Reminder on basic differential geometry

for the mastermath course of 2013

Charts

Manifolds will be denoted by M, N etc. One should think of a manifold as made out of points (while the elements of a vector space should be viewed as vectors). For each point $x \in M$, M looks, around x, like $\mathbb{R}^n$ (where n is the dimension of M). This is precisely what the local charts do. Such local charts will be denoted by

$$(U, \chi) = (U, \chi_1, \ldots, \chi_n),$$

which means that

$$\chi = (\chi_1, \ldots, \chi_n) : U \xrightarrow{\sim} \chi(U) \subset \mathbb{R}^n$$

go from the open $U \subset M$ to an open $\chi(U)$ in the model space $\mathbb{R}^n$. For $x \in U$,

$$(\chi_1(x), \ldots, \chi_n(x))$$

are called the coordinates of x with respect to the local chart (U, χ). Charts are used to check (and make sense of) smoothness in local coordinates. E.g., any function

$$f : M \to \mathbb{R},$$

can be represented in a local chart (U, χ) by a map from an open inside $\mathbb{R}^n$:

$$f_\chi := f \circ \chi^{-1} : \chi(U) \to \mathbb{R}.$$

Read:

$$f(x) = f_\chi(\chi_1(x), \ldots, \chi_n(x)) \quad \text{for } x \in \mathcal{U}.$$

The smoothness of f with respect to the chart (or: in the chart) (U, χ) is equivalent (by definition) with the smoothness of f_χ in the usual sense.

For two charts

$$(U, \chi),\ (U', \chi')$$

one considers the composition $\chi' \circ \chi^{-1}$ on the largest open on which it makes sense; it gives a map between two opens inside $\mathbb{R}^n$:

$$c_{\chi,\chi'} := \chi' \circ \chi^{-1} : \chi(U \cap U') \to \chi'(U \cap U'),$$

called the change of coordinates from (U, χ) to (U', χ') since

$$\chi_i^{'}(x) = c^i_{\chi,\chi'}(\chi_1(x), \ldots, \chi_n(x))$$

for all x in the domain and all i. The last formula is sometimes written in the abbreviated form:

$$\chi_i^{'} = c^i_{\chi,\chi'}(\chi_1, \ldots, \chi_n).$$

Recall that an atlas on M is a collection $\mathcal{A}$ of local charts whose domains cover M and such that any (U, χ), (U', χ') in $\mathcal{A}$ are compatible in the sense that $c_{\chi,\chi'}$ is smooth. This allows us to talk about smoothness on M. E.g., for functions $f : M \to \mathbb{R}$, its smoothness (w.r.t. $\mathcal{A}$) is equivalent (by definition) to the smoothness of f with respect to every chart in $\mathcal{A}$. The compatibility condition on the charts that belong to $\mathcal{A}$ ensure that the notion of smoothness is defined unambiguously. Moreover, for checking the smoothness (e.g. of a function f) it is enough to use a family of charts in $\mathcal{A}$ whose domains cover M.

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are called equivalent if any chart in $\mathcal{A}$ is compatible with any chart in $\mathcal{A}'$. Or equivalently: $\mathcal{A} \cup \mathcal{A}'$ is still an atlas. Or, equivalently (and better conceptually): $\mathcal{A}$ and $\mathcal{A}'$ define the same notion of smoothness. A smooth structure on M (making M into a manifold) is described (by definition) by a maximal atlas. Maximality is with respect to the inclusion; the condition is posed for conceptual reasons. In practice, one uses non-maximal (actually as small as possible) atlases to define smooth structures. Formally, that means that one associates to any atlas $\mathcal{A}$ the smooth maximal atlas $\mathcal{A}^{\max}$ that contains $\mathcal{A}$ (it consists of charts on M which are compatible with all the charts that belong to $\mathcal{A}$). For instance, the smooth structure on $\mathbb{R}^n$ is usually defined by the atlas which contains only one chart: $\mathbb{R}^n$ with the identity map; the associated maximal atlas consists of all (U, χ) with $\chi : U \to \chi(U)$ a diffeomorphism between two opens in $\mathbb{R}^n$. For the sphere however, one needs at least two charts to define an atlas (and it does work with two- remember the stereographic projection).

