Topologically distinct collision-free periodic solutions for the N-center problem.

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Abstract

This work concerns the planar N-center problem with homogeneous potential of degree $-\alpha$ ($\alpha \in [1, 2)$). The existence of infinitely many, topologically distinct, non-collision periodic solutions with a prescribed energy is proved. A notion of *admissibility* in the space of loops on the punctured plane is introduced so that in any admissible class and for any positive h it is proved the existence of a classical periodic solution with energy h for the N-center problem with $\alpha \in (1, 2)$. In case $\alpha = 1$ a slightly different result is shown: it exists either a non-collision periodic solution or a collision-reflection solution. The results hold for any position of the centres and it is possible to prescribe in advance the shape of the periodic solutions. The proof combines topological properties of the space of loops in the punctured plane with variational and geometrical arguments.

Keywords

N-center problem, Variational argument, Admissible classes, Self-intersections of curves

2000 Mathematics Subject Classification

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1 Introduction

The planar N-*center problem* with homogeneous potential of degree $-\alpha$, $\alpha \in [1, 2)$, consists in the equation

$$\ddot{q} = \nabla V(q), \quad q \in \mathbb{R}^2 \tag{1}$$

where the potential function is given by

$$V(x) := \frac{1}{\alpha} \sum_{j=1}^{N} \frac{m_j}{|x - c_j|^\alpha}, \qquad x \in \mathbb{R}^2 \setminus \mathcal{C}, \tag{2}$$

being $\mathcal{C} \stackrel{\text{def}}{=} \{c_1, \ldots, c_N\}$ the set of centres, $c_j \in \mathbb{R}^2$ and $m_j > 0$ for all $j = 1, \ldots, N$. Following the terminology usually adopted in celestial mechanics, a non collision periodic solution is a $\mathscr{C}^2$ function $q(t)$ so that $q(t + T) = q(t)$ for some period $T > 0$ and $q(t) \notin \mathcal{C}$ for all t.

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This paper studies the existence of non-collision, periodic solutions of (1) satisfying the energy constraint

$$\frac{1}{2}|\dot{q}|^2 - V(q) = h \tag{3}$$

for any $h > 0$.

The gravitational case, i.e. $\alpha = 1$, is surely the most interesting and widely studied case. When $N = 1$ it corresponds to the 2 body problem which is integrable and was solved by Newton. The 2-center problem, adopted as a model for one-electron molecules such as H_2^+ and $H\,He^{2+}$ [39], it was shown by Euler to be integrable in elliptic coordinates, [40]. Except for the cases $N = 1, 2$, the gravitational N-center problem is not integrable, [6, 7]. In the last decades a quite broad literature has been produced dealing with the case $N \geq 3$, both for planar and spatial motion, and a complete review of all the existing results is beyond the scope of this introduction. However we assemble some remarkable quantitative and qualitative results related to ours. In [6, 8, 9] it was shown that for $N \geq 3$ the system is non-integrable on non-negative energy levels, and it has positive entropy. The authors introduced a topological (global) regularization by applying the local KS-regularization around each center. It results that the dynamics restricted on the energy level sets is a reparametrization of the geodesic flow on a non compact complete manifold. Homological properties of the space of loops of the regularized manifold allow to estimate from below the geodesic and hence the topological entropy of the system. In [25] it was also stated the symbolic dynamics in the invariant Cantor set formed by bounded trajectories. The paper [27] concerns the non existence of analytic independent integral for the $N \geq 3$ centre problem in the space, when the energy is larger than a threshold E_{th}. In case of negative energy it has been proved the existence of chaotic dynamics for the 3-center problem on small negative energy level sets when one centre is far away from the others two [10] or when the mass of one of the centres is much smaller then the others [17]. Both the results rely on perturbative methods.

Most of the quoted works consider a regularisation of the singularities. The question about regularisation of singular potential problem and possible extension for the collision solutions have been proposed and discussed by different authors. We mention [28, 29] for $\alpha = 1$, [4, 30] for $\alpha \neq 1$, [12, 13, 35] in case of logarithmic potential ($\alpha = 0$).

A different approach to the N-center problem and in general to singular lagrangian systems is by means of variational arguments. In this context, the major difficulty is to prove that the minimiser is collision free. The *Strong force* case $\alpha \geq 2$, treated for instance in [1, 5, 20, 22, 31] is relatively easier, compared to the *weak force* case $\alpha \in (0, 2)$ where there is no control on the value of the functional whose critical points may correspond to trajectories passing through the singularities, see [3, 16, 32, 33, 36, 37].

A variational technique has been exploited in [34] to prove the existence of infinitely many non collision periodic solutions with small negative energy for the N-center problem with $\alpha \in [1, 2)$. Also, the symbolic dynamics of trajectories having a prescribed topological characterisation with respect to the centres is discussed.

Motivated by this work, our contribution aims at answering the following question: which requirements, in terms of the shape of the orbit, are necessary to assure the existence of a collision-free solution? How many and how strong conditions are required when prescribing the behaviour of a non collision periodic solution? By prescribing the shape of the solution we mean to select the homotopy class $[\gamma]$ in $\pi_1(\mathbb{R}^2 \setminus \mathcal{C})$ in which the solution belongs.

The answer of our question relies on Definition 2.15, where the concept of *Admissible class* is introduced, and on the notion of collision-reflection solution.

Definition 1.1. A continuous function $q : S \subset \mathbb{R} \to \mathbb{R}^2$ is a *collision-reflection* solution of the equation (1) if

- q has a finite number of collisions, i.e. there exists a finite set of instants $T_c = (\tau_1, \dots, \tau_K) \subset S$ such that $q(\tau_i) \in \mathcal{C}$ for any $i = 1, \dots, K$;
- the restriction $q|_{S \setminus T_c}$ is a classical solution of
$$\ddot{q}(t) = \nabla V(q(t));$$
- the energy is preserved through collisions;
- at any collision instant the trajectory is reflected:
$$q(t + \tau_i) = q(\tau_i - t), \qquad \forall\, i = 1, \dots, K.$$

We prove the following:

Theorem 1.2 (Kepler case, $\alpha = 1$)**.** *Let $\alpha = 1$ and let $\mathcal{C} = \{c_1, \dots, c_N\}$, $c_i \in \mathbb{R}^2$, be any set of centres. Then, for any admissible class $[\gamma]$ and any $h > 0$, at least one of the following holds:*

1. *it exists a non-collision periodic solution $x(s)$ for the N-center problem with energy h and $x(s)$ parametrizes a curve in $[\gamma]$,*
2. *it exists a collision-reflection solution.*

Theorem 1.3 ($\alpha \in (1, 2)$)**.** *Let $\alpha \in (1, 2)$ and let $\mathcal{C} = \{c_1, \dots, c_N\}$, $c_i \in \mathbb{R}^2$, be any set of centres. Then, for any admissible class $[\gamma]$ and any $h > 0$ it exists a non-collision periodic solution $x(s)$ for the N-center problem with energy h and $x(s)$ parametrizes a curve in $[\gamma]$.*

Before proceeding further we acknowledge that the result of Theorem 1.2 is not new and a different proof can be found in [25].

As it will be clear during the exposition, the conditions for an homotopy class $[\gamma]$ to be admissible are weak enough to assure, for any $h > 0$, the existence of infinitely many homotopically distinct periodic solutions with energy h whenever the number N of the centres is larger than 2. In case $N = 3$ it is admissible any homotopy class $[\gamma]$ where γ is for instance any of the path depicted in Figure 1.

