How Probable is it to Discard an Ace of Hearts?

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Dedicated to Roel de Vrijer on occasion of his 60th birthday.

1 Introduction

Binomial coefficients arise in the context of counting problems. For example, $\binom{n}{k}$ is the number of ways to select k balls from a bin of n balls, where the order of the selection is not important. Binomial coefficients have the following formula[1]:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

if $0 \leq k \leq n$.

A geometric arrangement of the binomial coefficients in a triangle is known as Pascal's triangle. Its construction is related to the binomial coefficients by Pascal's rule:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{1}$$

for any $n \geq 0$ and $0 \leq k \leq n$. A combinatorial interpretation of the formula (1) is as follows. The left-hand side counts the number of ways to select k balls (numbered with the corresponding number) from a bin of n balls. This is equivalent to the sum of the following two cases: (a) if the ball n is selected, the number of ways to select the remaining $k-1$ balls from $n-1$, and (b) if the ball n is not selected, the number of ways to select all k balls from $n-1$. This is often a good approach to understand binomial coefficient identities. The interpretation above is considered as a *combinatorial proof* of the identity (1) as it gives two different solutions to the same counting problem. The combinatorial identities and, in particular, the identities involving binomial coefficients have been studied in, e.g., [5, 9, 13, 15, 8, 12].

In this paper, we describe some identities involving binomial coefficients that are the result of a non-trivial counting problem. This paper is organized as follows. Section 2 describes the counting problem that we are interested in. Section 3 presents three answers to the stated problem. The first and the third answer are exact solutions to the counting problem, thus establishing a combinatorial proof of their equality. The second answer has been obtained by extensive experiments, its correctness was hitherto not proven. In Section 4, we give an algebraic proof of a identity of the second and the third solution, thereby proving its correctness and the equality of all solutions. Section 5 concludes the paper.

2 Problem statement

The problem presented below has been described already in the context of a gossip-based protocol [3]. Since Roel likes to play games in general, and is an occasional bridge player, in this paper we present it in terms of the initial drawing of cards before a game.

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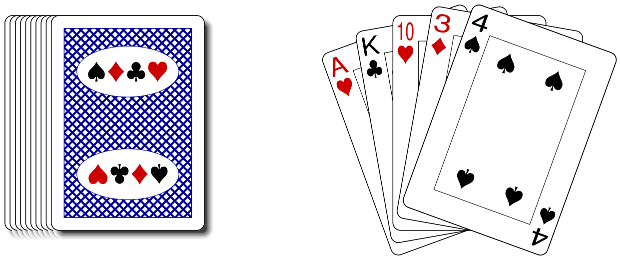
[1] A generalisation of the definition to negative integers is: $\binom{n}{k} = (-1)^k\binom{-n+k-1}{k}$ for all n and $k \geq 0$ (cf. [11]).



Most of the card games start with shuffling a deck of cards. Card games, such as blackjack, are played with multiple standard decks. There are card games that use multiple decks shuffled together depending on the number of players. Rules of the card games often require the deck(s) to be cut after cards have been shuffled by the dealer. Multiple decks and cutting the deck(s) are used to discourage card counting.

For more details on a theoretical work regarding shuffling and cutting deck(s), we refer to [4, 2]. Notably in [2], it has been shown that uniform randomness can be achieved by shuffling the deck of cards a certain number of times.

Our problem however can be described as a problem of the dealing cards, regardless of a card game played afterwards. A reason for different card dealing strategies is to discourage card counting. Furthermore, in e-card games, the dealing of a deck cannot be implemented as picking a random number, for instance from 1 to 52, since it allows a player to pick the same card more than once.

Suppose there are two identical card decks A and B, each with n different cards. Each of the decks is randomly shuffled itself. Initially, player N draws $c \leq n$ cards for its hand from the first deck A (C_A), and puts them on the table face-down. Without looking at the cards, N chooses randomly $s \leq c$ cards from its hand (S_A), and puts them aside. Next, N draws s new cards (S_B) from the second deck B, picks them up and adds them to $C_A \backslash S_A$. Since cards in hand have to be unique (i.e. presented at most once), the player must discard duplicates. Now the player has the cards $(C_A \backslash S_A) \cup S_B$ in hand and the stack S_A kept aside. If the number of cards in hand is less than c, the player N chooses the remaining cards for its hand at random among S_A that was put aside, though avoiding duplicates.

