)$ is an essential surface for $f = f_0 \circ \pi$.

Remark 1.4 *For several links, we can find a Seifert surface satisfying (i) and (ii) with f_0 the standard height function on S^3.*

Proof The critical points of f (resp. of $f|_C$) are of two types:

- the preimages under π of critical points of f_0 (resp. of $f_0|_{C_0}$);
- the preimages under π of critical points of $f_0|_\Gamma$. For f, such a point $x \in V$ is of index 1 or 2 accordingly as $f_0|_\Gamma$ has at $\pi(x)$ a minimum or maximum; for $f|_C$, such a point is always of index 1, by (ii). ■

B An example of a function having no essential surface (constructed with C. Lescop)

Example 1.5 *Let p, q be relatively prime integers with $0 \le q \le p-1$. The oriented lens space $L(p,q)$ possesses a "canonical" Morse function f which is ordered and has exactly one Morse critical point of each index 0,1,2,3. If this function has an essential surface C then either $q = 1$, or $q = p-1$, or q is odd and $p = 2(q \pm 1)$.*

Proof Let b be a regular value of f between critical values of index 1 or 2. We set $C_0 = C \cap \{f \le b\}$, $\Gamma = \partial C_0 \subset \{f = b\}$ and we denote by Θ the attaching curve of the handle of index 2 on the surface $\{f = b\}$, which is an oriented torus. Finally, we denote by μ an oriented meridian of this torus (μ bounds a disk in $\{f \le b\}$) and by λ the oriented curve determined by the 2 following conditions:

- the intersection number with μ is $+1$: $[\lambda] \cdot [\mu] = +1$;
- for a good orientation of Θ, $[\Theta] = q[\mu] + p[\lambda]$.

We distinguish two cases accordingly as C_0 is orientable or not.

(i) If C_0 is orientable, it's an annulus and the curve Γ has two isotopic components, Γ_0 and Γ_1, which cut μ once each. Orienting them appropriately, we have, for $i = 0, 1$:

$$[\Gamma_i] = m[\mu] + [\lambda],$$

therefore

$$[\Theta] \cdot [\Gamma_i] = pm - q, \quad \text{where } m \in \mathbb{Z}.$$

Thus, Γ cuts Θ in at least $2|pm - q|$ points; yet, since C exists, Θ cuts Γ in exactly two points, so $pm - q = 0, 1$ or -1. It follows that either $m = 0$ and $q = 1$ (unless $q = 0$ and $p = 1$) or $m = 1$ and $q = p - 1$.

(ii) If C_0 is not orientable, it's a Mobius strip and Γ is connected. With the appropriate orientation, we have $[\Gamma] = m[\mu] + 2[\lambda]$, where m is an odd integer.

The same argument as before shows that we must have $mp - 2q = 0, 2$ or -2; then, $m = 1$ and $p = 2(q \pm 1)$, with q odd since p and q are relatively prime. ∎

Remark 1.6 *Keeping in mind proposition I.4.5, this example shows that there exist vector fields which, for global reasons, do not preserve any contact structure.*

2 A general method for constructing essential surfaces

A Splitting along an essential surface

Definition 2.1 *Let S be a surface and Γ a closed curve on S, in general disconnected. We will say that Γ* divides S "equitably" *if we can recover S from* *two subsurfaces, in general non connected, which are both bounded by* Γ *and have the same Euler–Poincaré characteristic.*

Lemma 2.2 *Let* V *be a 3-dimensional manifold,* $f : V \longrightarrow [0, \infty)$ *a proper Morse function (with distinct critical values) and* C *an* f*-essential surface transversely orientable in* V*. Then:*

(i) C *separates* V *into handlebodies;*

(ii) C *cuts each regular level set of* f *along a curve which divides the level set equitably.*

We recall that a compact handlebody is a compact 3-dimensional manifold with boundary obtained by attaching to a ball handles of index 1; in the non-compact case, a handlebody is a direct limit of compact handlebodies.