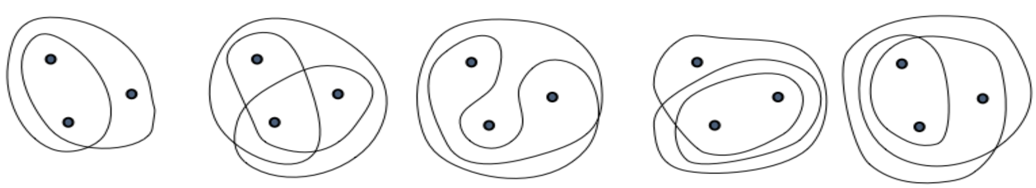


Figure 1: Examples of curves $\gamma \in \mathbb{R}^2 \setminus \mathcal{C}$ so that the homotopy class $[\gamma]$ is admissible.

Even more, we introduce a set of symbols to encode the behaviour of a loop γ, so that to any finite but arbitrarily long sequence of symbols satisfying a certain rule it corresponds an admissible homotopy class.

Following [34], the proofs of theorems 1.2, 1.3 are based on a constrained minimisation argument for the Maupertuis' functional (*the obstacle problem*). Similar techniques have been exploited in the literature to solve minimisation problem in presence of singular potentials, for instance in [11, 14, 15, 38]. The core of the method consists in the analysis, close to the singularities, of a particular minimising sequence $\{u_n\}_n$. In order to prove existence of collision-free orbits, it is required that any u_n does not admit any loop enclosing one of the centres.

The occurrence of such a loop can be avoided by imposing symmetry constraints [38], or by performing the minimisation on spaces of intersection-free paths [14], or on spaces of paths whose vectors of winding numbers with respect to the centres have the same parity [34]. In any of the above mentioned cases, the periodic solution results to be intersection free. Instead of these assumptions, in this work the minimisation is performed on selected homotopy classes and we allow the minimisers to have self intersections and hence to design much more complicated dynamics around the centres.

The presence of self intersections on the minimising sequence represents a technical hurdle when applying the obstacle problem argument. Moreover, no information on the number and location of the self intersections of a loop γ, that is a local property, can be inferred by the knowledge of the homotopy class of γ, that is rather a global property.

In spite of that, it is possible to control the change in number of the self-intersections of a curve, and intersections between different curves, along the homotopies. This result, although weak, reveals to be enough for our purposes.

The paper is organised as follows. In Section 2 we deal with the topological properties of the space of loops and we introduce the definition of Admissible class. Section 3 concerns the variational setting: we introduce the Maupertuis' functional $\mathscr{M}$ and the space of loops $H_{[\gamma]}$ where to look for minima. Also, useful properties of the Maupertuis' functional are discussed. In Section 4, by means of classical arguments we prove the existence of a minimiser for $\mathscr{M}$ on $H_{[\gamma]}$ and, combining topological and variational arguments, we study geometrical features of the minimal path. Section 5 concerns the obstacle problem. We show the major steps of the procedure, while referring to [34] and [38] for the details. In Section 6 we provide the proofs of the theorems (1.2) (1.3). Finally, in Section 7 we discuss a method to generate, in terms of symbols, the admissible classes.

We hasten to remark that the dichotomy presented in Theorem 1.2 is easily solvable: indeed if the support of a collision-reflection solution does not belong to the boundary of the homotopy class $[\gamma]$, then the periodic solution is collision-free.

On the other side, the same variational method does not allow to conclude the existence of non-collsion solutions in the *ultra-weak* case $\alpha \in (0,1)$, neither for the logarithmic potential, $\alpha = 0$.

2 Topological setting

Curves and intersections

Let $\Omega \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$ any interval. A curve in Ω is any continuos map $\gamma(t) : I \to \Omega$.

Definition 2.1. A *loop* is a closed curve, that is a curve $\ell : [a,b] \to \Omega$ such that $\ell(a) = \ell(b)$.

By identifying the endpoints of I, a loop is seen as a curve on $S_{a,b} = [a,b]_{/\{a,b\}}$.

Definition 2.2. Let $\gamma : I \to \Omega$ be a curve, where $I = [a,b]$ or $I = [a,b]_{/\{a,b\}}$. We say that

- γ is simple if γ is injective.

- γ has a *self-intersection of order k at the point p_** if there exist $t_1, \dots, t_k \in I$ such that $\gamma(t_j) = p_*$. If $k = 2$, for short, we simply refer to p_* with the name of *self-intersection*.

Definition 2.3. Let p_* be a self-intersection point for $\gamma \in \mathscr{C}^1(I, \Omega)$. We say that the intersection is *transversal* if γ is $\mathscr{C}^1$-transversal (in the sense of differential topology) at the point p_* as immersed manifold. We say that γ has a *self-tangency* at the point p_* if the tangent space of γ at p_* is 1-dimensional.

Remark 2.4. *If the path $\gamma : [a, b] \to \Omega$ is $\mathscr{C}^1$ with everywhere non-vanishing derivative then the point $p_* = \gamma(t_1) = \gamma(t_2)$ is a transversal self-intersection if*

$$\gamma'(t_1) \times \gamma'(t_2) \neq 0.$$

Otherwise p_* is a self-tangency point.

Let $\gamma : I \to \Omega$ be a curve and $I_1 \subseteq I$ any subinterval. In the following we denote by

- $\gamma|_{I_1}$ the restricted function, i.e. the curve $\gamma_1 : I_1 \to \Omega$, $\gamma_1(t) = \gamma(t)$, for all $t \in I_1$
- $\gamma(I_1)$ the image of $\gamma(t)$ for $t \in I_1$.

For later purposes, we introduce the operation of composition of curves: given two curves $\gamma : [a, b] \to \Omega$ and $\gamma_0 : [c, d] \to \Omega$ such that $\gamma(b) = \gamma_0(c)$, we denote by $\gamma\#\gamma_0$ the curve obtained by gluing γ with γ_0, that is

$$\gamma\#\gamma_0 : [0, (b-a) + (d-c)] \to \Omega$$

$$(\gamma\#\gamma_0)(t) = \begin{cases} \gamma(a+t) & t \in [0, b-a] \\ \\ \gamma_0(c + t - (b-a)) & t \in [b-a, (b-a) + (d-c)] \end{cases} . \tag{4}$$

Moreover we denote by $inv(\gamma)$ the curve

$$inv(\gamma)(t) = \gamma(b-t), \quad t \in [0, b-a].$$

When possible, in the following we adopt the notation γ^{-1} instead of $inv(\gamma)$.

Definition 2.5. Let $\gamma : S \to \Omega$ be a loop. A subloop of γ is any loop $\gamma' : S' \to \Omega$ such that $\gamma'(S') \subset \gamma(S)$. A subloop γ' of γ is said to be innermost if γ' does not admit any subloops. If γ is simple then we refer to γ itself as the innermost loop.