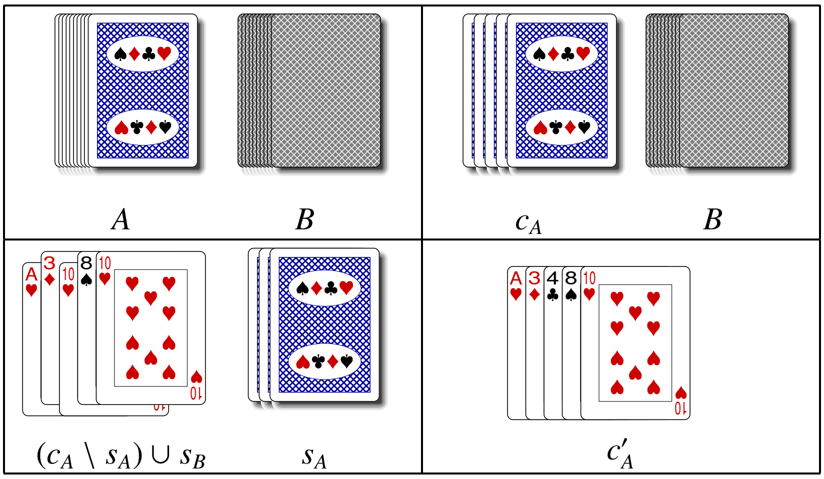


Figure 1: The example of dealing the cards with $c = 5$ and $s = 3$: (1) Using two card decks; (2) Drawing c cards from the first deck; (3) Replacing random s cards from the hand by s cards from the second one (replaced cards are kept aside); (4) Replacing dublicates in hands with the cards that are kept aside.

The question

What is the probability of a card in $S_A \setminus S_B$ to be discarded (that is, not present in the hand of N after the dealing procedure)?

3 Probability of discarding a card

In this section, we present three solutions for the described problem. These solutions reveal interesting combinatorial identities and, at the same time, are their combinatorial proof. Sections 3.1 and 3.2 repeat the analysis done in [3], while Section 3.3 contributes new material to this paper.

After selecting S_A and S_B we consider a card in $S_A \setminus S_B$ which we call Ace of Hearts. We are interested in the probability that after the dealing procedure N has no Ace of Hearts in hand, that is, the Ace of Hearts has been discarded. If $S_A = S_B$ then there is no such card and we define the probability of discarding to be 0.

3.1 Answer 1

Let k be the number of duplicates (as shown in Fig. 2), i.e. the cards of an intersection of the cards in C_A and the cards S_B drawn from the second deck B, that is, $S_B \cap C_A$. We use s for the number of cards in S_A and S_B, and c for the number of cards in C_A.

Consider Figs. 2 and 3. All cards in S_B are kept in C_A after our dealing procedure. Thus, all cards in $S_B \backslash C_A$ will replace those in $S_A \subseteq C_A$, and all cards in $S_B \cap C_A$ are kept in C_A. Hence, the probability that a card from S_A will be replaced is determined by the probability that a card from S_B is in C_A, but not in S_A. Namely, the cards from $S_B \backslash C_A$ are kept at the expense of the cards in $S_A \backslash S_B$. We would like to compute the probability P_{drop} that a card Ace of Hearts in $S_A \backslash S_B$ is discarded from C_A. The expected value of this probability depends on how many duplicates drawn from the second deck:

$$E[P_{drop}] = \begin{cases} \displaystyle\sum_{k=0}^{s} (P_{drop}^{|S_B \cap C_A| = k} \cdot P_{|S_B \cap C_A| = k}) & \text{if } s + c \leqslant n \\ \displaystyle\sum_{k=(s+c)-n}^{s} (P_{drop}^{|S_B \cap C_A| = k} \cdot P_{|S_B \cap C_A| = k}) & \text{otherwise} \end{cases}$$

where $P_{|S_B \cap C_A| = k}$ is the probability of having exactly k cards in $S_B \cap C_A$, and $P_{drop}^{|S_B \cap C_A| = k}$ is the probability that a card in $S_A \backslash S_B$ is discarded from C_A given k duplicates in $S_B \cap C_A$. The case distinction is because if $s + c > n$, then clearly there are at least $(s + c) - n$ cards in $S_B \cap C_A$.