Manifolds; their topology

Here is the formal definition of manifolds adopted in our course: a manifold is a Hausdorff, 2nd countable topological space M endowed with a maximal atlas. Recall that 2nd countable means that there exists a countable basis of opens that generate the entire topology. The role of 2nd countability is to make sure that manifolds do behave in agreement with our expectations/desires. In particular:

- manifolds are automatically paracompact (in general: Hausdorff, locally compact, 2nd countable spaces are paracompact). Paracompactness is about the existence of partitions of unity- which provide the main tool for passing from local to global. In some text-books one assumes that manifolds are paracompact instead of 2nd countable; however, under the assumption that the number of connected components is at most countable (which is implied by 2nd countability and is a very mild condition), then requiring 2nd countability is equivalent (but easier to formulate) to paracompactness.
- in particular, the topology of any manifold is metrizable.

(you may want to have a look at the lecture notes for the “Inldeiding Topologie” in Utrecht).

One more comment on the axioms:

- Hausdorffness: sometimes one does have to consider non-Hausdorff manifolds, but then one is very clear by calling them "non-Hausdorff manifolds".

- one often assume connectedness but, again, when this is so then it is very clear about it. In principle this is a very mild assumption, because one can always break a manifold into its connected components and study these components separately (they are themselves manifolds).

- one reason to allow non-connected manifolds is that one may then allow the various components to have different dimensions. This phenomena may arise in various interesting situations. For instance, if a finite group Γ acts on a manifold M, then one can show that the fixed point set M^Γ is itself a manifold (submanifold of M) but with components of possibly varying dimensions. For us, we will adopt in principle the convention that all the components have the same dimension (just to avoid notational complications).

Tangent vectors; the tangent space(s)

Given an n-dimensional manifold M then, for each $x \in M$, one has the tangent space T_xM of M at x. It is an n-dimensional vector space which should be though of as "the linear approximation of M around x". It can be formally defined/looked at either:

- As the space of "derivations at x", i.e. linear maps $\delta : C^\infty(M) \to \mathbb{R}$ (defined on the space $C^\infty(M)$ of real-valued smooth functions on M) satisfying

$$\delta(fg) = f(x)\delta(g) + g(x)\delta(f), \quad \forall\ f, g \in C^\infty(M).$$

  The advantage of this description is that it is sometimes easier to work with; in particular, the fact that T_xM is a vector space is clear. Note that the intuition that "one should be able to take derivatives of functions at x, in the direction of tangent vectors" is built in abstractly (tautologically).

- Or as the space of all possible speeds of curves passing through x. More precisely, look at smooth curves $\gamma : I \to M$ defined on some interval I containing 0 such that $\gamma(0) = x$. One says that two such curves define have the same speed at x if, in a (and equivalently: in any) coordinate chart, the two curves have the same derivative at x. This defines an equivalence relation on the set of all such curves and T_xM is the resulting quotient. The equivalence class of such a curve is denoted $\dot{\gamma}(0) \in T_xM$.

The bijection between the two description associates to an equivalence class $\dot{\gamma}(0) \in T_xM$ the derivation

$$C^\infty(M) \to \mathbb{R}, \quad f \mapsto \frac{d}{dt}|_{t=0} f(\gamma(t)).$$

A coordinate chart

$$(U, \chi) = (\chi_1, \ldots, \chi_n))$$

induces n tangent vectors at x:

$$\frac{\partial}{\partial \chi_1}(x), \ldots, \frac{\partial}{\partial \chi_n}(x) \in T_x M \tag{1}$$

defined either by the partial derivatives with respect to the corresponding variables (as derivations):

$$f \mapsto \frac{\partial f}{\partial \chi_1}(x) := \frac{\partial f_\chi}{\partial \chi_1}(\chi(x)),$$

or, using curves, they correspond to the curves in the direction of the coordinate axes:

$$t \mapsto \chi^{-1}(\chi(x) + te_i).$$

These tangent vectors form a basis of $T_x M$.