Lemma 2.6. *A loop γ is innermost if and only if it is simple.*

Proof. If γ is simple then, by definition, it is innermost. On the contrary, if $\gamma(b) = \gamma(c)$, $b, c \in S$ then there is at least a sub-loop $\gamma' = \gamma|_{[b,c]}$. □

Loops in the punctured plane

Denote by $\mathbb{R}^2_{\mathcal{C}} = \mathbb{R}^2 \setminus \mathcal{C}$ the punctured plane, $S := [0, 1]/\{0, 1\}$ the unit circle parameterized by $t \in [0, 1]$ and let $\gamma_1, \gamma_2 \in \mathscr{C}^0(S, \mathbb{R}^2_{\mathcal{C}})$ be two loops in $\mathbb{R}^2_{\mathcal{C}}$. The two loops are said freely homotopic, and we write $\gamma_1 \sim \gamma_2$, if there exists a continuous function $F : S \times [0, 1] \to \mathbb{R}^2_{\mathcal{C}}$ such that $F(t, 0) = \gamma_1(t)$, $F(t, 1) = \gamma_2(t)$ for all $t \in S$ and $F(S, s)$ is a loop for any $s \in [0, 1]$. The relation $\sim$ is an equivalence relation. Denote by Λ the set of equivalence classes and indicate with $[\gamma]$ an element of Λ.

$$\Lambda \overset{\text{def}}{=} \mathscr{C}^0(S, \mathbb{R}^2_{\mathcal{C}})/_{\sim}, \quad [\gamma] = \{\gamma_1 \in \mathscr{C}^0(S, \mathbb{R}^2_{\mathcal{C}}) : \gamma_1 \sim \gamma\}.$$

The space $\mathbb{R}^2_{\mathcal{C}}$ is path connected then, given a loop γ and a point x_0 in $\mathbb{R}^2_{\mathcal{C}}$, it always exists a loop $\tilde{\gamma}$ equivalent to γ and based in x_0. Because of this, we can define the composition of two classes $[\gamma], [\tau] \in \Lambda$: if γ and τ represent two equivalence classes, without loss of generality we can assume that γ and τ are based at the same point, so we may define the operation $[\gamma] * [\tau]$ as naturally done in $\pi_1(\mathbb{R}^2_{\mathcal{C}}, x_0)$. Indeed the elements of Λ correspond to conjugacy classes in the fundamental group $\pi_1(\mathbb{R}^2_{\mathcal{C}}, x_0)$, modulo changing of the base point x_0 and $(\Lambda, *)$ is a group. Since the space $\mathbb{R}^2_{\mathcal{C}}$ is not simply connected, the set Λ is not trivial. More precisely, since $\pi_1(\mathbb{R}^2_{\mathcal{C}}, x_0) \cong \pi_0(\mathscr{C}^0(S, \mathbb{R}^2_{\mathcal{C}}), x_0)$, for any $N \geq 1$ (N is the cardinality of the set of centers $\mathcal{C}$), the set Λ has cardinality $\mathbb{Z}^N$and it is isomorphic to the free group of N generators.

The unity of this group is given by the class of *null-homotopic* loops, that is those loops homotopic to a point.

For any loop $\gamma \in \mathscr{C}^0(S, \mathbb{R}^2_{\mathcal{C}})$, with the identification $\mathbb{R}^2 \cong \mathbb{C}$, it is well defined the index

$$Ind(\gamma) = \left(\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - c_1}, \dots, \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - c_N} \right) \in \mathbb{Z}^N.$$

The j-th component of $Ind(\gamma)$ is the index with respect to the centre c_j and it agrees with the winding number of γ with respect to c_j.

Remark 2.7. *If a path γ is null-homotopic in $\mathbb{R}^2_{\mathcal{C}}$ then the winding number of γ with respect to any center is zero, i.e. $Ind(\gamma) = 0$.*

Remark 2.8. *If $\gamma \sim \nu$ then $Ind(\gamma) = Ind(\nu)$. Indeed, let $\Phi(t, s)$ be an homotopy such that $\Phi(t, 0) = \gamma(t)$ and $\Phi(t, 1) = \nu(t)$. Denote*

$$\Psi(s) = Ind(\Phi(\cdot, s)) = \left(\frac{1}{2\pi i} \int_S \frac{dt}{\Phi(t, s) - c_1}, \dots, \frac{1}{2\pi i} \int_S \frac{dt}{\Phi(t, s) - c_N} \right).$$

The homotopy $\Phi(t, s)$ is continuos in both the variable and $\Phi(t, s) \neq c_j$ for all $t \in S$, $s \in [0, 1]$ and $j = 1, \dots, N$, thus the function $\Psi(s)$ is continuos. Since the index function takes values in $\mathbb{Z}$, it follows that $\Psi(s)$ is constant.

Remark 2.9. *The contrary is not true. The condition $Ind(\gamma) = Ind(\nu)$ does not imply $\gamma \sim \nu$. For instance, there exist closed curves that have zero winding number with respect each of the center without being null homotopic in $\mathbb{R}^2_{\mathcal{C}}$, see figure 2.*

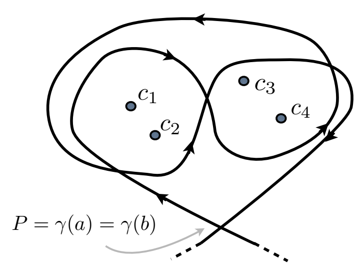


Figure 2: The loop $\gamma|_{[a,b]}$ has index zero with respect any of the centres. On the other side it is not null homotopic.

Definition 2.10. Let γ be a loop in $\mathbb{R}^2_{\mathcal{C}}$. We say that γ has a

i) *singular 1-gon* if there is a subinterval I of S such that γ identifies the end-points of I and $\gamma|_I$ defines a null-homotopic loop in $\mathbb{R}^2_{\mathcal{C}}$ (i.e. γ has a null-homotopic sub-loop);

ii) *singular 2-gon* if there are disjoint intervals I_1 and I_2 of S such that γ identifies the end-points of I_1 and I_2 and $\gamma|_{I_1}\#\gamma|_{I_2}$ defines a null-homotopic loop on $\mathbb{R}^2_{\mathcal{C}}$.

With some abuse of terminology, we will say that the loop $\gamma|_I$ is the singular 1-gon (in the first case) and $\gamma|_{I_1}\#\gamma|_{I_2}$ is the singular 2-gon.

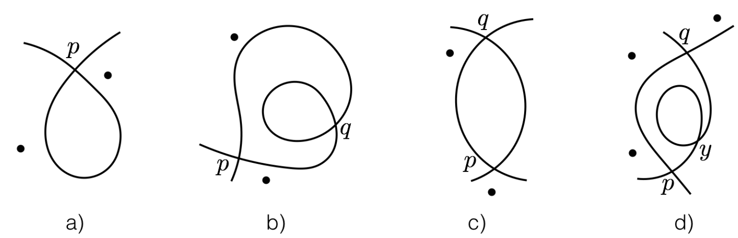


Figure 3: a) The curve has a singular 1-gon pointed in p. b) The curve has two singular 1-gons, one pointed in p, the other pointed in q. The second is an innermost loop. c) The curve admits a singular 2-gon with end-points p, q. d) The curve has a singular 2-gon and a singular 1-gon pointed in y. The singular 1-gon is also an innermost loop.

Definition 2.11.

- Define $Int([\gamma])$ the minimal number of self-intersections of the elements of $[\gamma]$.
- A loop γ is said *taut* if it is an immersion and the number of self-intersections of γ is equal to $Int([\gamma])$, that is, γ has the minimal number of self-intersections in its homotopy class. If the immersion γ is not taut then we say that γ exceeds the number of self-intersections in $[\gamma]$.

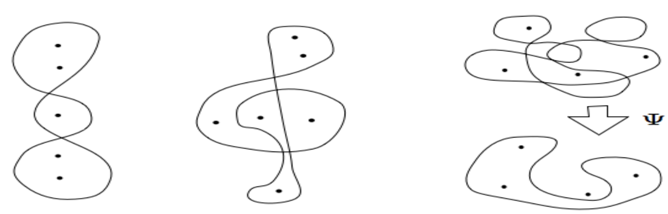


Figure 4: From the left, the first and the second loops are taut. The last figure on the top shows a loop that exceeds the number of self intersections. Below a taut representative in the same homotopy class.

A criteria to determine whether a given loop is taut is provided by the following theorem, [23].

Theorem 2.12. *If γ has excess self-intersections, then γ has a singular 1-gon or 2-gon.*

Figure 4 depicts examples of taut and non taut loops. Moreover, the following results hold, [24].

