# Mean-Field Analysis for the Evaluation of Gossip Protocols

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# Why mean-field approximation?

Observations

- Simulations/analysis of large-scale prob. systems is very costly
- Interactions between nodes
- Modelling of local behaviour of node
- Union of them results in a large (stochastic) model

Assuming "$N \to \infty$" instead of "large $N$"

- Small deterministic process [Le Boudec et al. 2007]
- Limit behaviour of complete system
- Simple matrix-vector multiplications
- Limit of the measure of interest

Gossip protocols naturally fit for mean-field analysis

- Operate in large-scale, decentralized network
- Symmetrical behaviour of nodes
- Data exchange in a random fashion

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# Mean-field approximation

$N$ interacting objects, identically defined

- gossiping nodes
- each with state space $S = \{0, 1, \dots K - 1\}$

Time is discrete $t \in \mathbb{N}$

## Occupancy measure

Fraction of nodes in state $i$ at time $t$

$$M_i^N(t) = \frac{1}{N} \sum_{n=1}^{N} 1_{\{X_n^N(t)=i\}}, \ i \in S,$$

where $X_n^N(t)$ - state of object $n$ at time $t$

## State transition probability

$P_{i,j}^N(\mathbf{m}) = \Pr\{X_n^N(t+1) = j \mid X_n^N(t) = i, M^N(t) = \mathbf{m}\}, i, j \in S, \ \mathbf{m} \in S_M^N.$

# Mean-field approximation

Limit behaviour of complete system

Approximation for the occupancy measure

**Theorem (Mean-field convergence [Le Boudec 2007])**

*Fixing* $M^N(0) = \mu(0)$, *define*

$$P(\mathbf{m}) = \lim_{N\to\infty} P^N(\mathbf{m}),\ \mathbf{m} \in \mathbb{R}^K.$$

*and the deterministic process*

$$\mu(t+1) = \mu(t) \cdot P(\mu(t)).$$

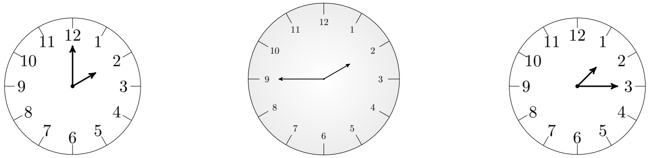
*Then*

$$\lim_{N\to\infty} M^N(t) = \mu(t),\ \textit{with probability 1},$$

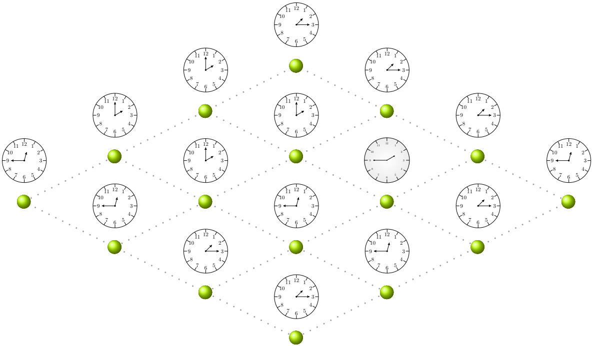
$\mu(t)$ is the deterministic limit occupancy measure for $N \to \infty$.

# Gossip Time Protocol

- Self-managing time synchronization protocol
- Synchronization of clocks in both time and frequency
- Nodes use peer-sampling service
- Presence of “time source”



# Gossip Time Protocol



- Network of nodes, each equipped with a local clock
- Nodes periodically exchange time info in random fashion
- Node with the worse-quality time adopts the higher-quality time of its peer

