INVARIANTS AND DYNAMICS IN SYMPLECTIC GEOMETRY

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1. Description of the proposed research

1.1. **Introduction.** The roots of symplectic geometry lie in the study of classical mechanical systems and the relationship between dynamical and geometric questions remains one of the most challenging and intriguing topics in the field. The essence of this interaction is effectively described by a number of symplectic invariants, whose construction relies on the variational principles underlying Hamiltonian dynamics. The aim of this introduction is to give a glimpse of the relationship and explain the main concepts.

Symplectic structures constitute the right mathematical formalism to extend the study of Hamiltonian systems beyond the classical setting of euclidean space. In their most familiar form, the Hamilton equations look as follows:

$$\dot{p} = \frac{\partial H}{\partial q}, \quad \dot{q} = -\frac{\partial H}{\partial p}$$

where H is the energy function. In the case of the motion of a planet or a particle, the vectors p and q represent velocity (momentum) and position of the moving object, respectively. Due to an energy conservation principle, solutions lie on a given *energy surface*, that is, a level set of the energy function. Even if we consider motions on a manifold with non-trivial topology (for instance, a surface with holes and handles) and with geometry differing from that of standard euclidean space, the space of positions and momenta is in a natural way a *symplectic manifold*. The symplectic structure associates to the function H a vector field, which prescribes direction and speed at each point for the solutions of the system of equations. In dimension 2 an area form suffices for this purpose, in higher dimension what we need is still a two-form, whose n-fold product is a volume form: this is what we call a *symplectic form*. If suitable assumptions are satisfied (for instance, convexity), the energy surfaces admit a special structure called a *contact form*.

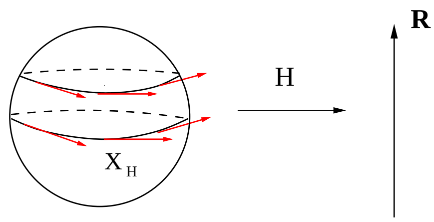

FIGURE 1. The 2-dimensional sphere with the field of velocities X_H associated to the function 'height' by the standard area form. The level sets of H are meridians on the sphere.

While every compact orientable surface admits an area form, in higher dimension not every manifold is symplectic. The dimension of the manifold needs to be even, but this condition is far from being sufficient: spheres of dimension greater than two are not symplectic. Contact geometry, on the other hand, can be regarded as the odd-dimensional counterpart of symplectic geometry. A symplectic form is a much richer object than a volume form: it was Gromov who clarified this point with his Non-Squeezing Theorem [Gro85], see also Fig. 2.

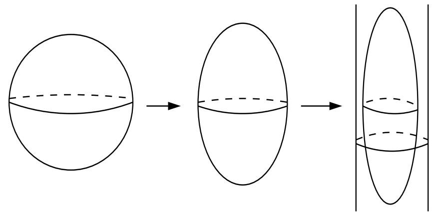

FIGURE 2. A $2n$-dimensional ball can be squeezed inside a cylinder of smaller radius in a volume-preserving way, but this cannot be done symplectically.

Several invariants have been introduced to adress the questions of existence and classification of symplectic and contact structures. A *symplectic invariant* is a simple numerical or algebraic quantity which can be associated to a given symplectic structure on a smooth manifold (analogously, contact invariants are associated to contact structures). In general, invariants do not uniquely identify a symplectic manifold, but they can distinguish different ones. In dimension 2, the total area of a closed surface is a symplectic invariant.

Symplectic geometry can also be thought of as a more flexible version of complex geometry: every symplectic manifold admits an *almost complex structure*, which gives a notion of complex multiplication for tangent vectors. In particular, one can always find almost complex structures which are *compatible* with the symplectic structure, meaning that the two can be combined to produce a Riemannian metric. It was again Gromov, in [Gro85], who realized that spaces of curves whose differential is linear with respect to such a compatible almost complex structure (*pseudo-holomorphic curves*) have good compactness properties and can be used to construct new invariants: this has led to theories such as *Gromov-Witten invariants*, *Floer homology* [Flo89], and, more recently, *contact homology* and *Symplectic Field Theory* [EGH00].