Theorem 2.13. *Let γ and γ_1 be homotopic immersed curves each minimizing the number of self intersections in their common homotopy class. Then there exists a homotopy Ψ from γ to γ_1 such that $\Psi_s = \Psi(\cdot, s)$ is self-transverse for all s and the number of self-intersections of Ψ_s is constant.*

Theorem 2.14. *Let γ be an immersed curve which does not minimize the number of self intersections in its homotopy class. Then there exists an homotopy Ψ from γ to a immersed curve γ_1 which is taut such that the number of self-intersections of the curve Ψ_s is not increasing with s. Ψ is a regular homotopy except for a finite number of times when a small loop in the curve shrinks to a point.*

We are now in the position to introduce the admissible classes.

Definition 2.15. Let $[\gamma] \in \Lambda$ be an homotopy class and let $\hat{\gamma}$ be a taut representative of $[\gamma]$. We say that the homotopy class $[\gamma]$ is Admissible if the following property holds:

[A] for any innermost loop ℓ_j of $\hat{\gamma}$ there exist at least two centers c_{j_1}, c_{j_2} so that ℓ_j is not null-homotopic both in $\mathbb{R}^2 \setminus c_{j_1}$ and in $\mathbb{R}^2 \setminus c_{j_2}$.

For Lemma 2.6, any innermost loop ℓ of $\hat{\gamma}$ is simple hence [A] simply means that at least two centers are in the region bounded by ℓ. The existence of a taut representative is implicit in the definition of minimal intersection number of a homotopy class. However, for any primitive element $[\gamma]$ in Λ we can construct a taut representative as follows. We can think of small open balls B_i in place of the centers c_i and the configuration space $\mathbb{R}^2_B = \mathbb{R}^2 \setminus (\bigcup B_i)$. Any free homotopy class of $\mathbb{R}^2_B$ contains exactly one geodesics and a primitive geodesics can not have excess self intersections, see c.f. [19].

Moreover, the definition is well posed and does not depend on the particular taut representative. If γ and γ_1 are two taut curves in $[\gamma]$, for Theorem 2.13 the sup-loops of the two curves are in one-to-one correspondence with the property of enclosing the same centers.

Remark 2.16. *If $[\gamma]$ is admissible then $[\gamma]$ is not null homotopic.*

Before proceeding, let us recall the definition of *tied* loop introduced by Gordon in [21]:

Definition 2.17. Let B be a closed non empty subset of $\mathbb{R}^2$. A loop γ in $\mathbb{R}^2 \setminus B$ is said to be tied to B if γ cannot be continuously moved off to infinity without either crossing B or having its arc length become infinite.

Lemma 2.18. *Let γ be a not null-homotopic loop in $\mathbb{R}^2_{\mathcal{C}}$. Then γ is tied to at last one of the center $c_j \in \mathcal{C}$.*

3 Variational setting

The natural setting, when addressing the existence of periodic solutions of the system (1) by means of the variational methods, is that of the Sobolev space

$$H^1(S, \mathbb{R}^2)$$

endowed with the scalar product and norm given, respectively, by

$$\langle u, v \rangle_1 = \int_0^1 \langle \dot{u}(t), \dot{v}(t) \rangle \, dt + \int_0^1 \langle u(t), v(t) \rangle \, dt$$

$$\|u\|_1^2 = \|\dot{u}\|_2^2 + \|u\|_2^2.$$

Introduce the space

$$H = \{u \in H^1(S, \mathbb{R}^2)\}.$$

and the open subspace $\widehat{H} \subset H$ of collision free periodic functions

$$\widehat{H} := \{u \in H : \; u(t) \in \mathbb{R}^2_{\mathcal{C}} \;\; \forall\, t \in [0,1]\}.$$

Since H is embedded into $\mathscr{C}^0(S, \mathbb{R}^2)$, the condition $u(t) \in \mathbb{R}^2 \setminus \mathcal{C}$ makes sense. Following the notations introduced by authors in [34], we define

$$\mathfrak{Coll} := \{u \in H : \exists\; t \in S : u(t) = \boldsymbol{c}_j, \text{ for some } j \in \{1, \ldots, N\}\}, \tag{5}$$

the set of all colliding paths in $H^1(S, \mathbb{R}^2)$; clearly

$$H = \mathfrak{Coll} \cup \widehat{H},$$

hence H is nothing but the weak H^1-closure of the space $\widehat{H}$.

From now on, suppose $h > 0$ is fixed. We introduce the Maupertuis' functional

$$\mathscr{M} : H \to \mathbb{R} \cup \{\infty\} \quad \text{given by} \quad \mathscr{M}(u) := \int_S \frac{1}{2}|\dot{u}(t)|^2\, dt \int_S (V(u) + h)\, dt\ .$$

Beside the space of periodic paths H and the functional $\mathscr{M}$, for any given couple of points p_1, p_2, let us introduce the spaces

$$\widehat{H}_{p_1,p_2}(a,b) \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} u \in H^1([a,b], \mathbb{R}^2) : & u(a) = p_1, u(b) = p_2, \\ & u(t) \in \mathbb{R}^2_{\mathcal{C}} \; \forall t \in (a,b) \end{array} \right\} \tag{6}$$

and $H_{p_1,p_2}(a,b)$ its weak H^1 closure. Let be defined the functional

$$\mathscr{M}_{p_1,p_2} : H_{p_1,p_2}(a,b) \to \mathbb{R} \cup \{\infty\} \;\; \text{given by} \;\; \mathscr{M}_{p_1,p_2}(u) := \int_a^b \frac{1}{2}|\dot{u}(t)|^2\, dt \int_a^b (V(u) + h)\, dt\ .$$

If $\mathscr{M}(u) > 0$ (or $\mathscr{M}_{p_1,p_2}(u) > 0$), denote

$$\omega^2 = \frac{\int_S (V(u) + h)\, dt}{\int_S \frac{1}{2}|\dot{u}(t)|^2\, dt}. \tag{7}$$

(with $[a,b]$ in place of S in the second case).

Let us now state some features of the Maupertuis' functional.

Lemma 3.1. *$\mathscr{M}(u)$ and $\mathscr{M}_{p_1,p_2}(u)$ are invariant under time rescaling of u.*

Proof. For any $\lambda \neq 0$ let $v(s) = u(\lambda s)$, $s \in [0, 1/\lambda]$. Then

$$\begin{aligned} \mathscr{M}(v) &= \int_0^{\frac{1}{\lambda}} \frac{1}{2}|v'(s)|^2\, ds \int_0^{\frac{1}{\lambda}} (V(v) + h)\, ds = \int_0^{\frac{1}{\lambda}} \frac{1}{2}\left|\frac{d}{ds}(u(\lambda s))\right|^2\, ds \int_0^{\frac{1}{\lambda}} (V(u(\lambda s)) + h)\, ds \\ &= \int_0^1 \frac{1}{2}\left|\dot{u}(t)\right|^2 \lambda\, dt \int_0^1 (V(u(t) + h)\frac{1}{\lambda}\, dt = \mathscr{M}(u). \end{aligned}$$

$\square$

Because of the invariance of Maupertuis' functional under time rescaling, the choice of the length of the time interval used in the definition of H and of $\mathscr{M}$ is irrelevant. In the above definition, the functional $\mathscr{M}$ is defined on paths u with $Supp(u) = S$. However, although v is a path with $Supp(v) \neq S$, with some abuse of notation we write $\mathscr{M}(v) = \int_{Supp(v)} \frac{1}{2}|\dot{v}|^2\,dt \int_{Supp(v)}(V(v) + h)\,dt$.