From the $\binom{n}{s}$ possible sets S_B, we compute how many have k cards in common with C_A. Firstly, there are $\binom{c}{k}$ ways to choose k such cards in C_A. Secondly, there are $\binom{n-c}{s-k}$ ways to choose the remaining $s - k$ cards outside C_A. So in total, $\binom{c}{k}\binom{n-c}{s-k}$ possible sets S_B have k cards in common with C_A. Hence,

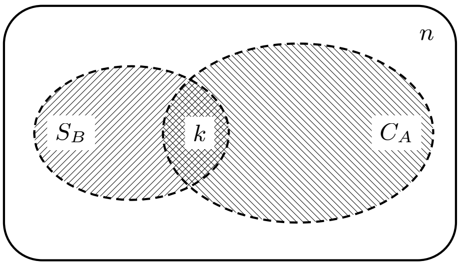


Figure 2: k cards in $S_B \cap C_A$

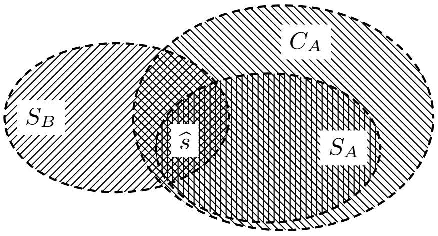


Figure 3: $\widehat{s}$ cards in $S_B \cap S_A$

$$P_{|S_B \cap C_A|=k} = \binom{c}{k}\frac{\binom{n-c}{s-k}}{\binom{n}{s}}.$$

The expected value of $P_{\mathit{drop}}^{|S_B \cap C_A|=k}$ is [1]:

$$E[P_{\mathit{drop}}^{|S_B \cap C_A|=k}] = \begin{cases} \displaystyle\sum_{\widehat{s}=0}^{k} P_{\mathit{drop}}^{|S_B \cap S_A|=\widehat{s}} \cdot P_{|S_B \cap S_A|=\widehat{s}} & \text{if } s+k \leqslant c \\ \displaystyle\sum_{\widehat{s}=(s+k)-c}^{k} P_{\mathit{drop}}^{|S_B \cap S_A|=\widehat{s}} \cdot P_{|S_B \cap S_A|=\widehat{s}} & \text{otherwise} \end{cases}$$

where $\widehat{s}$ is the number of cards in $S_B \cap S_A$ (see Fig. 3). The case distinction is because if $s+k > c$ (with k cards in $S_B \cap C_A$), then clearly there are at least $(s+k)-c$ cards in $S_B \cap S_A$.

Among the s cards in S_A, there are $\widehat{s}$ cards also in S_B, and thus only the $s-\widehat{s}$ cards in $S_A \backslash S_B$ can be discarded from C_A. $P_{\mathit{drop}}^{|S_B \cap S_A|=\widehat{s}}$ is the probability that a card in $S_A \backslash S_B$ is discarded from C_A, given $\widehat{s}$ cards in $S_B \cap S_A$:

$$P_{\mathit{drop}}^{|S_B \cap S_A|=\widehat{s}} = \begin{cases} 0 & \text{if } s = \widehat{s} \\ \frac{s-k}{s-\widehat{s}} & \text{otherwise} \end{cases}$$

$P_{|S_B \cap S_A|=\widehat{s}}$ is the probability of having exactly $\widehat{s}$ cards in $S_B \cap S_A$:

$$P_{|S_B \cap S_A|=\widehat{s}} = \binom{s}{\widehat{s}}\frac{\binom{c-s}{k-\widehat{s}}}{\binom{c}{k}}.$$

The intuition behind $P_{|S_B \cap S_A|=\widehat{s}}$ is similar to the one of $P_{|S_B \cap C_A|=k}$, and the one in, e.g. [10, p. 274].

