Mean-Field Analysis for the Evaluation of Gossip Protocols

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Why mean-field approximation?

Observations

- Simulations/analysis of large-scale prob. systems is very costly
- Interactions between nodes
- Modelling of local behaviour of node
- Union of them results in a large (stochastic) model

Assuming “$N \to \infty$” instead of “large $N$”

- Small deterministic process [Le Boudec et al. 2007]
- Limit behaviour of complete system
- Simple matrix-vector multiplications
- Limit of the measure of interest

Gossip protocols naturally fit for mean-field analysis

- Operate in large-scale, decentralized network
- Symmetrical behaviour of nodes
- Data exchange in a random fashion
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Mean-field approximation

$N$ interacting objects, identically defined

- gossiping nodes
- each with state space $S = \{0, 1, \ldots K - 1\}$

Time is discrete $t \in \mathbb{N}$

**Occupancy measure**

Fraction of nodes in state $i$ at time $t$

$$M_i^N(t) = \frac{1}{N} \sum_{n=1}^{N} 1\{X_n^N(t) = i\}, \ i \in S,$$

where $X_n^N(t)$ - state of object $n$ at time $t$

**State transition probability**

$$P_{i,j}^N(m) = \Pr\{X_n^N(t+1) = j \mid X_n^N(t) = i, M^N(t) = m\}, \ i, j \in S, \ m \in S_M^N.$$
Mean-field approximation

Limit behaviour of complete system

Approximation for the occupancy measure

**Theorem (Mean-field convergence [Le Boudec 2007])**

*Fixing $M^N(0) = \mu(0)$, define*

$$P(m) = \lim_{N \to \infty} P^N(m), \quad m \in \mathbb{R}^K.$$  

*and the deterministic process*

$$\mu(t + 1) = \mu(t) \cdot P(\mu(t)).$$

*Then*

$$\lim_{N \to \infty} M^N(t) = \mu(t), \quad with \ probability \ 1,$$

$\mu(t)$ is the deterministic limit occupancy measure for $N \to \infty$. 
Gossip Time Protocol

- Self-managing time synchronization protocol
- Synchronization of clocks in both time and frequency
- Nodes use peer-sampling service
- Presence of “time source”
Gossip Time Protocol

- Network of nodes, each equipped with a local clock
- Nodes periodically exchange time info in random fashion
- Node with the worse-quality time adopts the higher-quality time of its peer
Gossip Time Protocol

A initiates a gossip with random peer B
  • A is active
  • B is passive

In **Basic GTP**
  • time sample based on hop count metric
  • sample with higher hop count is rejected
  • update is enforced if last-update timer expires

In **Gradual GTP**
  • node may adjust gossip frequency
GTP Analysis

Node’s behaviour := state + occupancy measure

The state of a node is a triple \((g, l, h)\)
- gossip delay \(g\) \% time to next contact
- last update counter \(l\) \% time that node did not synchronize
- hop count \(h\) \% distance to time source

Occupancy measure \(m_{(g,l,h)}\)
- fraction of nodes in state \((g, l, h)\)

**Gossip delay function**

\[
G : \{0, \ldots, H, \infty\} \mapsto \{0, \ldots, G_{\text{max}}\}
\]
GTP Analysis

We distinguish

- Time sources, $h = 0$
- Active nodes, $g = 0$
- Passive nodes, $g > 0$

For active and passive nodes

$\Rightarrow$ State transition probabilities
- successful update
  - $(0, \cdot, h \mid h > 0) \rightarrow (G(h' + 1), L, h' + 1)$
- clock did not update
  - $(0, \cdot, h \mid h > 0) \rightarrow (G(h), \cdot, h)$

$\Rightarrow$ Updates
- enforced
- optional
GTP Analysis

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\[ \Rightarrow \text{State transition probabilities} \]

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  • \((0, \cdot, h \mid h > 0) \rightarrow (G(h' + 1), L, h' + 1)\)

• clock did not update
  • \((0, \cdot, h \mid h > 0) \rightarrow (G(h), \cdot, h)\)

\[ \Rightarrow \text{Updates} \]

• enforced
• optional
GTP Analysis

A time source
- has state \((g, L, 0)\)
- is independent of the environment

Transition probabilities

\[
P_N^{(g,l,0|g>0),(g-1,L,0)}(m) = 1
\]
\[
P_N^{(0,l,0),(G(0),L,0)}(m) = 1
\]
GTP Analysis