Unlike others functionals, as for instance the Lagrange action functional or the Jacoby length functional often used in addressing similar problems, the Maupertuis' functional is not additive. Indeed if $u(t) = u_1 \# u_2$ we may have $\mathscr{M}(u) \neq \mathscr{M}(u_1) + \mathscr{M}(u_2)$. It is known that if $u(t)$, $t \in [a,b]$, is a minimiser for an additive functional then u minimises locally. It means that any restriction $u_1 = u|_{[c,d]}$ ($[c,d] \subset [a,b]$) is the minimiser of the restricted functional among all paths that satisfy the end-points constraints. Nevertheless, although the Maupertuis' functional is not additive, the minimisers still admit the same property.

Lemma 3.2. *Let $u \in H$ be a non constant minimiser for $\mathscr{M}$. Let $[c,d] \subset S$ and set $p_1 = u(c)$, $p_2 = u(d)$. Then $u|_{[c,d]}$ is a minimiser for $\mathscr{M}_{p_1,p_2}$ in $H_{p_1,p_2}(c,d)$. Moreover suppose v is any minimiser for $\mathscr{M}_{p_1,p_2}$ on $H_{p_1,p_2}(c,d)$. Then the path $w = v\#u|_{S\setminus[c,d]}$ is a minimiser of $\mathscr{M}$ on H.*

Proof. Denote by $u_1 = u|_{[c,d]}$, $u_2 = u|_{S\setminus[c,d]}$ so that $u = u_1\#u_2$ and write

$$\begin{aligned}\mathscr{M}(u) &= \left(\int_{[c,d]} \frac{1}{2}|\dot{u}_1|^2\,dt + \int_{S\setminus[c,d]} \frac{1}{2}|\dot{u}_2|^2\,dt\right)\left(\int_{[c,d]} V(u_1) + h\,dt + \int_{S\setminus[c,d]} V(u_2) + h\,dt\right)\\ &= (K_{u_1} + K_{u_2})(V_{u_1} + V_{u_2}).\end{aligned} \tag{8}$$

Assume by contradiction that u_1 is not a minimiser for $\mathscr{M}_{p_1,p_2}$ and let $v_1 \in H_{p_1,p_2}(c,d)$ be such that $\mathscr{M}_{p_1,p_2}(v_1) < \mathscr{M}_{p_1,p_2}(u_1)$. As before, let us write explicitly the kinetic and potential parts

$$\mathscr{M}_{p_1,p_2}(v_1) = \int_c^d \frac{1}{2}|\dot{v}_1|^2\,dt \int_c^d V(v_1) + h\,dt = K_{v_1}V_{v_1}.$$

Hence $K_{v_1}V_{v_1} < K_{u_1}V_{u_1}$. Consider now $\eta(t) = v_1(\lambda t)$, for a proper λ, so that $K_\eta = K_{u_1}$. Since $\mathscr{M}_{p_1,p_2}$ is invariant under rescaling, $K_\eta V_\eta = \mathscr{M}_{p_1,p_2}(\eta) = \mathscr{M}_{p_1,p_2}(v_1) < K_{u_1}V_{u_1}$ giving $V_\eta < V_{u_1}$. Up to time rescaling, define $\tilde{w} = \eta\#u_2$ so that $w \in H$. Then $\mathscr{M}(w) < \mathscr{M}(u)$, giving the contradiction.

To prove the second part of the statement, let us assume that $v_1 \in H_{p_1,p_2}(c,d)$ is a minimiser for $\mathscr{M}_{p_1,p_2}$ different from u_1. Hence $K_{v_1}V_{v_1} = \mathscr{M}_{p_1,p_2}(v_1) = \mathscr{M}_{p_1 p_2}(u_1) = K_{u_1}V_{u_1}$. As before denote by η a proper time rescaling of v_1 so that $K_\eta = K_{u_1}$. It follows that $V_\eta = V_{u_1}$ and, denoting $\tilde{w} = \eta\#u_2$, we have $\mathscr{M}(\tilde{w}) = \mathscr{M}(u)$. Again for the invariance of $\mathscr{M}$ under time rescaling, the path $w \in H$, obtained from $\tilde{w}$ by rescaling, is such that $\mathscr{M}(w) = \mathscr{M}(u)$.

□

The Maupertuis' functional is differentiable over $\widehat{H}$ and, up to reparametrization, its critical points are solutions to of the N-center problem.

Theorem 3.3. *Let $u \in \widehat{H}$ be a critical point of $\mathscr{M}$ at positive level and ω as in (7). Then $u \in \mathscr{C}^2(S)$ and solves the equation*

$$\omega^2\ddot{u}(t) = \nabla V(u(t)), \quad \forall t \in S \tag{9}$$

and the path $x(t) \stackrel{\text{def}}{=} u(\omega t)$ is a periodic classical solution of (1)-(3).

Similarly, let $v \in \widehat{H}_{p_1,p_2}(a,b)$ be a critical point of $\mathscr{M}_{p_1,p_2}$ at positive level and ω as in (7) *(with $[a,b]$ in place of S). Then $v \in \mathscr{C}^2((a,b))$ and solves the equation* (9) *for any $t \in (a,b)$ and the function $x(t) \stackrel{\text{def}}{=} v(\omega t)$ is a classical solution of* (1)-(3).

Proof. Let u be a critical point for $\mathscr{M}$, that is $\delta\mathscr{M}(u) = 0$. Then $\frac{1}{\varepsilon}(\mathscr{M}(u+\varepsilon v) - \mathscr{M}(u)) \to 0$ as $\varepsilon \to 0$ for any $v \in \mathscr{C}_0^\infty(S)$. Explicitly

$$\begin{aligned}\mathscr{M}(u+\varepsilon v) - \mathscr{M}(u) &= \int_S \frac{1}{2}|\dot{u} + \varepsilon\dot{v}|^2\,dt \int_S V(u+\varepsilon v) + h\,dt - \int_S \frac{1}{2}|\dot{u}|^2\,dt \int_S V(u) + h\,dt \\ &= \varepsilon\left(\int_S \frac{|\dot{u}|^2}{2}\,dt \int_S \nabla V(u)\cdot v\,dt + \int_S \dot{u}\cdot\dot{v}\,dt \int_S V(u) + h\,dt\right) + o(\varepsilon^2) \\ &= \varepsilon\left(\omega^2 \int_S \dot{u}\cdot\dot{v}\,dt + \int_S \nabla V(u)\cdot v\,dt\right) + o(\varepsilon^2)\ .\end{aligned}$$

Integrating by parts and letting ε to go to zero, it follows

$$\int_S (\omega^2\ddot{u} - \nabla V(u))\cdot v\,dt = 0, \quad \forall v \in \mathscr{C}_0^\infty(S)$$

hence equation (9).

From $x'(s) = \omega\dot{u}(\omega s)$ it descends (1). The energy $E(s) = \frac{1}{2}|x'(s)|^2 - V(x(s))$ is constant and it is equal to $E = \frac{1}{2}\omega^2|\dot{u}|^2 - V(u)$. Therefore

$$\int_S E\,dt = \omega^2 \int_S \frac{1}{2}|\dot{u}|^2\,dt - \int_S V(u)\,dt = \int_S h\,dt$$

and $E = h$.

Same arguments apply for the fixed-end problem.

□

By regularity theory it turns out that a critical point $u \in \widehat{H}$ of $\mathscr{M}$ is indeed a classical solution (at least $\mathscr{C}^2$) of the differential equation in (1) .

According to the previous lemma, we seek for periodic solutions of the N-center problem as minimisers of the action functional $\mathscr{M}$ in $\widehat{H}$. The existence of minimisers will be proved following the direct method of calculus of variations, therefore rather than $\widehat{H}$ which is not weakly closed, a suitable closed subspaces $\mathcal{H} \subset H$ must be selected. Aiming at classical solutions, some topological constraints are required on the elements of $\mathcal{H}$ so to role out the occurrence of collisions on the minimising paths.