Let us assume $s+c \leq n$ and $s+k \leq c$. Then, substituting in the expression for $E[P_{\mathit{drop}}]$ in case $s+c \leq n$, and noting that in the summand $k=s$ the factor $P_{\mathit{drop}}^{|S_B \cap S_A|=s}$ is equal to zero, we get:

$$\begin{aligned} E[P_{\mathit{drop}}] &= \sum_{k=0}^{s-1}\binom{c}{k}\frac{\binom{n-c}{s-k}}{\binom{n}{s}}\sum_{\widehat{s}=0}^{k}\frac{s-k}{s-\widehat{s}}\binom{s}{\widehat{s}}\frac{\binom{c-s}{k-\widehat{s}}}{\binom{c}{k}} \\ &= \frac{n-c}{\binom{n}{s}}\sum_{k=0}^{s-1}\binom{(n-c)-1}{(s-k)-1}\sum_{\widehat{s}=0}^{k}\frac{\binom{c-s}{k-\widehat{s}}\binom{s}{\widehat{s}}}{s-\widehat{s}} \end{aligned} \tag{2}$$

3.2 Answer 2

It is possible to approach the problem differently: from the simplifying assumption derive a simple and approximate solution. This solution has an error, which we reconstruct by extensive numerical experiments. Together with the derived correction factor, the solution becomes exact.

We re-examine the relationships between the k duplicates drawn from the second deck, the $\widehat{s}$ cards of the overlap $S_A \cap S_B$, and P_{drop}. Let's estimate $P_{\mathit{drop}}^{|S_B \cap C_A|=k}$ by considering each card from S_B separately, and calculating the probability that the card is a duplicate (i.e., is also in C_A). The probability of a card from S_B to be a duplicate (also present in C_A) is $\frac{c}{n}$. In view of the uniform distribution, the cards in deck are a random sample from the universe of n cards; so all cards in S_B have the same chance to be a duplicate. Thus, the expected number k of cards in $S_B \cap C_A$ can be estimated by $s \cdot \frac{c}{n}$. The expected number $\widehat{s}$ of cards in $S_B \cap S_A$ can be estimated by $k \cdot \frac{s}{c}$, because only the k cards in $S_B \cap C_A$ may end up in $S_B \cap C_A$; $\frac{s}{c}$ captures the probability that a card from C_A is also chosen to be in S_A. It follows that the probability of a card in $S_A \backslash S_B$ to be discarded from C_A after the dealing is

$$P_{\mathit{drop}} = \frac{s-k}{s-\widehat{s}} = \frac{s - s \cdot \frac{c}{n}}{s - s \cdot \frac{c}{n} \cdot \frac{s}{c}} = \frac{n-c}{n-s}.$$

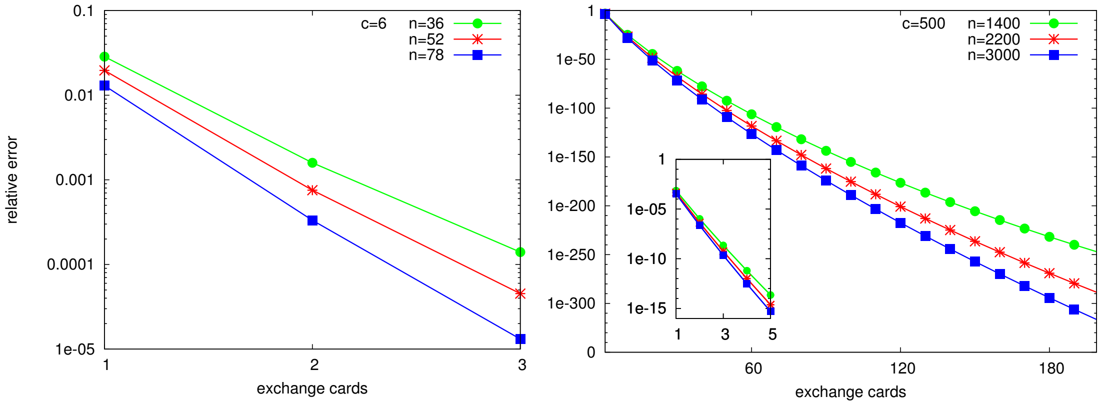


Figure 4: The relative error of the difference of the accurate P_{drop} and its approximation, for $c = 6$ (left) and $c = 500$ (right) and different values of n.