The spaces encountered in symplectic geometry can display *symmetries*. If this is the case, it can be extremely useful to identify equivalent points and study the resulting *quotient space*: of particular interest are the cases in which this also carries a symplectic structure. If we think of the original symplectic manifold as the phase space of a Hamiltonian dynamical system, this quotient is known in classical mechanics as the *reduced phase space*. The properties of the quotient space depend on the type of identification: if the symmetries have no fixed points, the resulting space will be a *smooth manifold*, otherwise it will carry *singularities*.

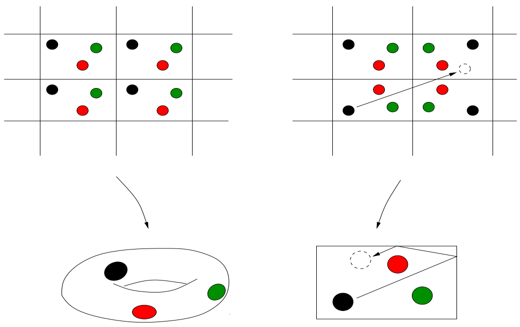

FIGURE 3. Identification of points in $\mathbb{R}^2$ corresponding under horizontal and vertical translation yields a 'torus' (the surface of a doughnut). Identifying points by reflection along horizontal and vertical axes, we will obtain a 'billiard table', a rectangle with reflecting edges.

A space with singularities like the reflecting edges of the billiard table in Fig. 3 is called an *orbifold*. Orbifolds are a very natural generalization of the concept of manifold: they have interesting topological and geometric properties and appear very often in physics.

My proposal deals with different types of symplectic invariants and their application to Hamiltonian dynamics. Schematically, I will address two main problems: the construction of invariants for manifolds lacking the usual compactness assumptions and invariants of resolutions of spaces with singularities arising from symplectic reduction.

1.2. **Non-compact contact homology and the Weinstein Conjecture.** Given a symplectic manifold (M, ω) and a smooth Hamiltonian function $H : M \to \mathbb{R}$, one is interested in the existence of *periodic solutions* of the associated Hamiltonian system of differential equations

$$\dot{x}(t) = X_H(x(t)) \text{ and } x(0) = x(T),\ T > 0,$$

on a given regular *energy surface*, that is, some level set $S = H^{-1}(c)$ of the function H. This question has generated some of the most interesting recent developments in Hamiltonian dynamics and symplectic topology. The *Hamiltonian vector field* X_H is defined using the symplectic structure on M by $i_{X_H}\omega = \omega(X_H, -) = dH$.

The first pioneering existence results were proved by Rabinowitz and Weinstein for starlike, respectively convex, hypersurfaces in the standard symplectic euclidean space $\mathbb{R}^{2n}$. Subsequently Weinstein introduced the notion of a hypersurface of *contact type*, a symplectically invariant generalization of convex and starlike, and formulated his famous Conjecture [Wei79]: every compact hypersurface of contact type must carry periodic orbits. Viterbo, in 1986, proved the Weinstein Conjecture for compact hypersurfaces in $\mathbb{R}^{2n}$ [Vit87]. In [vdBPV09] we were able to formulate a set of geometric and topological assumptions (implying in particular the contact type condition) that led to a proof of the analogue of the Weinstein Conjecture for the case of non-compact *mechanical hypersurfaces*, that is, hypersurfaces arising as level sets of Hamiltonian functions consisting of kinetic and potential energy.

The contact type condition for a hypersurface S is equivalent to the existence of a *Liouville vector field* Y, defined in a neighbourhood of S and everywhere transverse to it: then $i_Y\omega$ is a contact form and the closed characteristics of the Hamiltonian vector field on S coincide with the closed trajectories of the Reeb vector field, so the Weinstein Conjecture for hypersurfaces of contact type can be restated as a conjecture on the existence of periodic orbits of the Reeb flow on contact manifolds. These closed Reeb orbits can be detected using *contact homology*: Hofer used contact homology (in other words, he considered moduli spaces of J-holomorphic curves, as introduced by Gromov, in the *symplectization* of a contact manifold) to prove the Weinstein Conjecture for $\mathbb{S}^3$ [Hof93].