The tangent bundle, vector fields

The tangent bundle TM is the disjoint union of all the tangent spaces $T_x M$

$$TM = \{(x, V_x) : x \in M, V_x \in T_x M\}.$$

We often denote the elements of TM by V_x, omitting the first component, but indicating it in the notation of the tangent vector. There is a canonical projection

$$p : TM \to M, \ (x, V) \mapsto x.$$

A section of TM is a section of this projection, i.e. a map

$$X : M \to TM \quad \text{satisfying} \ \ p \circ X = \mathrm{Id}_M.$$

In other words, it is a map

$$M \ni x \mapsto X(x) \in T_x M. \tag{2}$$

There is a natural way to make sense of "smoothness" of such an X. The point is that any chart (U, χ) induces, as above,

$$\frac{\partial}{\partial \chi_i} : U \ni x \mapsto \frac{\partial}{\partial \chi_i}(x)$$

which define (at each x) a basis of $T_x M$. Hence an arbitrary X is determined, on U, by the resulting coefficients:

$$X(x) = f_1(x)\frac{\partial}{\partial \chi_1}(x) + \ldots + f_n(x)\frac{\partial}{\partial \chi_n}(x),$$

where $f_1, \ldots, f_n : U \to \mathbb{R}$. With these, we declare X to be smooth if, for any coordinate chart (U, χ), the resulting f_i's are smooth.

A vector field on M is such a smooth section of TM. We denote by $\mathcal{X}(M)$ the space of all vector fields on M.

The smoothness of X, (2), can be characterized more algebraically as follows. Given X and $f \in C^\infty(M)$, one can define

$$L_X(f) : M \to \mathbb{R}, \quad L_X(f)(x) = X_x(f)$$

(the Lie derivative of f in the direction of X). The smoothness of X is then equivalent to the fact that $L_X(f)$ is smooth, for all smooth f.

Note that, in particular, any vector field induces a map (Lie derivative in the directions of X)

$$L_X : C^\infty(M) \to C^\infty(M).$$

It has the property that it is a derivation, i.e. it is linear and satisfies:

$$L_X(fg) = L_X(f)g + fL_X(g) \quad \forall\ f, g \in C^\infty(M).$$

One of the first results on manifolds is that any derivation on $C^\infty(M)$ is necessarily of this type; for this reason, one sometimes encounters this purely algebraic description (and even definition in some places) of vector fields. The advantage of this is clearly shown when looking at the Lie bracket of two vector fields (see below).

It is interesting to keep in mind the full algebraic structure of the space $\mathcal{X}(M)$ of vector fields:

- it is clearly a vector space.
- actually, it is naturally a $C^\infty(M)$-module: one can multiply any $X \in \mathcal{X}(M)$ by any function $f \in C^\infty(M)$ to obtain a new vector field fX given by $(fX)(x) = f(x)X(x)$. Hence the module operation

$$C^\infty(M) \times \mathcal{X}(M) \to \mathcal{X}(M),\ (f, X) \mapsto fX.$$

- a less obvious structure present on $\mathcal{X}(M)$ is that of Lie algebra: we have an operation (Lie bracket):

$$\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

which is antisymmetric and satisfies the Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \quad \forall\ X, Y, Z \in \mathcal{X}(M).$$

The easiest way to define this operation is by using the fact that vector fields can be interpreted as derivations: it is a simple check to see that, for any two vector fields X and Y,

$$[L_X, L_Y] := L_X \circ L_Y - L_Y \circ L_X : C^\infty(M) \to C^\infty(M)$$

is a derivation, hence it comes from another vector field on M- and that is precisely $[X, Y]$.

With the last structure in mind, the basic Lie derivative operation

$$\mathcal{X}(M) \times C^\infty(M) \to C^\infty(M), \quad (X, f) \mapsto L_X(f)$$

makes $C^\infty(M)$ into a Lie module of $\mathcal{X}(M)$.

Note also that, regarding the smoothness of vector fields, one can define a smooth structure on TM (making it into a manifold) so that the smoothness of a map X as in (2) is equivalent to the smoothness of X as a map from the manifold M to the manifold TM. The charts of TM are induced by charts (U, χ) of M. More precisely, given (U, χ), we mentioned that any $v \in T_xM$ are determined by its coefficients $\lambda_i(v)$ from the writing

$$v = \sum_i \lambda_i(v) \frac{\partial}{\partial \chi_i}(x).$$

With this, the induced chart $(\tilde{U}, \tilde{\chi})$ of TM is

$$\tilde{U} = \{(x, X_x) : x \in U, \ \tilde{\chi}(x, X_x) = (\chi_1(x), \ldots, \chi_n(x), \lambda_1(X_x), \ldots, \lambda_n(X_x)).$$

Reminder: differential forms

And here is a short reminder on differential forms on a manifold M. Start with 1-forms. A 1-form on M is a map ω:

$$M \ni x \stackrel{\omega}{\mapsto} \omega_x \in T_x^*M$$

(or a collection $\omega = \{\omega_x\}$ of elements $\omega_x \in T_x^*M$ as above), which is smooth in the following senses.

