We consider the projection $\pi : SO(3) \to S^2$, where we send a matrix to its first row vector. Since matrices in $SO(3)$ are such that all the row and column vectors are of unit length, we see that this map is well-defined.
(a) We will show that π is a principal S^1 bundle. For this, we first need to define the action of S^1 on $SO(3)$. We define it as follows:

$$\begin{pmatrix} a \\ v \\ w \end{pmatrix} \cdot (z_1 + iz_2) := \begin{pmatrix} a \\ z_1 v + z_2 w \\ -z_2 v + z_1 w \end{pmatrix},$$

where we denote with a, v, w row-vectors and $z_1 + iz_2 \in S^1 \subset \mathbb{C}$ an arbitrary element. Note that using that we can characterize $z_1 + iz_2 = e^{i\theta}$, we find that

$$\begin{pmatrix} a \\ \tilde{A} \end{pmatrix} \cdot e^{i\theta} = \begin{pmatrix} a \\ \widetilde{A \cdot e^{i\theta}} \end{pmatrix}, \qquad \text{with } \widetilde{A \cdot e^{i\theta}} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tilde{A}.$$

Note that we have the following computations:

$$\det\left(\begin{pmatrix} a \\ v \\ w \end{pmatrix} \cdot (z_1 + iz_2) \right) = \det \begin{pmatrix} a \\ z_1 v + z_2 w \\ -z_2 v + z_1 w \end{pmatrix} = z_1^2 \det(A) + z_2^2 \det(A) = \det(A) = 1,$$

with some standard rule of taking the determinant,

$$\begin{aligned}
&\langle a, z_1 v + z_2 w \rangle = z_1 \langle a, v \rangle + z_2 \langle a, w \rangle = 0 + 0 = 0, \\
&\langle a, -z_2 v + z_1 w \rangle = -z_2 \langle a, v \rangle + z_1 \langle a, w \rangle = -0 + 0 = 0, \\
&\langle z_1 v + z_2 w, z_1 v + z_2 w \rangle = z_1^2 \langle v, v \rangle + z_2^2 \langle w, w \rangle + 0 = 1, \\
&\langle -z_2 v + z_1 w, -z_2 v + z_1 w \rangle = z_2^2 \langle v, v \rangle + z_1^2 \langle w, w \rangle + 0 = 1.
\end{aligned}$$

These computation show that $A \cdot (z_1 + iz_2) \in SO(3)$. Let us check that this is indeed an action on $SO(3)$. Firstly, if $e^{i\theta} = 1$, then we matrix multiply with the identity matrix so that $A \cdot 1 = A$ for all A. Secondly, if we have $g, h \in S^1$, represented by θ_g, θ_h, then:

$$\begin{aligned}
(A \cdot g) \cdot h &= \begin{pmatrix} a \\ \begin{pmatrix} \cos(\theta_g) & \sin(\theta_g) \\ -\sin(\theta_g) & \cos(\theta_g) \end{pmatrix} \tilde{A} \end{pmatrix} \cdot h \\
&= \begin{pmatrix} a \\ \begin{pmatrix} \cos(\theta_h) & \sin(\theta_h) \\ -\sin(\theta_h) & \cos(\theta_h) \end{pmatrix} \begin{pmatrix} \cos(\theta_g) & \sin(\theta_g) \\ -\sin(\theta_g) & \cos(\theta_g) \end{pmatrix} \tilde{A} \end{pmatrix} \\
&= \begin{pmatrix} a \\ \begin{pmatrix} \cos(\theta_h + \theta_g) & \sin(\theta_h + \theta_g) \\ -\sin(\theta_h + \theta_g) & \cos(\theta_h + \theta_g) \end{pmatrix} \tilde{A} \end{pmatrix} \\
&= A \cdot (g \cdot h),
\end{aligned}$$