# Gossip Time Protocol

*A* initiates a gossip with random peer *B*

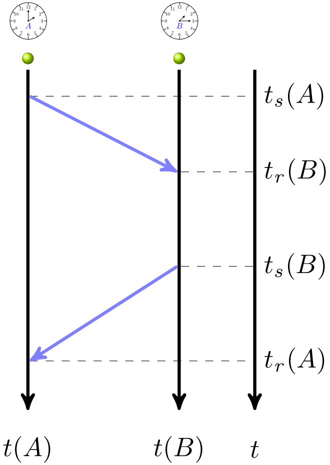
- *A* is *active*
- *B* is *passive*

In *Basic GTP*

- time sample based on hop count metric
- sample with higher hop count is rejected
- update is enforced if last-update timer expires

In *Gradual GTP*

- node may adjust gossip frequency



# GTP Analysis

Node's behaviour := state + occupancy measure

The state of a node is a triple $(g, l, h)$

- gossip delay $g$ % time to next contact
- last update counter $l$ % time that node did not synchronize
- hop count $h$ % distance to time source

Occupancy measure $\mathbf{m}_{(g,l,h)}$

- fraction of nodes in state $(g, l, h)$

## Gossip delay function

$$G : \{0, \ldots, H, \infty\} \mapsto \{0, \ldots, G_{\max}\}$$

# GTP Analysis

We distinguish

- Time sources, $h = 0$
- Active nodes, $g = 0$
- Passive nodes, $g > 0$

For active and passive nodes

$\Rightarrow$ State transition probabilities
- successful update
  - $(0, \cdot, h \mid h > 0) \rightarrow (G(h' + 1), L, h' + 1)$
- clock did not update
  - $(0, \cdot, h \mid h > 0) \rightarrow (G(h), \cdot, h)$

$\Rightarrow$ Updates
- enforced
- optional

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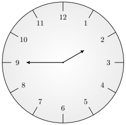
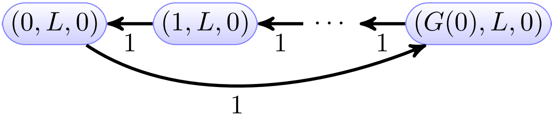
# GTP Analysis

A *time source*

- has state $(g, L, 0)$
- is independent of the enviroment

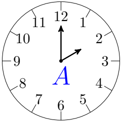



## Transition probabilities

$$P^N_{(g,l,0|g>0),(g-1,L,0)}(\mathbf{m}) = 1$$
$$P^N_{(0,l,0),(G(0),L,0)}(\mathbf{m}) = 1$$

# GTP Analysis




An *active node*

- has state $(0, l, h)$
- initiates gossip with random peer

Update $A \leftarrow (G(h_B + 1), L, h_B + 1)$ is

- enforced, if $l_A = 0 \wedge h_B \neq \infty$
- optional, if $l_A \neq 0 \wedge h_A > h_B$

## Transition probabilities

Probability of successful update:

$$P^N_{s(l=0),s'}(\mathbf{m}) = \mathbf{m}_{(g',l',h'|g'>0)} \cdot \frac{N}{N-1} \cdot \mathrm{noc}^N(\mathbf{m}), \quad \forall h' < \infty,$$

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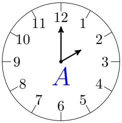


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# GTP Analysis

## Limits

The "no-collision" probability $\mathrm{noc}^N(\mathbf{m})$ converges for $N \to \infty$:

$$\mathrm{noc}(\mathbf{m}) = \lim_{N\to\infty} \mathrm{noc}^N(\mathbf{m}) = \lim_{N\to\infty} \left(\frac{N-3}{N-1}\right)^{\mathbf{m}_{(0,l,h)} \cdot N - 1} = e^{-2\cdot \mathbf{m}_{(0,l,h)}}.$$

Moreover,

$$\lim_{N\to\infty} \frac{N}{N-1} = 1$$

## Example

$$P^N_{s(0),s'}(\mathbf{m}) = \mathbf{m}_{(g',l',h' \mid g'>0)} \cdot \frac{N}{N-1} \cdot \mathrm{noc}^N(\mathbf{m})$$

# GTP Analysis

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Emulation results taken from [Iwanicki]

