Home-work exercises for week 13

Introduction to Contact Topology, Fall 2014

Exercise 1. Let (M, ξ) be a $2n-1$ dimensional contact manifold and L a Legendrian sphere in M and let $\mathcal{U}$ be a neighborhood of L, contactomorphic to a standard model (by the Legendrian neighborhood theorem). We want to construct a symplectic manifold (W, ω) by attaching a *handle* (of index n in this case) to the symplectization of $M - \mathcal{U}$. This is how to construct the handle: consider the function

$$H : \mathbb{R}^{2n} \to \mathbb{R}, \qquad (x, y) \mapsto |x|^2 - \frac{1}{2}|y|^2$$

and check the following facts:

1. the vector field $Y = \nabla H$ is Liouville;
2. H has an index n critical point and no other critical points;
3. the level set $H = c \neq 0$ is a contact type submanifold of $\mathbb{R}^{2n}$ and the induced contact form is
$$\lambda = \Sigma_{i=1}^{n} 2x_i dy_i + y_i dx_i,$$
restricted to the tangent space to $H = c$;
4. each regular level set H=c intersects the stable and unstable manifold of 0 for the flow of Y in a Legendrian sphere, which we will denote by L^c;
5. let $\Psi = \{\psi_t\}$ be the flow of Y: if c and d are both positive, then Ψ induces a contactomorphism between the level sets $H = c$ and $H = d$;
6. $L^- = \{x = 0, |y|^2 = 2\}$ and $L^+ = \{|x|^2 = 1, y = 0\}$ are Legendrian spheres in $H = -1$ and $H = 1$, respectively;
7. if we consider the neighborhood of L^- defined by
$$N_\delta^- = \left\{ |x|^2 - \frac{1}{2}|y^|2 = -1 \text{ and } |x| \leq \delta \right\},$$
there exists a neighborhood N_δ^+ of L^+ such that Ψ induces a diffeomorphism
$$N_\delta^- - L^- \xrightarrow{\sim} N_\delta^+ - L^+.$$

The region between N_δ^- and N_δ^- and bounded by the flow liens of Y through the boundary of N_δ^- will be the local model that we glue in to the symplectization of $M - \mathcal{U}$.