Second round of home-work exercises
(to be handed in before or on November 6)

Exercise 1 (Symplectic form on cotangent bundles)**.** Let L be an n-dimensional manifold. Given any chart $(U, \chi) = (U, x_1, \ldots, x_n)$ on L , the differentials $(dx_1)_q, \ldots (dx_n)_q$ form a basis of T_q^*L at any $q \in U$, that is, if $\xi \in T_q^*L$, then $\xi = \sum_{i=1}^n \xi_i (dx_i)_q$ for some real coefficients $\xi_1, \ldots, \xi_n$. Hence we get an induced chart $(T^*U, \tilde{\chi})$ on T^*L given by

$$\tilde{\chi}(q, \xi) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n).$$

(i) In these coordinates the *tautological* 1*-form* λ_{can} on T^*L is defined by

$$\lambda_{\text{can}} = \sum_{i=1}^n \xi_i \, dx_i.$$

Show that this definition does not depend on the choice of coordinate chart.

(ii) Show that the 1-form λ_{can} can be written in the (coordinate-free) form

$$\lambda_p(\eta) = \xi(d\pi(p)(\eta)), \quad p = (q, \xi) \in T^*L, \quad \eta \in T_p(T^*L),$$

where $\pi : T^*L \to L$ denotes the bundle projection.

(iii) Recall that we can think of a 1-form on L as a smooth map from L to T^*L. Show that the pullback of λ_{can} under any 1-form $\sigma : L \to T^*L$ is σ itself, i.e., $\sigma^* \lambda can = \sigma$. Prove that λcan is uniquely determined by this property.

(iv) Prove that $\omega = d\lambda_{\text{can}}$ is a symplectic form on T^*M and that ω is in standard form in the charts $(T^*U, \tilde{\chi})$.

(v) If $\Lambda \subset TM$ is a collection of lattices (one in each tangent space), and $\Lambda^\vee$ the dual lattices, show that Λ is an integral affine structure if and only if $\Lambda^\vee$ is a Lagrangian submanifold of T^*M.

Exercise 2 (The Levi-Civita connection for a hypersurface)**.** Let $M \subset \mathbb{R}^{n+1}$ be an oriented hypersurface, with $\mathbf{n} : M \to \mathbb{S}^n$ the oriented normal. Let X and Y be tangent vector fields to M and define

$$\nabla_X Y = \nabla_X^o Y - \langle \nabla_X^o Y, \mathbf{n} \rangle \mathbf{n},$$

where ∇^o is the standard flat connection on $\mathbb{R}^{n+1}$. Show that ∇ is a connection on M and that it is symmetric an compatible with the metric induced on M by its inclusion into $\mathbb{R}^{n+1}$ (equipped with the standard Euclidean metric).