An active node
- has state \((0, l, h)\)
- initiates gossip with random peer

Update \(A \leftarrow (G(h_B + 1), L, h_B + 1)\) is
- enforced, if \(l_A = 0 \land h_B \neq \infty\)
- optional, if \(l_A \neq 0 \land h_A > h_B\)

Transition probabilities

Probability of successful update:

\[
P^N_{s(l=0), s'}(m) = m(g', l', h' | g' > 0) \cdot \frac{N}{N - 1} \cdot noc^N(m), \quad \forall h' < \infty,
\]

\[
P^N_{s(l>0), s'}(m) = m(g', l', h' | g' > 0) \cdot \frac{N}{N - 1} \cdot noc^N(m), \quad \forall h' < h.
\]

\(s(\cdot) = (0, \cdot, h | h > 0), s' = (G(h' + 1), L, h' + 1)\)
An *active node* 

- has state \((0, l, h)\)
- initiates gossip with random peer

Update \(A \leftarrow (G(h_B + 1), L, h_B + 1)\) is

- enforced, if \(l_A = 0 \land h_B \neq \infty\)
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### Transition probabilities

Probability of successful update:

\[
P^N_{s(l=0),s'}(m) = m_{(g',l',h'|g'>0)} \cdot \frac{N}{N - 1} \cdot noc^N(m), \quad \forall h' < \infty,
\]

\[
P^N_{s(l>0),s'}(m) = m_{(g',l',h'|g'>0)} \cdot \frac{N}{N - 1} \cdot noc^N(m), \quad \forall h' < h.
\]

\(s(\cdot) = (0, \cdot, h \mid h > 0), s' = (G(h' + 1), L, h' + 1)\)
## Limits

The “no-collision” probability $\text{noc}^N(m)$ converges for $N \to \infty$:

$$\text{noc}(m) = \lim_{N \to \infty} \text{noc}^N(m) = \lim_{N \to \infty} \left( \frac{N - 3}{N - 1} \right)^{m(0,l,h) \cdot N - 1} = e^{-2 \cdot m(0,l,h)}.$$ 

Moreover,

$$\lim_{N \to \infty} \frac{N}{N - 1} = 1$$

## Example

$$P^{N}_{s(0),s'}(m) = m_{(g,l',h'|g'>0)} \cdot \frac{N}{N - 1} \cdot \text{noc}^N(m)$$
Limits

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$$\text{noc}(m) = \lim_{N \to \infty} \text{noc}^N(m) = \lim_{N \to \infty} \left( \frac{N - 3}{N - 1} \right)^{m_{(0,l,h)} \cdot N - 1} = e^{-2 \cdot m_{(0,l,h)}}.$$ 

Moreover,

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Example

$$\lim_{N \to \infty} P_{s(0), s'}^N(m) = m_{(g', l', h' | g' > 0)} \cdot 1 \cdot e^{-2 \cdot m_{(0,l,h)}}$$
Mean-field vs. Emulation

Emulation results taken from [Iwanicki]

Setup

- $N = 1500$, 1 time source
- $G_{\text{max}} = 25$ sec, $L = 25$ sec, $H = 15$

![Graph showing number of nodes discovering the time source over time](image)

**Figure:** # nodes discovering the time source over time
Mean-field vs. Emulation

Emulation results taken from [Iwanicki]

Setup

- \( N = 1500 \)
- \( G_{\text{max}} = 25 \text{ sec}, \) \( L = 25 \text{ sec}, \) \( H = 15 \)

Figure: Average hop count for different \# time sources
Mean-field vs. Emulation

Emulation results taken from [Iwanicki]

Setup

• $N = 1500$, 1 time source
• $G_{\text{max}} = 25$ sec, $L = 25$ sec, $H = 15$

Figure: Hop count distribution after stabilization
Further Measurements

- $G_{\text{max}} = 25 \text{ sec}, \ L = 25 \text{ sec}, \ H = 15$

Figure: Hop count distribution over the first 10 min
Static gossip delay

Gossip delay function

\[ G : \{0, \ldots, H, \infty\} \mapsto \{0, \ldots, G_{\text{max}}\} \]

- \( G_{\text{max}} = 25 \text{ sec}, \; L = 25 \text{ sec}, \; H = 15 \)

Figure: Average hop count for different gossip delays
Dynamic gossip delay

Adaptation of gossip delay $G(h)$
- nodes with “bad quality time” to gossip more
- $G_{\text{min}} = \text{minimal gossip delay}$

Figure: Average hop count for different $G_{\text{max}}, G_{\text{min}}$ in first 10 min