SYMPLECTIC STRUCTURES: AT THE INTERFACE OF ANALYSIS, GEOMETRY, AND TOPOLOGY

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1. Description of the proposed research

1.1. **Introduction.** Symplectic structures made their first appearance in the study of classical mechanical systems, describing, for example, the motion of planets around the sun. In the last few decades symplectic topology has developed into an extremely active field of research, with deep connections to numerous other areas of mathematics and physics. The proposal will be centered around three different problems, which can be schematically said to be of analytical, topological, and geometric nature, but which all involve symplectic structures. The aim of this introduction is to give a glimpse of a diversity of areas and of the way they interact with symplectic geometry.

Suppose we consider a *manifold* M (think of spheres or hypersurfaces). At first we are interested in the *topology* of the manifold ("rubber" geometry). At some point we may also want to introduce some additional structure: a *Riemannian structure* (or metric) g, for example, prescribes at every point a way of multiplying two tangent vectors to get a real number (an *inner product*): this operation is symmetric and gives us the notions of distance, length, and angles. A *symplectic structure* ω defines an anti-symmetric product of vectors: this necessarily vanishes on all 1-dimensional subspaces, so instead of 1-dimensional measurements we have 2-dimensional measurements.

Let us explain some concepts in the case of dimension 2, by looking at *compact surfaces*. This is an extremely simplified case, but hopefully it will provide some intuition. If we consider closed surfaces from a purely topological point of view, the only characterising feature is the *genus*, or number of holes in the surface, denoted by G (see Fig. 1).

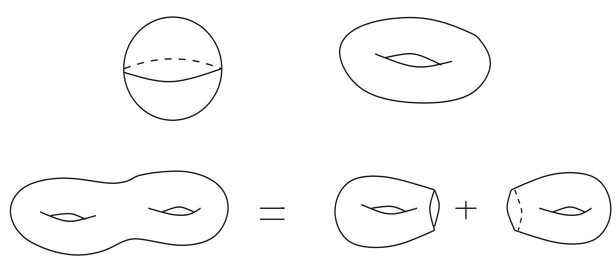

FIGURE 1. Topology only sees the number of holes in each surface.

Using a subdivision into triangles or squares, we can compute the *Euler number* e of the surface as follows:

$$e = (\text{number of faces}) - (\text{number of edges}) + (\text{number of vertices})$$

The Euler number is independent of the subdivision: it only depends on the genus and can be expressed by the relation $e = 2(1-G)$, see Fig. 2. It is the first example of a *topological invariant*, which means it does not change under *continuous deformations*.

Surfaces always admit both a metric and a symplectic structure. For a pair of tangent vectors v and w, the metric g measures the length of the vectors and the angle between them. The symplectic form ω measures the area of the parallelogram spanned by v and w. We can introduce another operation, denoted by J, which rotates tangent vectors counterclockwise by $\pi/2$, giving us a notion of *complex multiplication* (see Fig. 3). Such a rotation leaves all angle, length and

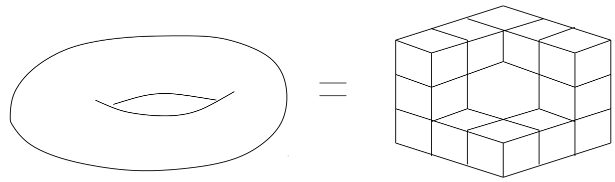

FIGURE 2. The Euler number of a torus is 0, as can be seen both by using a subdivision into squares (with 48 faces, 102 edges and 54 vertices) and the formula involving the genus.

area measurements invariant, so we say that it is *compatible* with the metric and the symplectic structure. The relation between g, ω and J is $\omega(v, Jw) = g(v, w)$.