where we use goniemetric identities and that S^1 is additive with respect to the θ. So indeed, this is an action.
Now let us check whether this action makes $SO(3)$ into a principal bundle over S^2. First of all, π is surjective. Given an element $a \in S^2 \subset \mathbb{R}^3$, we can use Gramm-Schmidt to get an orthonormal basis $\{a, v, w\}$ of $\mathbb{R}^3$ such that a is the first basis elements. Then, we find that either $\begin{pmatrix} a \\ v \\ w \end{pmatrix}$ or $\begin{pmatrix} a \\ v \\ -w \end{pmatrix}$ has determinant 1, so that at least one of these two are elements of $SO(3)$. Note that π

maps them exactly to a.
Secondly, we find that π is S^1-invariant by the following computation:

$$\pi(A\cdot g) = \pi\left(\begin{matrix} a \\ \begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}\tilde{A}\end{matrix}\right) = a = \pi\begin{pmatrix} a \\ \tilde{A}\end{pmatrix} = \pi(A)$$

Finally, we will show that locally we can construct a section of π, which is equivalent with the local triviality statement. First we let $U = S^2\setminus\{\pm e_3\}$. If $a\in U$, we know that $a_3\neq\pm 1$. Hence we can define $\tilde{v}_a := e_3 - a_3 a$ and $v_a := \frac{\tilde{v}_a}{|\tilde{v}_a|}$ using Gramm-Schmidt. Note that since $a_3\neq\pm1$, we find that v_a depends smoothly on a. Now, for any $a_0\in U$, we pick the unique w_0 such that $\begin{pmatrix} a_0\\ v_{a_0}\\ w_0\end{pmatrix}\in SO(3)$. We do this by construction, pick any element $w\notin \text{span}(a_0, a_{v_0})$ and use Gramm-schmidt to make an orthonormal basis $\{a_0, v_{a_0}, w_0\}$. Then the determinant of $\begin{pmatrix} a_0\\ v_{a_0}\\ w_0\end{pmatrix}$ is either plus or minus one, so after switching w_0 with $-w_0$ when the determinant was negative, gives the wanted w_0.
Now, we define $\tilde{w}_a := w_0 - \langle w_0, a\rangle a - \langle w_0, v_a\rangle v_a$ and $w_a := \frac{\tilde{w}_a}{|\tilde{w}_a|}$. Let us see whether we can actually do this. We see that $\tilde{w}_a = 0$ if and only if $w_0 = \langle w_0, a\rangle a + \langle w_0, v_a\rangle v_a$. Since this is not the case for $a = a_0$ and the innerproduct and v_a all depend smoothly on a, we see that there exists some small neighborhood V around a_0 such that also w_a is well-defined and hence smoothly depending on a. Now note that we can choose V to be connected, so that by continuity of the determinant and since it has only values ±1 on $O(3)$:

$$\det\begin{pmatrix} a\\ v_a\\ w_a\end{pmatrix} = \det\begin{pmatrix} a_0\\ v_{a_0}\\ w_0\end{pmatrix} = 1.$$

We conclude that on V, we have the section which sends $a\mapsto\begin{pmatrix} a\\ v_a\\ w_a\end{pmatrix}$. Of course we can do the same for all points inside $S^2\setminus\{\pm e_2\}$, so we conclude that around all points there exists a small neighbourhood and a local section on it. We conclude that $SO(r)\to S^2$ is a principal S^1-bundle.

(b) Here we will prove that $E := E(SO(3),\mathbb{R}^2)$ is isomorphic to TS^2. We use the representation $\rho : S^1\to GL_2; g\mapsto\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$. Note that $\rho(g)^{-1} = \rho(g)^T$, what we will use later on. Consider the following isomorphism:

$$\Psi : E\to TS^2; [A,\xi]\mapsto \xi^T\tilde{A}.$$

Here we view ξ^T as a row vector and we use normal matrix multiplication. Let us first check whether everything is well-defined. So, we have to check whether $[A\cdot g, \rho(g)^{-1}\xi]$ gives the same answer. Here we find that

$$\begin{aligned}\Psi([A\cdot g,\rho(g)^{-1}\xi]) &= (\rho(g)^{-1}\xi)^T\cdot\widetilde{A\cdot g} = \xi^T\cdot(\rho(g)^{-1})^T\rho(g)^{-1}\tilde{A}\\ &= \xi^T\rho(g)\rho(g)^{-1}\tilde{A} = \xi^T\tilde{A} = \Psi([A,\xi]).\end{aligned}$$