• For any coordinate chart (U, χ) of M we have the induced basis (1) on the tangent spaces T_xM (for $x \in U$) and we consider the induced dual basis of T_x^*M, denoted

$$(d\chi_1)_x, \ldots, (d\chi_n)_x.$$

Then ω_x is determined by the coordinates λ_i with respect to this basis obtained by writing

$$\omega_x = \lambda_1(x)(d\chi_1)_x + \ldots + \lambda_n(x)(d\chi_n)_x.$$

Each λ_i is a real-valued function on U (associated to ω and depending on the chart ϕ- hence a more suggestive notation would be $\lambda_i^\phi(\omega)$), which can also be described directly as

$$\lambda_i = \omega(\frac{\partial}{\partial \chi_i}).$$

With this, the smoothness of ω means that all the λ_i are smooth, for any coordinate chart ϕ.

• As for the tangent bundle, one can introduce the cotangent bundle which is the disjoint union of all the cotangent spaces T_x^*M:

$$T^*M = \{x, \xi) : x \in M, \xi \in T_x^*M.$$

As before, it comes with a projection

$$p : T^*M \to M, (x, \xi) \mapsto x$$

and T_M^* has a canonical smooth structure. With these, a 1-form is a smooth section of this projection.

- An equivalent, more algebraic (but less intuitive), way to describe the smoothness of ω is to note that any ω can be evaluated on any vector field X giving rise to a function

$$\omega(X) : M \to \mathbb{R}, \ x \mapsto \omega_x(X(x));$$

the smoothness of ω is equivalent to the fact that $\omega(X)$ is smooth for any vector field X. This brings us to yet another way of looking at 1-forms: as linear maps

$$\omega : \mathcal{X}(M) \to C^\infty(M)$$

defined on the space $\mathcal{X}(M)$ of vector fields on M, which are also $C^\infty(M)$-linear:

$$\omega(fX) = f\omega(X) \quad \forall\, f \in C^\infty(M), X \in \mathcal{X}(M).$$

Indeed, it is a basic result in differential geometry that any $C^\infty(M)$-linear map is induced by a 1-form (and the two descriptions are equivalent).

We denote by $\Omega^1(M)$ the space of all 1-forms on M. The basic example of 1-forms are the differentials of functions: for $f \in C^\infty(M)$, one has $df \in \Omega^1(M)$ characterized by

$$(df)(X) = L_X(f)$$

for all $X \in \mathcal{X}(M)$.

A similar discussion applies to p-forms for arbitrary p. They can be seen either as:

- Maps ω:
$$M \ni x \overset{\omega}{\mapsto} \omega_x \in \Lambda^p T_x^* M$$
(or a collection $\omega = \{\omega_x\}$ of elements ω_x as above) which are smooth (see below).

- Maps
$$\omega : \underbrace{\mathcal{X}(M) \times \ldots \mathcal{X}(M)}_{p \text{ times}} \to C^\infty(M)$$
which are $C^\infty(M)$-multilinear and antisymmetric.

With the first point of view, smoothness can be described either:

- by considering local charts (U, χ), the induced basis $(d\chi_i)_x$ of T_x^*M and then the induced basis
$$\{(d\chi_{i_1})_x \wedge \ldots \wedge (d\chi_{i_p})_x : 1 \le i_1 < \ldots < i_p \le n\}$$
of $\Lambda^p T_x^* M$ (see the linear story of the previous section); then we require that the coefficients of ω with respect to this basis are smooth (on U) for all coordinate charts (U, χ).

- or by requiring that for any $X_1, \ldots, X_p \in \mathcal{X}(M)$,
$$\omega(X_1, \ldots, X_p) : M \to \mathbb{R}$$
is smooth (i.e. the induced multilinear map does take values in $C^\infty(M)$).

The last point also explains the relation between the two descriptions of forms. With these, we obtain the space $\Omega^p(M)$ of p-forms on M.