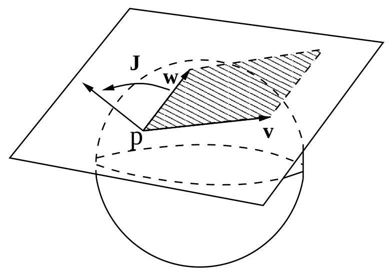

FIGURE 3. The tangent space at the point p consists of vectors representing the velocity of some curve lying on the surface and going through p. Using g, ω and J we can speak of lengths, angles, areas, complex multiplication.

Looking at the metric, we can detect local differences such as *curvature*. By using angle measurements we can tell a flat from a (positively) curved surface, see Fig. 4.

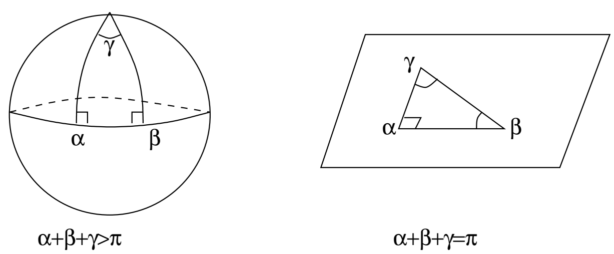

FIGURE 4. Great circles on the sphere play the same role as straight lines in the plane. By looking at the sum of the interior angles of a triangle we are able to distinguish an elliptic from a euclidean metric.

If we look at symplectic structures, only global differences can be detected (e.g., total area). To understand their local properties it is enough to consider the "standard" symplectic manifold $\mathbb{R}^{2n}$. This is expressed by saying that all symplectic manifolds are *locally isomorphic*.

Manifolds in dimension higher than 2 always admit a metric but, due to anti-symmetry, no symplectic structure can be defined if the dimension is odd. If a $2n$-dimensional manifold admits a symplectic structure, ω^n must be a volume form. This implies that not all even-dimensional manifolds admit symplectic structures: spheres of dimension $2n$, $n > 1$, for example, do not.

The symplectic structure can be used to define a set of integer numbers: the *Chern numbers* of the manifold. *Symplectic geography* investigates the possible combinations of numbers which arise in this way and compares them for the different classes of manifolds. In dimension 2 there is just one Chern number and it coincides with the Euler number described for surfaces.

We now come to the relation between symplectic topology and analysis or, more precisely, dynamical systems. Many interesting physical phenomena are described by the solutions of a system of *Hamiltonian differential equations*: these solutions live in the so called *phase space* (for example, the space of positions and velocities in the case of the motion of a planet or a particle), which is in a natural way a symplectic manifold. Illustrating how these problems historically originated in celestial mechanics, the solutions of the Hamiltonian equations are usually called *orbits* and one is interested in the existence of *periodic* orbits. In particular, due to a *conservation of energy* principle, each solution can be seen to lie on a given *energy surface* (think of the surface of the unitary ball in $\mathbb{R}^{2n}$). Using the symplectic structure one can define a special geometric property of the energy surfaces, which guarantees the existence of periodic orbits on them (*hypersurfaces of contact type*).

Apart from topological and symplectic manifolds, in this proposal we will encounter spaces that display other interesting geometrical features: for instance cone points, as shown in Fig. 5, or reflecting edges, as in a billiard table. These spaces arise very naturally, both in mathematics and in physics: they are called *orbifolds* and some admit symplectic structures.

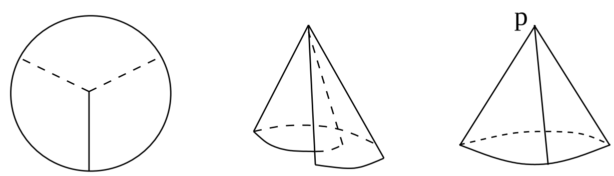

FIGURE 5. A small neighbourhood of the cone point p is obtained by wrapping a disk around itself k times or, in more technical terms, taking the quotient of the disk by a group of rotations.

I will now introduce the individual problems on which I shall focus in my research, indicating the way I plan to tackle them.