These estimates are valid for general $s \leq c \leq n$.

We plot the difference of the accurate P_{drop} and the simplification, for the number of cards in hand $c = 6$ and $c = 500$ (Fig. 4). These figures show that the simplification gives a close approximation of the formula (2) (note the log y-scale).

This brings us to the question how closely the simplified formula of P_{drop}, that is $\frac{n-c}{n-s}$ (here referred to as $S(n, c, s)$) approximates formula (2) (here referred to as $E(n, c, s)$). We compared the difference between these two formulas using a Java package based on common fractions, which provides loss-less calculation [6]. We observe that the inverse of the difference of the inverse values of both formulas, i.e.

$$e_{c,s}(n) = \left(E(n, c, s)^{-1} - S(n, c, s)^{-1}\right)^{-1},$$

exhibits a certain pattern for different values of n, c and s.

For $s = 1$ and arbitrary values of n and c, $E(n, c, 1) = \frac{n-c}{n}$, whereas $S(n, c, 1) = \frac{n-c}{n-1}$. This leads us to investigate the correction factor θ as in $E(n, c, s) = \frac{n-c}{(n-s)+\theta}$. For $s = 1$ clearly the factor $\theta = 1$. Yet, for $s > 1$ the situation is more complicated. We therefore calculate the first, the second and other (forward) differences[2] over n.

For $s = 1$, the result of the first difference of the function $e_{c,1}(n)$ is 1. However, for $s = 2$ the first difference is, e.g. if $c = 4$: $e_{4,2}(7) - e_{4,2}(6) = 3.5$, $e_{4,2}(8) - e_{4,2}(7) = 4$, $e_{4,2}(9) - e_{4,2}(8) = 4.5$ and so on. Thus, we observe that the second difference of $e_{c,2}(n)$ is $\frac{1}{2}$. By calculating higher differences for $s > 2$, we conclude that the s-th difference of the function $e_{c,s}(n)$ is always $\frac{1}{s}$.

Moreover, at the point $n = 0$ the first, …, s-th differences of the function $e_{c,s}$ exhibit a pattern similar to the Pascal triangle [14]. That is, for $d \geq 1$ the d-th difference is $(\Delta^d\, e_{c,s})(0) = \frac{1}{s\cdot\binom{s-1}{d}}$ (assuming $\binom{a}{b} = 0$, whenever $b > a$). Knowing the initial difference at point $n = 0$, we were able to use the Newton forward difference equation [1] to derive the following formula for $n > 0$:

$$E[P_{drop}] = \frac{n-c}{(n-s)+\frac{1}{\gamma}}, \quad \text{where} \quad \gamma = \sum_{d=0}^{s-1} \frac{\binom{n}{d}}{s\cdot\binom{s-1}{d}} = \frac{\binom{n}{s}\cdot\sum\limits_{d=0}^{s-1}\frac{1}{\binom{n-d}{(s-1)-d}}}{(n-s)+1} \tag{3}$$

In this equation the sum is finite because due to the observation that the s-th difference is constant $\frac{1}{s}$, all higher differences are 0.

Thus, the correction factor is $\theta = \frac{1}{\gamma}$. Extensive numerical experiments with Mathematica and Matlab indicate that $\frac{n-c}{(n-s)+\frac{1}{\gamma}}$ and formula (2) indeed coincide. We can also see from Fig. 4 that the correction factor is small.

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[2] A forward difference of a discrete function $f : \mathbb{Z} \to \mathbb{Z}$ is a function $\Delta f : \mathbb{Z} \to \mathbb{Z}$ with $\Delta f(n) = f(n+1) - f(n)$ (cf. [1]).