On important operation here is that of wedge-products

$$\Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M), \quad (\omega, \eta) \mapsto \omega \wedge \eta$$

which is obtained by applying pointwise the linear wedge product (see the previous section):

$$(\omega \wedge \eta)_x := \omega_x \wedge \eta_x.$$

Viewing forms as multilinear maps, $\omega \wedge \eta$ is given by the same formula as in the linear case. Clearly, the main properties of the wedge product (associativity and graded antisymmetry) still hold in this context.

Another important operation with forms is the DeRham differential

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

(which was mentioned above for $p = 0$). One can describe d explicitly by the algebraic Koszul-formula:

$$(d\omega)(X_1, \ldots, X_{p+1}) = \sum_i (-1)^{i+1} L_{X_i}(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})) +$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1})$$

(where the hat sign indicates that the symbol under it is missing). The main properties of the DeRham differential (which can actually be used to characterize d uniquely) are:

- it satisfies the graded derivation property

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta) \quad \forall\, \omega \in \Omega^p(M), \eta \in \Omega^q(M).$$

- $d \circ d = 0$.
- it is a local linear operator.

One also has some terminology related to the DeRham operator: one says that a p-form ω is

- closed if $d\omega = 0$.
- exact if $\omega = d\eta$ for some $(p-1)$- form η.

The fact that $d \circ d = 0$ says that all exact forms are closed (actually one should think of the exact forms as the only closed forms that "come for free"). The "difference" between the two notion gives a very important invariant of manifolds: DeRham cohomology. More precisely, one defines the DeRham cohomology in degree p as the quotient vector space:

$$H^p(M) := \frac{\{\omega \in \Omega^p(M) : \omega \text{ is closed}\}}{\{\omega \in \Omega^p(M) : \omega \text{ is exact}\}}.$$

It is remarkable that these vector spaces are finite dimensional when M is compact. They also provide important topological invariants. For instance, for compact M,

$$\chi(M) := \sum_p (-1)^p \dim(H^p(M))$$

is the Euler characteristic of M (which can be computed using "triangulations" as the number of vertices $-$ the number of edges+ number of faces $- \ldots$). But ... that is another story.

Cartan's magic formula $L_X = di_X + i_X d$

Any vector field X induces two basic operations on differential forms:

- The interior product operations (for all p)

  $$i_X : \Omega^p(M) \to \Omega^{p-1}(M)$$

  simply by plugging in X as the first variable:

  $$i_X(\eta)(X_1, \ldots, X_{p-1}) = \eta(X, X_1, \ldots, X_{p-1}).$$

  These operations are odd derivations, in the sense that they satisfy the graded Leibniz derivation identity:

  $$i_X(\eta \wedge \theta) = i_X(\eta) \wedge \theta + (-1)^{|\eta|} \eta \wedge i_X(\theta),$$

  where $|\eta|$ is the degree of η.

- The Lie derivative operations

  $$L_X : \Omega^p(M) \to \Omega^p(M)$$

  given explicitly by the formula:

  $$L_X(\eta)(X_1, \ldots, X_p) := L_X(\eta(X_1, \ldots, X_p)) - \sum_i \eta(X_1, \ldots, [X, X_i], \ldots, X_n).$$

  The best way to think of $L_X(\eta)$ is: as the variation of η along X; this can be made sense of (and taken as definition of L_X) by using the flow of X. These operations are derivations:

  $$L_X(\eta \wedge \theta) = L_X(\eta) \wedge \theta + \eta \wedge L_X(\theta).$$

  There are various formulas that one can write down here but by far the most important one is the one relating these two operations with DeRham differential:

  $$di_X + i_X d = L_x,$$

  also known as "Cartan's magic formula".

Flows of vector fields

Ok, this should have been mentioned earlier. Let X be a vector field. Its flow arises by looking at all the possible integral curves of X. An integral curve of X is a smooth curve $\gamma : I \to M$ defined on some open interval I satisfying

$$\dot{\gamma}(t) = X(\gamma(t)) \quad \forall\, t \in I.$$

Given x_0 in M, an integral curve of X through x_0 is any γ as above with the property that $0 \in I$ and

$$\gamma(0) = x_0.$$

The basic properties of integral curves are:

- there exists an integral curve through each point.
- two integral curves which coincide at one point, must coincide on their common domain.