Recalling the definition of $[\gamma]$, let be introduced

$$\widehat{H}_{[\gamma]} = \widehat{H} \cap [\gamma] = \{u \in \widehat{H} : u \sim \gamma\}$$

the space of paths in $\widehat{H}$ that belong to the homotopy class $[\gamma]$ and $H_{[\gamma]}$ the closure of $\widehat{H}_{[\gamma]}$ in the weak H^1 topology.

Remark 3.4. *We might be tempted to explicitly characterise the set $H_{[\gamma]}$, that is, to describe the elements of $\mathfrak{Coll}_{[\gamma]} \subset \mathfrak{Coll}$ so that*

$$H_{[\gamma]} = \widehat{H}_{[\gamma]} \cup \mathfrak{Coll}_{[\gamma]}\ .$$

As suggested in [34], for $j \in \{1, \ldots, N\}$ define

$$\mathfrak{Coll}^j_{[\gamma]} = \{u \in H \,:\, u \sim \gamma \text{ in } \mathbb{R}^2_{\mathcal{C}} \cup c_j, \text{ and it exists } t \,:\, u(t) = c_j\}$$

the set of paths that collide in the centre c_j and that are homotopic to γ in $\mathbb{R}^2$ minus the other centres. Similarly, for any $J = (j_1, \ldots, j_m) \subseteq \{1, \ldots, N\}$ let be defined

$$\mathfrak{Coll}^J_{[\gamma]} = \{u \in H \, : \, u \sim \gamma \text{ in } \mathbb{R}^2_{\mathcal{C}} \bigcup_{i=1\ldots m} c_{j_i}, \text{ and there exist } t_1, \ldots, t_m \, : \, u(t_m) = c_{j_m}\}.$$

and

$$\mathfrak{Coll}_{[\gamma]} = \bigcup_{J \in \mathcal{P}(\{1,2,\ldots,N\})} \mathfrak{Coll}^J_{[\gamma]}.$$

Although reasonable, this construction leads to a set $H_{[\gamma]}$ that is not the weak H^1 closure of $\widehat{H}_{[\gamma]}$, as the counterexample depicted in figure (5) shows.

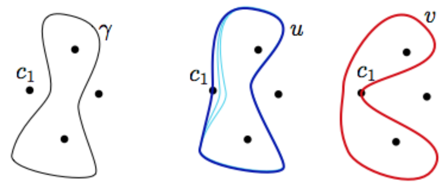


Figure 5: Both the paths $u, v \in \mathfrak{Coll}^1_{[\gamma]}$. While u belongs to the boundary of $\widehat{H}_{[\gamma]}$, v does not.

4 The solution curve

4.1 Existence of minimizers

Theorem 4.1. *Let $[\gamma] \in \Lambda$ be a not null homotopic class. Then, for any $h > 0$, $\mathscr{M}$ achieves the minimum in $H_{[\gamma]}$.*

Proof. Since $\int_S (V(u) + h)\, dt > C > 0$, if $||\dot{u}||_2 \to \infty$ also $\mathscr{M}(u) \to \infty$. Also, for Lemma 2.18 and according to Gordon [21], it exists at least one center $c \in \mathcal{C}$ so that the elements of $[\gamma]$ are tied to c. For the Cauchy-Schwarz inequality,

$$\text{arclength}(u) = \int_0^1 |\dot{u}| dt \leq \left(\int_0^1 |\dot{u}(t)|^2 dt \right)^{1/2} \leq C_1 \sqrt{\mathscr{M}(u)}$$

for a positive constant C_1, therefore $||u||_2 \to \infty$ implies $\mathscr{M}(u) \to \infty$.

Let $(u_n)_n \subset H_{[\gamma]}$ be a sequence $u_n \rightharpoonup u$ weakly in H^1. Then $\dot{u}_n \rightharpoonup \dot{u}$ weakly in L^2 and, being u_n a bounded sequence in H^1, for the Sobolev embedding, up to a subsequence, $u_n \to u$ uniformly. The lower semicontinuity of the L^2 norm and the Fatou's lemma imply

$$\mathscr{M}(u) \leq \liminf_{n \to \infty} \mathscr{M}(u_n).$$

Hence $\mathscr{M}$ is coercive and lower semicontinuous. By application of the direct method of calculus of variation $\mathscr{M}$ attains its minimum in $H_{[\gamma]}$. $\square$

Denote by $\bar{u}$ a minimiser of $\mathscr{M}$ on $H_{[\gamma]}$:

$$\bar{u} \; : \; \mathscr{M}(\bar{u}) = \min_{u \in H_{[\gamma]}} \mathscr{M}(u).$$

4.2 Properties of the minimisers

In this section some analytical and geometrical properties of the minimiser are described. Concerning the occurrence of collisions, introduce the set of collision times:

$$T_c(\bar{u}) \stackrel{\text{def}}{=} \{t \in S : \bar{u}(t) = c_j, \text{ for some } j \in \{1, 2, \ldots, N\}\}$$

Since V is infinite on collisions and being the value $\mathscr{M}(\bar{u})$ finite, it follows that $T_c(\bar{u}) = 0$ is closed and of null measure. Hence the complement $S \setminus T_c(\bar{u})$ is the union of a finite or countable number of open intervals.

Given a path $u(t)$, define the inertia of u with respect to the j-th center as

$$I_j(u)(t) \stackrel{\text{def}}{=} |u(t) - c_j|^2.$$

Lemma 4.2. *Suppose $u(t)$ is a solution of (9) possibly with collisions. Then for any $j \in \{1, 2, \ldots, N\}$ there exists $R_j > 0$ such that $\ddot{I}_j(u)(t) \geq 0$ whenever $|u(t) - c_j| \leq R_j$, $u(t) \neq c_j$.*

Proof. For Theorem 3.3,

$$\frac{\omega^2}{2}\ddot{I}_j = 2h + \left(\frac{2}{\alpha} - 1\right)\frac{m_j}{|u(t) - c_j|^\alpha} + \sum_{k \neq j} \frac{m_k}{|u(t) - c_k|^\alpha}\left(\frac{2}{\alpha} - \frac{(u(t) - c_j)\cdot(u(t) - c_k)}{|u - c_k|^2}\right).$$

In the neighbourhood of c_j the first and the last term of the right hand side are bounded, while the second term tends to $+\infty$ as $|u(t) - c_j| \to 0$. Hence it exists R_j so that $\ddot{I}_j(u)(t) > 0$ for any t so that $|u(t) - c_j| \leq R_j$. □

The convexity of the inertia functionals in the neighborhood of the centers allows to prove the following, see for instance [38].

Lemma 4.3. *The cardinality of $T_c(\bar{u})$ is finite.*

Let $K = |T_c(\bar{u})|$ be the number of collisions of $\bar{u}$ and denote by $0 < \tau_1 < \tau_2 < \cdots < \tau_K < 1$ the collision times. Denote by $\mathcal{J}(\bar{u}) = (j_1, \ldots, j_K) \in \{1, \ldots, N\}^K$ the ordered sequence of the labels of the centers hit by $\bar{u}$, that is $\bar{u}(\tau_i) = c_{j_i}$. Out of the collision instants, the minimiser $\bar{u}$ provides a classical solution for the N-center problem. More precisely, we have the following

Proposition 4.4. *For any $(a, b) \subset S \setminus T_c(\bar{u})$, the restriction $\bar{u}|_{(a,b)}$ is a $\mathscr{C}^2$ path and solves the equation*

$$\omega^2\ddot{u}(t) = \nabla V(u).$$

Proof. As in the proof of Theorem 3.3, it is a consequence of the minimality of u with respect to variations with compact support in (a, b). □

In what follows the geometric properties of the minimiser are analysed, with particular emphasis on the possible occurrence of 1-gons and 2-gons. For this, let us first recall the notion of homotopy relative to the boundary.