1.2. **Periodic orbits on non-compact energy surfaces.** As explained in the introduction, given a symplectic manifold (M, ω) and a smooth Hamiltonian function $H : M \to \mathbb{R}$, one is interested in the existence of *periodic solutions* of the associated Hamiltonian system of differential equations, namely

$$\dot{x}(t) = X_H(x(t)) \text{ and } x(0) = x(T),\ T > 0,$$

on a given regular *energy surface*, that is, some level set $S = H^{-1}(c)$ of the function H. The vector field X_H is defined using the symplectic structure on M by $i_{X_H}\omega = \omega(X_H, -) = -dH$. The answer to the existence problem for a particular energy surface can be seen to be independent of the choice of Hamiltonian function describing it, so in 1978 Weinstein formulated:

Conjecture (Weinstein): Every compact hypersurface of contact type in a symplectic manifold M carries at least one closed orbit.

Hypersurfaces of *contact type* (see Fig. 6) appear for instance in the case of the geodesic flow on a Riemannian manifold N, where the unit cotangent bundle of the manifold N is the energy surface of contact type and the periodic solutions correspond to closed geodesics on N.

Problem: Weinstein's conjecture deals with compact energy surfaces, as does most of the research which has been done on this problem. For compact hypersurfaces, several important results have been proved: in particular, Weinstein's conjecture has been shown to hold for hypersurfaces of contact type in $\mathbb{R}^{2n}$ (Viterbo, [12]) and, under an additional assumption, in cotangent bundles (Hofer and Viterbo, [4]).

The *noncompact* case, which includes for instance the geodesic flow on a noncompact Riemannian manifold, is at least equally interesting but still widely open. We would like to mention one result in this case, namely [7]. It is easy to see that Weinstein's conjecture does not hold without the compactness assumption, as the example $S = \{H = |p|^2 = 1\} \cong S^{n-1} \times \mathbb{R}^n$ shows. Periodic solutions of the given Hamiltonian system of equations are critical points of a suitable *action*

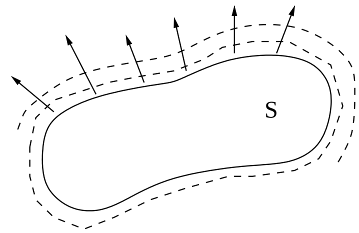

FIGURE 6. The contact type condition for a hypersurface S is equivalent to the existence of a *Liouville vector field*, defined in a neighbourhood of S and everywhere transverse to it. Such a neighbourhood is foliated by hypersurfaces which are diffeomorphic to S: the dynamics is the same on each hypersurface.

functional. Special *minimax techniques* have been successfully applied in the compact case by Rabinowitz [9] and others. Their proofs make extensive use of compactness, so that a fundamentally new approach is required in order to find the appropriate conditions to replace this assumption.

Plan: The difficulties arising from the lack of compactness are twofold: on the one hand, additional obstacles are encountered as one tries to verify the Palais-Smale condition, so that suitable geometric conditions need to be introduced to overcome them. On the other hand, one needs to find a candidate critical value for the action functional and in this the topology of the energy surface can be seen to play an extremely important role. With J. B. van den Berg and R. Vandervorst I have already done some work on the existence problem for *mechanical energy surfaces* in $\mathbb{R}^{2n}$, that is, hypersurfaces given as the 0-level set of a Hamiltonian function of the form $H(p,q) = \frac{1}{2}|p|^2 + V(q)$. In this special case we have achieved the appropriate geometric and topological conditions [11] that imply existence of closed orbits. One important element is that the topology of the energy surface can be related to the topology of its projection onto the trajectory space $\mathbb{R}^n$, see Fig. 7.

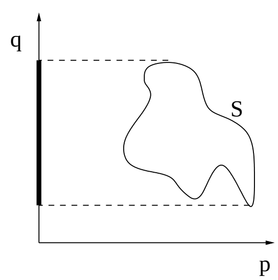

FIGURE 7. A mechanical energy surface and its projection onto the q-coordinate plane.