Proof We choose a transvese orientation on C and we take two regular values of f, $b_0 < b_1$, between which f takes exactly one critical value. For $i = 0, 1$, we set $V_i = \{f \leq b_i\}$, $C_i = C \cap V_i$, $S_i = \{f = b_i\}$ and $\Gamma_i = C \cap S_i$. Thus, V_1 (resp. C_1) is obtained from V_0 (resp. from C_0) by attaching a handle H (resp. $K \subset H$: see the discussion of II.1.B). We observe that K separates H into two components; we denote them H^- and H^+, K being transversally oriented from H^- towards H^+.

If the critical value of f between b_0 and b_1 is the absolute minimum of f, C_1 is a disk which separates the ball V_1 into two balls (with corners). Moreover, Γ_1 is circle and therefore divides the sphere S_1 equitably.

We now suppose that V_0 is the union of two handlebodies, possibly disconnected and with corners, which intersect each other exactly along C_0. We denote them by V_0^- and V_0^+, C_0 being transversally oriented from V_0^- towards V_0^+. We suppose additionally that Γ_0 divides S_0 equitably.

As the attachment of K to C_0 must respect the transverse orientation, C_1 separates V_1 into $V_1^- = V_0^- \cup H^-$ and $V_1^+ = V_0^+ \cup H^+$. Thus C separates V into two submanifolds V^- and V^+.

To show (i) and (ii), we observe that the boundary of $V_i^\pm$, for $i = 0, 1$, decomposes into two parts: C_i and $S_i^\pm = V_i^\pm \cap S_i$. By hypothesis, S_0^- and S_0^+ have the same Euler characteristic. Yet:

- If H has index $j = 0, 1$, $V_i^\pm$ is obtained from $V_0^\pm$ by attaching a handle of index j. Similarly, $S_i^\pm$ is obtained from $S_0^\pm$ by attaching a handle of index j, hence:
$$\chi(S_i^\pm) = (-1)^j + \chi(S_0^\pm),$$
therefore
$$\chi(S_1^+) = \chi(S_1^-).$$
- If H has index $j = 2, 3$, $V_i^\pm$ is homeomorphic to $V_0^\pm$: we simply glue a ball along a disk contained in the boundary. However, $S_i^\pm$ is obtained from

$S_0^{\pm}$ by a "half-surgery" of index j (a surgery along an arc or a disc lying on the boundary of $S_0^{\pm}$). We then have:

$$\chi(S_i^{\pm}) = (-1)^j + \chi(S_0^{\pm}),$$

therefore, as previously,

$$\chi(S_1^{+}) = \chi(S_1^{-}).$$

■

B The principal construction

Lemma 2.3 *Let S be a closed surface, Γ_0 a closed curve in S, not necessarily connected, and α a simple arc joining in S two points of Γ_0 without other intersection. Then there exists a Morse function $f : S \times [0,1] \longrightarrow [0,1]$ satisfying the following conditions:*

(i) f has exactly two ordered critical points with respective indices 1 and 2; moreover, for t near 0 or 1, $f|_{(S\times t)} = t$;

(ii) f has an essential surface which cuts $S \times \{0\}$ along Γ_0 and $S \times \{1\}$ along the curve Γ_1, drawn in figure 7 and obtained as follows: we add a small closed component Γ', on one side or the other of α and we perform surgery on Γ_0 along α in a neighbourhood of α avoiding Γ'.

(iii) If Γ_0 divides S equitably, C is transversely orientable, and Γ_1 divides S equitably also.

Proof The method is the following: we realise $S \times [0,1]$ by attaching successively a handle of index 1 to $S \times [0,\epsilon]$, then a handle of index 2 in elimination position; simultaneously, we attach two handles of index 1 to $\Gamma_0 \times [0,\epsilon]$ so as to obtain the desired essential surface.

