First round of home-work exercises (to be handed in by October 9)

Exercise 1. Fix two positive integers p, q and consider $n = p + q$.

A product decomposition (of dimension (p, q)) of an n-dimensional vector space V is a pair (V_1, V_2) consisting of two vectors subspaces of dimensions p and q, respectively, such that

$$V = V_1 \oplus V_2.$$

Do the usual theory for such structures (as in the course, i.e. describe the isomorphisms, the group, the special frames).

Remark 1. *You see that, if you see the structure, one can compute the group. The other way around is not so clear (and not even so well-defined). Think e.g. that, in the previous exercise, I gave you the G that you found and I asked you to describe the actual structure. Trey to learn something from this (intuition?). Then do the following:*

Exercise 2. Let now $G \subset GL_n(\mathbb{R})$ be the group of invertible upper triangular matrices. Describe geometrically what a G-structure on a vector space is.

Exercise 3. Show that a skew-symmetric bilinear form ω on a $2k$-dimensional vector space V, is symplectic if and only if

$$\omega^k = \underbrace{\omega \wedge \ldots \wedge \omega}_{k \text{ times}}$$

is a volume form.

Exercise 4. Show that, on an orientable connected manifold, there are precisely two orientations.

(You may want to remember that, if M is connected, then the only non-empty subset of M which is both open and closed is M itself. So, if you want to prove that something holds at all points of M, you just prove that the subset where it does hold ...)