The aim of my research would be to extend these results to higher order Lagrangian problems and, ultimately, general Hamiltonians. Also the next logic step would be to investigate multiplicity of solutions.

1.3. **Topology and invariants of symplectic manifolds.** Symplectic geography deals with one particular kind of numerical invariants, which have already appeared in the introduction, namely the *Chern numbers*. For a given symplectic manifold, the Chern numbers are determined by the homotopy class of *compatible almost complex structures* associated to the symplectic form. These symplectic invariants, though, are very far from classifying symplectic structures. It is relatively easy to produce examples of manifolds carrying inequivalent symplectic forms, that is, not related by any sequence of isomorphisms or deformation equivalences, which are not distinguished by their Chern numbers (see for instance the eight-dimensional construction sketched in the introduction of [10]).

Problem: Although most people believe Chern numbers in dimension larger than 4 should not be determined by the topology of the underlying smooth manifold, examples of inequivalent forms on the same smooth manifold inducing different systems of Chern numbers are still missing.

Plan: I believe an example which settles the problem (and thus gives more insight into the issue of the topology of symplectic structures) could be found already in dimension 6 or 8. An interesting comparison to be made here is with the case of complex three-folds, studied by LeBrun [5]: he shows that infinitely many different values of a Chern number can be achieved by different complex structures on the same 6-manifold. A successful approach to the problem in the symplectic case will require a good knowledge of symplectic geography and of the topology of the manifolds filling the geography picture in these dimensions. This is in general no easy task, but at least in special cases one should be able to get very clear results, because the constructions in [3] and [8] give good control of the homological data and in dimension 6, for instance, there are results classifying smooth manifolds in terms of these same data [13]. Given this, one might hope to be able to detect a smooth manifold realising two different combinations of Chern numbers, that is, carrying two different symplectic structures, distinguished by the Chern numbers.

1.4. **Resolution of symplectic orbifold singularities.** Symplectic quotients are an important source of new symplectic manifolds: they appear naturally in the context of Hamiltonian actions and the associated moment maps (*symplectic reduction*). In general such quotients will turn out to be *symplectic orbifolds.*

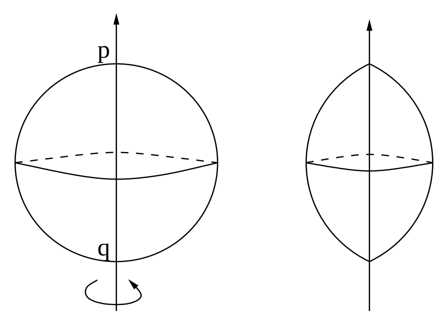

FIGURE 8. The orbifold arising as the quotient of S^2 under a finite cyclic group action (generated by rotation by $2\pi i/k$ around the z-axis). The fixed points p and q of the action give rise to orbifold singularities.

If this is the case, one can still look for a symplectic *resolution* of the orbifold, that is, a smooth symplectic manifold which is isomorphic to the orbifold outside the singular points.

Problem: The problem of resolving symplectic singularities appeared for the first time in [2], where it was suggested that all such singularities could be solved by repeated blow-ups. This is not true in general [6], but blow-up should still be the appropriate tool to solve symplectic singularities in many interesting cases. A technique involving quotients and resolutions via blow-ups is presented in [1]. We are therefore led to the question: Under which conditions can symplectic (orbifold) singularities be resolved?

Plan: I have, in collaboration with Klaus Niederkrüger, started to generalize the construction in [1], that is, a symplectic resolution of isolated orbifold singularities, as a first step in the direction of finding resolutions for more general types of singularities. Roughly speaking, we expect to be able to symplectically resolve all isolated singularities, provided a suitable local expression for the action can be found. Although this plan is far less detailed than the other points included in my proposal, I am nevertheless convinced that it is an interesting and challenging problem to work on, with applications to the construction of new examples of symplectic manifolds having specific properties (e.g., Chern numbers and geography, symplectic fillings of contact manifolds...)

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