3.3 Answer 3

There is the third possible solution to the discarding problem. As stated in the problem, S_A is a uniform selection from C_A, which, in turn, is a uniform selection from the entire deck A with n cards. Consequently, S_A is a uniform selection from A, and S_B is a uniform selection of B by definition. We revise the probabilities in Section 3.1, considering S_A and S_B as a uniform sample from A and B, respectively. Thus, the probability $P_{|S_B \cap S_A| = \widehat{s}}$ of having exactly $\widehat{s}$ cards in $S_B \cap S_A$ is:

$$P_{|S_B \cap S_A| = \widehat{s}} = \frac{\binom{s}{\widehat{s}}\binom{n-s}{s-\widehat{s}}}{\binom{n}{s}},$$

and the probability of discarding a card is:

$$E[P_{drop}] = \sum_{\widehat{s}=0}^{s} (P_{drop}^{|S_B \cap S_A| = \widehat{s}} \cdot P_{|S_B \cap S_A| = \widehat{s}}).$$

Recall the dealing procedure. The player uses the cards in S_A to fill its hand $(C_A \backslash S_A) \cup S_B$ up to c cards. That is, exactly $|(C_A \backslash S_A) \cap S_B|$ cards are randomly chosen from $S_A \backslash S_B$. Basically, the cards in $S_A \backslash S_B$ (that are also in C_A) are replaced by the cards in $S_B \backslash C_A$. Since the numbers of cards in $S_B \backslash S_A$ and in $S_A \backslash S_B$ are equal, a one-to-one correspondence can be arranged by random selection of pairs from the sets. If a card in $S_B \backslash S_A$ is also in C_A, the pair card from $S_A \backslash S_B$ is kept, and discarded, otherwise. Thus, the probability of a card in $S_A \backslash S_B$ to be discarded depends only on the probability that a pair card in $S_B \backslash S_A$ is not in C_A. In view of the uniform selection, every card in $S_A \backslash S_B$ has a fair chance of being replaced by a card in $S_B \backslash C_A$. Moreover, a card in $S_B \backslash S_A$ has a chance to be any card in $A \backslash S_A$. Hence, the probability that the card is not in C_A is $\frac{n-c}{n-s}$. Thus, the probability $P_{drop}^{|S_B \cap S_A| = \widehat{s}}$ of a card in $S_A \backslash S_B$ to be discarded from C_A, given $\widehat{s}$ cards in $S_B \cap S_A$, is:

$$P_{drop}^{|S_B \cap S_A| = \widehat{s}} = \begin{cases} 0 & \text{if } s = \widehat{s} \\ \frac{n-c}{n-s} & \text{otherwise} \end{cases}$$

Putting the pieces together we obtain a formula $E[P_{drop}]$ for every $s \leq c \leq n$:

$$\begin{aligned} E[P_{drop}] &= \sum_{\widehat{s}=0}^{s-1} \frac{\binom{s}{\widehat{s}}\binom{n-s}{s-\widehat{s}}}{\binom{n}{s}} \cdot \frac{n-c}{n-s} = \frac{n-c}{(n-s)\binom{n}{s}} \left(\sum_{\widehat{s}=0}^{s} \binom{s}{\widehat{s}} \binom{n-s}{s-\widehat{s}} - 1 \right) \\ &=^{3} \frac{n-c}{n-s} \cdot \left(1 - \frac{1}{\binom{n}{s}} \right) \end{aligned} \tag{4}$$

4 The binomial identities

In the previous section, we presented a combinatorial proof of the identity:

$$\frac{n-c}{\binom{n}{s}} \sum_{k=0}^{s-1} \binom{(n-c)-1}{(s-k)-1} \sum_{\widehat{s}=0}^{k} \frac{\binom{c-s}{k-\widehat{s}}\binom{s}{\widehat{s}}}{s-\widehat{s}} = \frac{n-c}{n-s} \cdot \left(1 - \frac{1}{\binom{n}{s}} \right),$$

for $s + c \leq n$ and $s + k \leq c$. This identity can be rewritten as:

$$\sum_{k=0}^{s-1} \binom{(n-c)-1}{(s-k)-1} \sum_{\widehat{s}=0}^{k} \frac{\binom{c-s}{k-\widehat{s}}\binom{s}{\widehat{s}}}{s-\widehat{s}} = \frac{1}{n-s} \cdot \left(\binom{n}{s} - 1 \right) \tag{5}$$

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[3] Vandermonde's identity states that $\sum_{i=0}^{k} \binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k}$ holds for binomial coefficients.