So indeed, it is well-defined on the quotient. Now let us check whether the image is in TS^2. For this to hold, we need that $\Psi([A,\xi])$ is orthogonal to a, with $A = \begin{pmatrix} a\\ \tilde{A}\end{pmatrix}$ and $\tilde{A} = \begin{pmatrix} v\\ w\end{pmatrix}$. We compute that:

$$\langle a, \Psi([A,\xi])\rangle = \langle a, \xi^T\tilde{A}\rangle = \langle a, \xi_1 v + \xi_2 w\rangle = \xi_1\langle a, v\rangle + \xi_2\langle a, w\rangle = \xi_1 0 + \xi_2 0 = 0.$$

Here we use that $A \in SO(3)$, so that the row columns form an orthonormal basis. We need to check a couple more things to conclude that Ψ is an isomorphism. Note that it is clear that Ψ is smooth, since it is just polynomial. Secondly, it is linear, since:

$$\Psi([A,\xi]+[A,\eta]) = \Psi([A,\xi+\eta]) = (\xi+\eta)^T\tilde{A} = \xi^T\tilde{A} + \eta^T\tilde{A} = \Psi([A,\xi]) + \Psi([A,\eta]).$$

And by the construction of E and the independence of choice of representative of the quotient class, this is all we need to check. We also find that this is indeed a vectorbundle morphism, that is, $\pi \circ \Psi = \pi$ by construction.
So we are left to check whether all Ψ_a are isomorphisms as linear maps. We start with injectivity. Suppose that $\Psi([A,\xi]) = 0$, this means that $\xi^T\tilde{A} = 0 \in \mathbb{R}^3$. Now suppose without loss of generality that $\xi_1 \neq 0$. Then we find that since $0 = \xi_1 v + \xi_2 w$, that $v = -\frac{\xi_2}{\xi_1} w$. Note that immediately implies that $\xi_2 \neq 0$, or else $v = 0$ which is not possible since it has to have length 1. Now we find that:

$$0 = \langle v, w\rangle = -\frac{\xi_2}{\xi_1}\langle w, w\rangle = -\frac{\xi_2}{\xi_1} \neq 0.$$

Hence we find a contradiction, so that $\xi = 0$ and hence $[A,\xi] = 0$.
Now let us prove surjectivity. Let $x = (x_1, x_2, x_3) \in T_aS^2$. That is, $\langle a, x\rangle = 0$. Now let $v = \frac{x}{|x|}$ and use Gramm-Schmidt once more to find an w such that $\{a, v, w\}$ is an orthonormal basis. Once more, we might have to change w into $-w$, so that $A = \begin{pmatrix} a \\ v \\ w \end{pmatrix} \in SO(3)$. Now finally, let $\xi = (|x|, 0)$. Then:

$$\Psi([A,\xi]) = \xi^T\tilde{A} = |x| \cdot \frac{x}{|x|} + 0 = x.$$

We conclude that Ψ is an isomorphism.

(c) Proposition 4.43 tells us that $SO(3)$ is isomorphic to a $SO(2)$ structure on S^2. Now let us consider what this exactly means. We know that a $O(n)-$structure is the same as a metric, and that a $GL^+(n)-$structure is the same as an orientation. Since $O(n) \cap GL^+(n) = SO(n)$, we find that finding a $SO(n)$-structure is the same as picking a metric and an orientation at the same time. Here use that in the linear case, a $SO(n)$ structure is a choice of basises of V which are orthogonal and change of basis between two of them has positive determinant, so that the orthogonal basises induce an orientation. We conclude that S^2 is a oriented riemannian manifold.