Setup

- $N = 1500$, 1 time source
- $G_{\max} = 25$ sec, $L = 25$ sec, $H = 15$

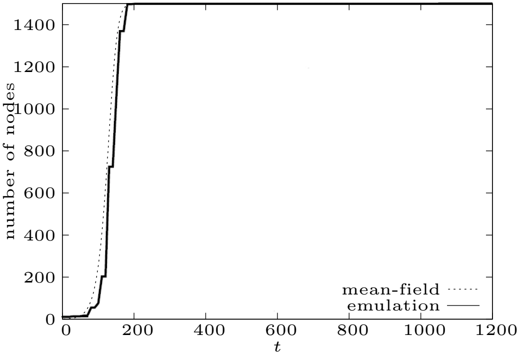


Figure: # nodes discovering the time source over time

# Mean-field vs. Emulation

Emulation results taken from [Iwanicki]

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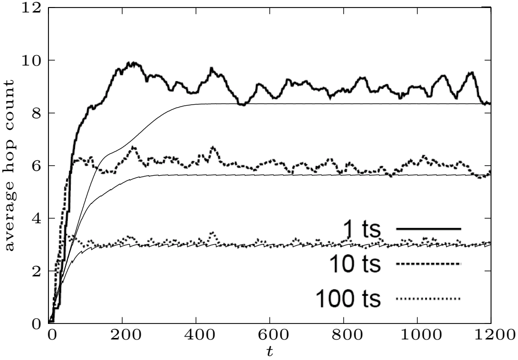


Figure: Average hop count for different # time sources

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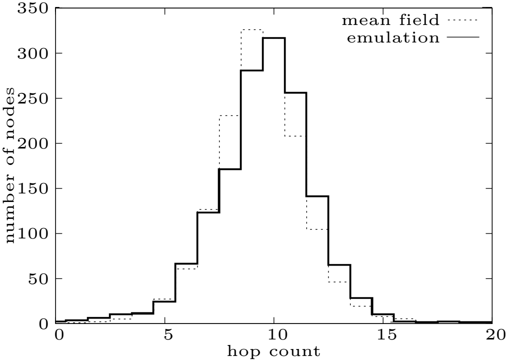


Figure: Hop count distribution after stabilization

# Further Measurements

- $G_{\text{max}} = 25$ sec, $L = 25$ sec, $H = 15$

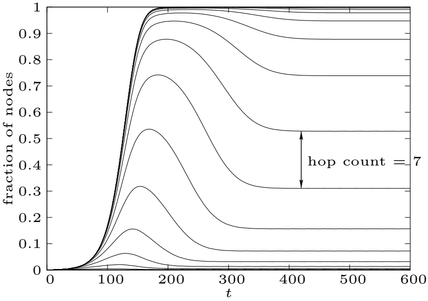


Figure: Hop count distribution over the first 10 min

# Static gossip delay

## Gossip delay function

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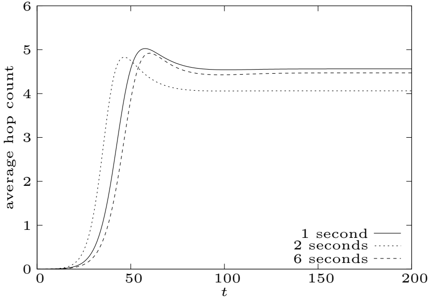


Figure: Average hop count for different gossip delays

# Dynamic gossip delay

Adaptation of gossip delay $G(h)$

- nodes with "bad quality time" to gossip more
- $G_{\text{min}}$ = minimal gossip delay

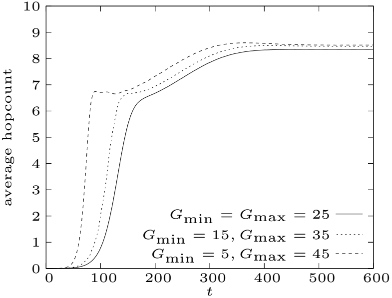


Figure: Average hop count for different $G_{\text{max}}, G_{\text{min}}$ in first 10 min