For the completeness, it remains to prove the equality of (3) and (4):

$$\frac{n-c}{(n-s)+\frac{s}{\sum\limits_{d=0}^{s-1}\frac{\binom{n}{d}}{\binom{s-1}{d}}}} = \frac{n-c}{n-s}\cdot\left(1-\frac{1}{\binom{n}{s}}\right) \tag{6}$$

The equality (6) is a consequence of the following identity.

Proposition 1 *For $n>0, s>0$ and $s\leq n$, the following equality holds:*

$$\sum_{d=0}^{s-1}\frac{\binom{n}{d}}{\binom{s-1}{d}} = \frac{s}{n-s}\left(\binom{n}{s}-1\right) \tag{7}$$

Proof. We first modify the fraction of the binomials as follows:

$$\frac{\binom{n}{d}}{\binom{s-1}{d}} = \frac{\binom{n}{d}}{\binom{s-1}{d-1}\cdot\frac{1}{d(s-1-d)}} = \frac{s-1-d}{d}\cdot\frac{\binom{n}{d}}{\binom{s-1}{d-1}} \tag{8}$$

Instead of computing the sum on the left hand side of (7), Gosper's algorithm [7, 13] allows us to get a result of the sum by solving the first-order linear recurrence equation

$$q_d\cdot f(d+1) - r_{d-1}\cdot f(d) = p_d \tag{9}$$

where $f(d)$ is an unknown rational function of d, and q_d, r_d, p_d are polynomials in d.

Let $a_d = \frac{\binom{n}{d}}{\binom{s-1}{d}}$. The summand a_d in (7) is a hypergeometric term. Thus, we apply Gosper's algorithm to see if the sum on the left hand side of (7) can be expressed as a hypergeometric term. That is, given the hypergeometric term a_d, to find a hypergeometric term $Z_s = \frac{r_{s-1}\cdot f(s)}{p_s}\cdot a_s$, such that:

$$\sum_{d=0}^{s-1} a_d = Z_s - Z_0$$

The term ratio $\frac{a_{d+1}}{a_d} = \frac{n-d}{s-1-d}$, also defined by Gosper as $\frac{a_{d+1}}{a_d} = \frac{p_{d+1}}{p_d}\frac{q_d}{r_d}$, is rational in d as expected. To find a nonzero rational solution $f(d)$ of the first-order recurrence (9), we choose $q_d = -(n-d),\ r_d = -(s-d-1),\ p_d = 1$. The equation

$$-(n-d)\cdot f(d+1) + (s-d)\cdot f(d) = 1$$

has a nonzero solution $f(d) = -\frac{1}{n-s}$. We substitute the solution $f(d)$, the summand a_s, and the polynomials r_s and p_s into the expression for Z_s:

$$Z_s \overset{(8)}{=} \frac{1}{n-s}\binom{n}{s}\frac{s}{\binom{s-1}{s-1}} = \frac{s}{n-s}\binom{n}{s}, \quad \text{and} \quad Z_0 = \frac{s}{n-s}\ .$$

Hence,

$$\sum_{d=0}^{s-1}\frac{\binom{n}{d}}{\binom{s-1}{d}} = Z_S - Z_0 = \frac{s}{n-s}\left(\binom{n}{s}-1\right)$$

□

5 Conclusion

This paper describes a non-trivial counting problem in the context of dealing of cards. We showed that the question related to this problem can be answered in three different ways, allowing us to discover interesting binomial identities. The first and the third answer are exact solutions of the counting problem, constituting a combinatorial proof of their equality. The second answer has been derived by extensive numerical experiments. We prove its equality to the third answer (and transitively, the first one) algebraically by applying the well-known Gosper algorithm [7, 13].

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