Definition 4.5. Two elements $\gamma_1, \gamma_2 \in \widehat{H}_{p_1,p_2}$ are said to be homotopic relative to the boundary, $\gamma_1 \sim_{rb} \gamma_2$, if there exists a continuous function $\Phi(t, s) : [a, b] \times [0, 1] \to \mathbb{R}^2$ such that

- $\Phi(t, 0) = \gamma_1(t)$, $\Phi(t, 1) = \gamma_2(t)$ $\forall t \in [a, b]$
- $\Phi(0, s) = p_1$ and $\Phi(1, s) = p_2$ for any $s \in [0, 1]$
- $\Phi(s, t) \in \mathbb{R}^2_{\mathcal{C}}$ for any $s \in [0, 1]$, $t \in (a, b)$.

Denote by

$$\Omega_{p_1,p_2} = \widehat{H}_{p_1,p_2}/_{\sim_{rb}}$$

the space of equivalence classes of $\widehat{H}_{p_1,p_2}$ with respect to the relative homotopy.

As for the space Λ, the space Ω_{p_1,p_2} has infinitely many elements. Using the same notation adopted in case of loops, for an equivalence class $[\gamma] \in \Omega_{p_1,p_2}$ we denote $\widehat{H}_{[\gamma]} = \widehat{H}_{p_1,p_2} \cap [\gamma]$ and $H_{[\gamma]}$ its closure in the weak H^1 topology. Also, since $H_{[\gamma]}$ is weakly closed and $\mathscr{M}_{p_1,p_2}$ is coercive and weakly lower semicontinuous, the direct method of calculus of variation assures the existence of a minimiser for $\mathscr{M}_{p_1,p_2}$ in $H_{[\gamma]}$.

We remark that, up to time reparameterization, trajectories of the N-body problem with energy h are geodesics of the Jacobi-Maupertuis metric $g_h(u, \dot{u}) = 2(V(u)+h)|\dot{u}|^2$ on $\mathbb{R}^2 \setminus \mathcal{C}$. For $h > 0$ the gaussian curvature $\mathcal{K}$ of g_h is negative, hence by the Gauss-Bonnet theorem, it descends that the minimiser $\bar{u}$ is unique, eventually with collisions, in any non trivial homotopy class $[\gamma] \in \Lambda$. Since the minimiser $\bar{u}$ is unique in the closure of its homotopy class $H_{[\gamma]}$, it follows that any arc $\bar{u}|_{[a,b]}$, joining $p_1 = \bar{u}(a)$ to $p_2 = \bar{u}(b)$, p_1, p_2 possibly singular points, is the unique minimiser for $\mathscr{M}_{p_1,p_2}$ in the homotopy class $[\bar{u}|_{[a,b]}] \in \Omega_{p_1,p_2}$.

The minimality condition implies that the solution curve can only have transversal self intersections and the number of intersections is minimised, as stated in the next proposition.

Proposition 4.6. *The minimizer $\bar{u}$ satisfies the following properties:*

1. *Let $t_a \in (\tau_i, \tau_{i+1})$ and $t_b \in (\tau_j, \tau_{j+1})$ be two non-collision instants. If $\bar{u}$ has a self tangency at point $p = u(t_a) = u(t_b)$, then $\bar{u}$ has a tangency at any $t \in (\tau_i, \tau_{i+1})$ and $t \in (\tau_j, \tau_{j+1})$.*

2. *The loop $\bar{u}$ does not admit any singular 1-gon neither singular 2-gon in $R^2_{\mathcal{C}}$.*

Proof.

1. Denote by $\bar{u}_1 = \bar{u}|_{[\tau_i,\tau_{i+1}]}$ and $\bar{u}_2 = \bar{u}|_{[\tau_j,\tau_{j+1}]}$. For Propositon 4.4 there exist $v_1(t)$ and $v_2(t)$, suitable reparametrization of $\bar{u}_1$, $\bar{u}_2$, both solutions of the N-center problem with energy h. For the uniqueness of solution of initial value problem $\ddot{u}(t) = \nabla V(u)$, $u(0) = p$, $\dot{u}(0) = \nu$ and the invariance of the differential equation under time reverse, it follows that v_1 and v_2 are tangent at any point.

2. Suppose that $\bar{u}(t_a) = \bar{u}(t_b)$ and that $\bar{u}([t_a, t_b])$ is a singular 1-gon. Consider the path $v = \bar{u}|_{S\setminus(a,b)}$ obtained from $\bar{u}$ by removing the singular 1-gon. Then $v \in H_{[\gamma]}$ and, being $(V(x) + h) > 0$, $\mathscr{M}(v) < \mathscr{M}(\bar{u})$. That contradicts the minimality of $\bar{u}$.

   Suppose by contradiction that $\bar{u}$ has a singular 2-gon. Denote by γ_1 and γ_2 the two geodesic arcs and by $\alpha, \beta > 0$ the two angles formed at the intersection points. The region A bounded by the singular 2-gon is homeomorphic to a disc, hence for the Gauss-Bonnet theorem

$$0 < \int_A K d = 2\pi - \pi + \alpha - \pi + \beta > 0.$$

□

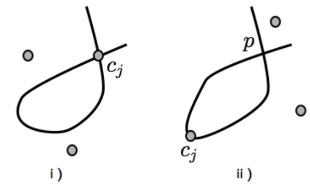
Beside the singular 1-gon and singular 2-gon, we now discuss the possibility for the minimiser $\bar{u}$ to admits 1-gons and 2-gons whose edges intersect some of the centres.

Lemma 4.7. *Let $\bar{u}$ be the minimal path for $\mathscr{M}$ in $H_{[\gamma]}$. Then $\bar{u}$ does not admit any sub loop ℓ that intersects c_j and that is a singular 1-gon in $\mathbb{R}^2_{\mathcal{C}} \cup \{c_j\}$ for one $c_j \in \mathcal{C}$.*

Proof. Arguing by contradiction, assume that $\ell = \bar{u}|_{[a,b]}$ is a singular 1-gon in $\mathbb{R}^2_{\mathcal{C}} \cup c_j$ and one of the following occurs:

i) $\bar{u}(a) = \bar{u}(b) = c_j$;

ii) $\bar{u}(a) = \bar{u}(b) = p$, $p \neq c_j$ and it exists $\tau \in (a,b)$ so that $\bar{u}(\tau) = c_j$.



In the case $i)$ consider $\tilde{u} = \bar{u}|_{S\setminus[a,b]}$. Since ℓ is homotopic to the point c_j, it follows that $\tilde{u} \in H_{[\gamma]}$ and $\mathscr{M}(\tilde{u}) < \mathscr{M}(\bar{u})$, hence the contradiction.

In the case $ii)$ denote by $\gamma_1 = \bar{u}|_{[a,\tau]}$ and $\gamma_2 = \bar{u}|_{[\tau,b]}$. Let $\widehat{H}_{[\gamma_1]} \in \Omega_{p,c_j}$ be the homotopy class containing γ_1 and, up to time reversion, γ_2. Let v_1 be the unique minimiser of $\mathscr{M}_{p,c_j}$ on $H_{[\gamma_1]}$. Thus, for Lemma 3.2, the path $\bar{u}$ is given by $\bar{u} = v_1 \# (v_1)^{-1} \# u|_{S\setminus[a,b]}$.

But $\bar{u}$ is not of class C^1 at p, hence the contradiction.

□

The following corollaries concern a generalisation of the situations described in point $i)$ and $ii)$ of the previous lemma, see also Figure 6.