Problem: Contact homology and the related theory of symplectic homology are not well developed in the absence of suitable compactness assumptions. A hint to a possible generalization in this direction is contained in [EGH00] and [BEH$^+$03], but so far it has not been worked out. The variational method used in [vdBPV09] can be extended to obtain the analogous existence result for closed characteristics on non-compact mechanical hypersurfaces in cotangent bundles over arbitrary Riemannian manifolds. Although the result is strong, the method relies essentially on the topological implications of the mechanical assumption and is therefore not suitable for application to arbitrary Hamiltonian systems.

Plan: I believe contact homology remains the most efficient 'tool' to address the Weinstein Conjecture, also in the non-compact case. As a first step, it will be necessary to extend the definition of this invariant from compact to (a suitable class of) non-compact manifolds. More precisely, one must find suitable additional geometric conditions to ensure that the spaces of curves involved are sufficiently well behaved, so we can still carry out the construction of this invariant. The next and even more challenging step, would be the computation of the non-compact contact homology for some significant class of examples. For instance, it would be very interesting to extend the concept of Stein fillability to the non-compact case and find out to what extent the contact homology of a non-compact Stein-fillable contact manifold is determined by the homology of a Stein filling, as is shown to be the case for compact manifolds in [Yau04].

1.3. **Orbifold resolutions and their invariants.** Symplectic quotients are an important source of new symplectic manifolds and appear naturally in the context of *moment maps* and *symplectic reduction*. A given Hamiltonian function on a symplectic manifold can generate in a natural way an action of $\mathbb{R}$ or $\mathbb{S}^1$ on the manifold: in this context, symplectic reduction amounts to taking the quotient of a regular level set of the Hamiltonian by the group action (*Marsden-Weinstein quotients*). Moment maps can be thought of as a natural generalization of this concept and arise in connection with actions of more general compact Lie groups.

The case of Hamiltonian torus actions is of particular interest: in this situation, the target of the moment map is $\mathbb{R}^n$ and one can do symplectic reduction by taking the quotient of the preimage of 0. If $\mathbb{T}^n$ acts freely and 0 is a regular level set of the moment map μ, the quotient or *reduced space* $\mu^{-1}(0)/\mathbb{T}^n$ will be again a symplectic manifold. If we do not require the torus action to be free, then $\mathbb{T}^n$ will still act semi-freely, and the quotient will turn out to be a symplectic orbifold (see [Wei77]): the points fixed by some elements of the group will become *singular points*, like the points on the reflecting edges mentioned in the introduction.

If the space arising from doing symplectic reduction at a regular level set of a moment map is an orbifold, one can still look for a *symplectic resolution*, that is, a smooth symplectic manifold which is isomorphic to the orbifold outside the singular points. In [NP07] and [NP09], Klaus Niederkrüger and I worked out a method to construct resolutions for symplectic quotients of semi-free Hamiltonian torus actions at a regular level set. As a corollary, this allows us to desingularize generic symplectic quotients for compact Lie group actions.

Problem: In order to fully exploit the potential of this construction, one needs to understand the corresponding behaviour of the invariants, for instance: how are Chern classes or Gromov-Witten invariants of the resolution related to those of the original orbifold? Even more interesting would be the following question: given a symplectic Kähler orbifolds, what is the relationship between the invariants of its symplectic and complex resolution?

Plan: I think the right context to pose these questions would be that of *generalized complex geometry*: if symplectic and complex geometry are two (antipodal) examples of such a structure, can we desingularize generalized Kähler orbifolds and can we find a continuous deformation from the symplectic to the complex resolution? In order to do this, we would need to extend our desingularization construction to the generalized complex setting.

Other interesting questions related to (resolution of) orbifold singularities would be:

- Does our resolution of orbifold singularities, combined with Kirwan's partial desingularization method [Kir85], provide resolutions of singularities of spaces arising from doing reduction at an arbitrary level (not necessarily a regular one)?
- Can we decrease the rigidity in certain problems (for instance, *symplectic packing* problems [Bir01]), by allowing orbifold singularities?

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