These imply that, for any $x \in M$, there exists an unique maximal integral curve through x:

$$\gamma_x : I_x \to M$$

for some open interval I_x containing the origin. One reorganize everything by considering:

- the domain of the flow of X:

  $$\mathcal{D}(X) := \{(x,t) \in M \times \mathbb{R} : t \in I_x\}$$

  (the set of points (x,t) for which $\gamma_x(t)$ is defined). It is an open subset of $M \times \mathbb{R}$.

- the flow of X:

  $$\Phi_X : \mathcal{D}(X) \to M,\ \Phi_X(t,x) := \gamma_x(t).$$

It is a smooth map. One often also uses the notation:

$$\Phi_X^t(x) := \Phi_X(x,t)$$

which suggests the fact that, when fixing the time t, one thinks of the flow as giving a map:

$$\Phi_X^t : M \to M.$$

Strictly speaking this is not correct unless $\mathcal{D}(X) = M \times \mathbb{R}$. In general, Φ_X^t defines a diffeomorphism defined on an open subset of M.

Hence the defining property of the flow is:

$$\Phi_X^0(x) = x,\ \frac{d}{dt}\Phi_X^t(x) = X(\Phi_X^t(x))$$

for all (x,t) for which the flow is defined. Its basic property is that:

$$\Phi_X^t \Phi_X^s(x) = \Phi_X^{t+s}(x)$$

(again, when all these expressions are defined).

Of course, the nicest case is when $\mathcal{D}(X) = M \times \mathbb{R}$. Vector fields with this property are called complete vector fields. They are the nicest since they give rise to true diffeomorphisms Φ_X^t of M; and this provides the main tool to produce diffeomorphisms on manifolds! And there are also lots of them. On any manifold, all vector fields with compact support are complete. In particular, on a compact manifold all vector fields are complete.

The flows are also used to make sense of the Lie derivative ("variation") of various objects along vector fields. For instance, for a differential form η, one would define

$$L_X(\eta) := \frac{d}{dt}|_{t=0}(\Phi_X^t)^*(\eta).$$

This coincides with the operation discussed above. Applied to a vector field Y instead of η, the similar formula gives

$$L_X(Y) = [X, Y].$$

Time dependent vector fields

Sometimes when one wants to use vector fields to produce diffeomorphisms, just vector fields are not enough and one has to consider time-dependent vector fields, i.e. smooth families $\{X_t\}$ of vector fields parametrized (smoothly) by the real parameter t, running in an interval containing the origin. There are many ways to characterize smoothness (starting with the local coordinates), and any of the sensible guesses is equivalent to all the others. Formally, we are talking about smooth section of vector bundle over $M \times \mathbb{R}$ (or M times a smaller interval) which is the pull-back of TM via the first projection. What is the flow of such a time dependent vector field? Of course, one can take the flow of each X_t (for each t) and we obtain expressions of type $\Phi_{X_t}^s$; we can also take $s = t$ to make everything depend on one t only. However, that is not it.

So, let us discuss the flow Φ of a time dependent vector field X_t (we will omit X from the notation): it is characterized by

$$\frac{d}{dt}\Phi^t(x) = X_t(\Phi^t(x)), \quad \Phi^0(x) = x.$$

Of course, one can go through the entire theory like for vector fields. However, there is a simple trick to reduce everything to vector fields: pass to $M \times \mathbb{R}$ and the vector field $\tilde{X}$ given by

$$\tilde{X}(x, s) := \tilde{X}_s(x) = \frac{d}{ds}$$

and consider its flow $\tilde{\Phi}$. With this, one defines the flow Φ by: $\Phi^t(x) =$ the first component of $\tilde{\Phi}^t(x, 0)$. This is not arbitrary: indeed, a careful look at how the flow $\tilde{\Phi}$ looks like (e.g. the second component is immediately computable, while to understand the first one one uses the flow property), we find that all the information is contained in Φ; actually:

$$\tilde{\Phi}^t(x, s) = (\Phi^{t+s} \circ (\Phi^s)^{-1}(x), t + s)$$

(but be aware that $(\Phi^s)^{-1}$ is no longer Φ^{-s}).

Finally, one should be aware that for a time dependent X_t as above and any smooth family η_t of forms on M, one has the variation formula

$$\frac{d}{dt}(\Phi^t)^*(\eta_t) = (\Phi^t)^*(L_{X_t}(\eta_t) + \frac{d}{dt}\eta_t).$$

Partitions of unity

To be continued.