Let α_0, α_1 be elements of Γ_0 at the endponints of α. For $i = 0,1$ we choose on α_i a basis (v_i, w_i) for the tangent space to S having the following properties:

(i) v_0 and v_1 are tangent to Γ_0 and on the same side of α;

(ii) w_0 and w_1 are tangent to α and go inside α.

We attach then a handle of index 1, $H_1 = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, y^2+z^2 \leq 1\}$, to $S \times \{\epsilon\}$, as follows: we send $(i,0,0)$ to α_i, $(\partial/\partial y)(i,0,0)$ to v_i and $(\partial/\partial z)(i,0,0)$ to w_i. More precisely, the points $(i,y,0)$ with $-1 \leq y \leq 1$ go to Γ_0 and the points $(i,0,z)$ with $0 \leq z \leq 1$ go to α. We thus attach $K_1 = H_1 \cap \{z = 0\}$ to $\Gamma_0 \times \{\epsilon\}$. We denote by C_1 the surface obtained and by Γ its upper boundary: $\Gamma = \partial C_1 \backslash \Gamma_0$.

For $i = 0,1$ we now denote by α'_i the α image of $(i,0,1)$ and α' the sub-arc of α joining α'_0 and α'_1. In the lateral boundary of H, $H \cap \{y^2+z^2 = 1\}$, we choose an arc α'' transverse to the circles $\{x = \text{const.}\}$, isotopic to the fixed endpoints of the segment $\{(x, 0, 1) \mid 0 \leq x \leq 1\}$ and which cuts the set $\{(x, \pm 1, 0) \mid 0 \leq x \leq 1\}$ in two points (see Figure 8).

We then attach a handle of index 2 along $\Theta = \alpha' \cup \alpha''$. As α'' is transverse to the circles $\{x = \text{const.}\}$, the manifold thus obtained is diffeomorphic to $S \times [0, 1]$ by the elimination lemma of S. Smale [Mi]. Moreover, by construction, Θ cuts Γ in two points in such a way that we can attach (in a unique way) a handle of index 1 to C_1. We then see painlessly that the surface C obtained satisfies the stated conditions. ∎

Example 2.4 *If S is the sphere S^2 and if Γ_0 is a circle, the curve Γ_1 given by lemma 2.2 is composed of three nested circles (i.e. for which the complement is a disjoint union of two disks and two annuli).*

Corollary 2.5 *There exists a Morse function $g : S^2 \times [0, 2] \longrightarrow [0, 2]$ satisfying the following conditions:*

(i) for t near 0 or 2, $g|_{S^2 \times \{t\}} = t$;

(ii) g possesses an essential surface C which cuts $S^2 \times \{0\}$ and $S^2 \times \{2\}$ along a circle, and which meets $S^2 \times \{1\}$ along three nested circles.

Proof Let $f : S^2 \times [0, 1] \longrightarrow [0, 1]$ be "the" function given by lemma 2.3 taking for Γ_0 a circle. We obtain g by gluing f with the function: $S^2 \times [1, 2] \longrightarrow [1, 2]$, $(x, y) \mapsto (2 - f(x, 2 - t))$. ∎

Corollary 2.6 *(Convex version of the Lutz modifictaion [Lu]). Any 3-dimensional manifold which carries a convex contact structure carries a convex overtwisted contact structure.*

Remark 2.7 *This corollary shows how to deduce corollary III.1.5 from theorem III.1.3.*

Proof Let V be the manifold. If there exists on V a convex contact structure, then there exists, by proposition I.4.5, a proper Morse function $f : V \longrightarrow [0, \infty)$ possessing an essential surface C. For a regular value b of f, slightly larger than the absolute minimum and for ϵ sufficiently small, the set $\{b - \epsilon \leq f \leq b + \epsilon\}$ is a product cobordism $W \cong S^2 \times [0, 1]$ which C cuts along a cylinder with circular base $\Gamma \times [0, 1]$. Corollary 2.5 allows us to replace f by a proper Morse function $f' : V \longrightarrow [0, \infty)$ with an essential surface C' which cuts $S = \{f' = b\} \cong S^2$ along three nested circles. Theorem III.1.2 gives a positive contact structure ξ' on V which is invariant under a pseudo-gradient X' of f' admitting C' as characteristic surface. Proposition II.3.1 shows that then, up to admissible isotopy, the characteristic foliation of S has two limit cycles, each bounding by a disk with exactly one singularity in its interior. ∎

C An existence theorem

Theorem 2.8 *On any 3-dimensional manifold, there exists a positive proper Morse function which admits a transversally orientable essential surface.*

Remark 2.9 *Theorem 2.7, with theorem III.1.2, immediately implies theorem III.1.3.*

Proof Let V be the manifold, and $f : V \longrightarrow [0, \infty)$ a proper Morse function, having only one maximum if V is closed and no maximum if V is open. Let b_0 and b_1 be two regular values of f between which f takes only one critical value a. We set $V_i = \{f \leq b_i\}$ for $i = 0, 1$ and $S = \{f = b_0\}$.

If a is the absolute minimum of f, $f|_{V_1}$ possesses a transversally orientable essential surface. We therefore now suppose that $f|_{V_0}$ has a transversally orientable essential surface C_0, with boundary Γ_0, and we distinguish three cases, according to the index of the critical value a.

Index 1. V_1 is obtained from V_0 by attaching a handle H of index 1. Changing the attachment of H by isotopy, we can simultaneously attach along C_0 a handle of index 1 so as to have, for $f|_{V_1}$, a transversally orientable essential surface.

Index 2. Byy lemma 2.2, Γ_0 divides S equitably. It follows that the attaching curve Θ and the handle H of index 2 cuts Γ_0 in an even number $2r$ of points. If $r = 1$, we can attach to C_0 a handle of index 1, $K \subset H$, which gives for $f|_{V_1}$ a transversally orientable essential surface. If $r = 0$, we move Θ by an isotopy to create two intersection points. Now if $r > 1$, we apply lemma 2.3 to a sub-arc α of Θ which joins two consecutive intersection points with Γ_0. We thus eliminate these two points replacing $f|_{V_0}$ with a function f' which has two more critical points with respective indices 1 and 2; we then have a new transversally orientable essential surface C'_0 whose boundary Γ'_0 still divides the surface $\{f' = b_0\} = \{f = b_0\}$ equitably. Repeating this operation several times, we reduce to the case where $r = 1$.

Remark 2.10 *For compact manifolds with boundary, the proof is finished; for open and non-compact manifolds, we finish with a classic direct limit argument.*

Index 3. As f has a single maximum, the surface $S = \{f = b_0\}$ is a sphere. If $\Gamma_0 \subset S$ is a circle, we can, attaching a handle of index 3, reglue a disk to C_0, which gives the sought transversally orientable essential surface. Now, if Γ_0 is not connected, we proceed as follows.

By lemma 2.2, Γ_0 divides S equitably. Then, there exists a component Γ of Γ_0 which satisfies the following properties:

(i) Γ does not bound a disk of $S \backslash \Gamma_0$;

(ii) in one of the hemispheres of S bounded by Γ, each component of Γ_0 bounds a disk of $S \backslash \Gamma_0$; we denote by S' this hemisphere and S'' the other.

(To see that Γ exists, we observe that, if no component of Γ satisfies (i), $S \backslash \Gamma_0$ is composed on the one hand of a disjoint union of disks, and on the other of a disk with holes. Consequently, Γ_0 does not divide S equitably. To obtain (ii), we choose a component Γ satisfying (i) "minimally".)

Now, we take a component Γ' of Γ_0 in S' and, in S'', we choose a component Γ'' which we can connect to Γ by an arc α^* without recutting Γ_0. The inverse construction to that of lemma 2.3 allows us to eliminate Γ' by doing connected sum of Γ and Γ'' along α^*. The curve thus obtained again divides S equitably and has two fewer components. Repeating this operation, we render Γ_0 connected which ends the proof. ∎