Corollary 4.8. *Let $\bar{u}$ be the minimal path for $\mathscr{M}$ in $H_{[\gamma]}$. Then $\bar{u}$ can not admit any sub loop $\ell = \bar{u}|_{[a,b]}$ with the properties, see Figure 6 (a):*

- $\bar{u}(a) = \bar{u}(b) = c_{j_0}$
- *there exist $a < \tau_1 < \ldots \tau_m < b$ such that $\bar{u}(\tau_i) = c_{j_i}$, $c_{j_i} \neq c_{j_k}$ for all $0 \leq k \neq i \leq m$*
- *it exists $\tilde{\ell} \in \widehat{H}_{c_{j_0},c_{j_0}}$ such that $\tilde{\ell}$ is null homotopic in $\widehat{H}_{c_{j_0},c_{j_0}}$ and the path $\bar{u}|_{S^1\setminus[a,b]} \# \tilde{\ell}$ belongs to $H_{[\gamma]}$.*

Proof. By assumption both $\bar{u}$ and $\bar{u}|_{S\setminus(a,b)} \# \tilde{\ell}$ belong to $H_{[\gamma]}$. Since $\tilde{\ell}$ is homotopic to c_{j_0}, the path $\tilde{u} = \bar{u}|_{S\setminus(a,b)}$ belongs to $H_{[\gamma]}$ and $\mathscr{M}(\tilde{u}) < \mathscr{M}(\bar{u})$. □

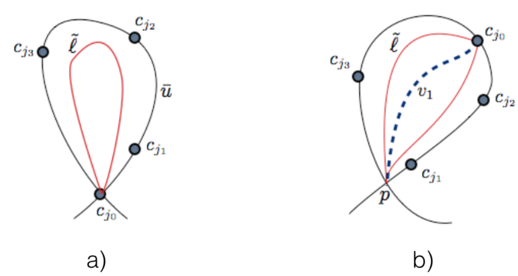


Figure 6: The figure on the left and on the right depicts the situation discussed in Corollary 4.8 and Lemma 4.9 respectively.

Lemma 4.9. *Let $\bar{u}$ be the minimal path for $\mathscr{M}$ in $H_{[\gamma]}$. Then $\bar{u}$ can not admit any sub loop $\ell = \bar{u}|_{[a,b]}$ with the properties, see Figure 6 (b):*

- $\bar{u}(a) = \bar{u}(b) = p$, *$p \notin \mathcal{C}$ and it exists $\tau_0 \in (a,b)$ such that $\bar{u}(\tau_0) = c_{j_0}$;*
- *there exist $a < \tau_1 < \ldots \tau_m < b$ such that $\bar{u}(\tau_i) = c_{j_i}$, $c_{j_i} \neq c_{j_k}$ for all $0 \leq k \neq i \leq m$*
- *there exist $\gamma_1 \in \widehat{H}_{p,c_{j_0}}$ and $\gamma_2 \in \widehat{H}_{c_{j_0},p}$ such that $\tilde{\ell} = \gamma_1 \# \gamma_2$ is a singular 1-gon in $\mathbb{R}^2_{\mathcal{C}} \cup \{c_{j_0}\}$ and the path $\bar{u}|_{S^1\setminus[a,b]} \# \tilde{\ell}$ belongs to $H_{[\gamma]}$.*

Proof. Let $\widehat{H}_{[\gamma_1]} \in \Omega_{p,c_{j_0}}$ be the homotopy class that contains γ_1 and denote by v the minimiser of $\mathscr{M}_{p,c_{j_0}}$ in the closure of $\widehat{H}_{[\gamma_1]}$. Hence arguing as in the proof of point *ii)* of Lemma 4.7, we obtain the contradiction. □

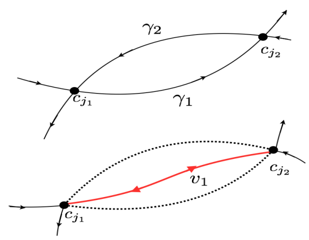
Lemma 4.7 and related corollaries concern the presence of 1-gon on minimal path. In practice, under certain assumptions, no 1-gon can be present on the minimal path.

The remaining of the section focuses on the occurrence of 2-gons. Under similar assumptions as above, we prove that the 2-gons as well can not be present in the minimal paths. Indeed the 2-gon is replaced by a geodesic arc covered twice.

Lemma 4.10. *Let $[a,b],[c,d] \subset S$ and denote $\gamma_1 = \bar{u}|_{[a,b]}, \gamma_2 = \bar{u}|_{[c,d]}$.*

Suppose there exist c_{j_1}, c_{j_2} (possibly equal) such that

- $\bar{u}(a) = \bar{u}(d) = c_{j_1}, \bar{u}(b) = \bar{u}(c) = c_{j_2}$
- $\gamma_1((a,b)) \subset \mathbb{R}^2_{\mathcal{C}}$, $\gamma_2((c,d)) \subset \mathbb{R}^2_{\mathcal{C}}$
- $\gamma_1 \sim_{rb} \gamma_2$ *(up to a time rescale).*



Then $\gamma_1([a,b]) = \gamma_2([c,d])$.

Proof. Let v_1 be the minimiser of $\mathscr{M}_{c_{j_1},c_{j_2}}$ on $H_{[\gamma_1]}$. Therefore it must be $\bar{u}|_{[a,b]} = v_1$ and $\bar{u}|_{[c,d]} = v_1^{-1}$. □

We now generalise the previous result in the case when some centres are crossed by one or both the arcs γ_1, γ_2, see Figure 7.

Corollary 4.11. *Let $\bar{u}$ be a minimizer for $\mathscr{M}$ in $H_{[\gamma]}$ and suppose there exist $a < b \le c < d$ and c_{j_1}, $c_{j_2} \in \mathcal{C}$, (possibly equal), so that $\bar{u}(a) = \bar{u}(d) = c_{j_1}$ and $\bar{u}(b) = \bar{u}(c) = c_{j_2}$.*

Then $\bar{u}$ can not admit any 2-gon $\gamma = \gamma_1 \# \gamma_2$ with the following properties:

- $\gamma_1 = \bar{u}|_{[a,b]}$, $\gamma_2 = \bar{u}|_{[c,d]}$
- $\gamma_1(a,b) \cap \mathcal{C} = \mathcal{C}_1$, $\gamma_2(c,d) \cap \mathcal{C} = \mathcal{C}_2$
- *there exist $\eta_1 \in \widehat{H}_{c_{j_1},c_{j_2}}(a,b)$, $\eta_2 \in \widehat{H}_{c_{j_2},c_{j_1}}(c,d)$ such that $\eta_1 \sim_{rb} \eta_2$ in $\mathbb{R}^2_{\mathcal{C}}$ (up to reparameterization)*
- *the loop $\eta_1 \# \bar{u}|_{[b,c]} \# \eta_2 \# \bar{u}|_{S\setminus[a,d]}$ belongs to $H_{[\gamma]}$.*

Proof. Let v_1 be the unique minimizer of $\mathscr{M}_{c_{j_1},c_{j_2}}$ in $H_{[\eta_1]}$. Then v_1^{-1} is the unique minimizer of $\mathscr{M}_{c_{j_2},c_{j_1}}$ in $H_{[\eta_2]}$. Therefore the minimiser $\bar{u}$ is given by $\bar{u} = v_1 \# \bar{u}|_{[b,c]} \# v_1^{-1} \# \bar{u}|_{S\setminus[a,d]}$. In particular the two arcs γ_1, γ_2 cover the same path back and forth and intersect the same set of centres. □

The above lemma concerns the case the when the two arcs γ_1 and γ_2 join the points c_{j_1} and c_{j_2} in the opposite direction. By straighforward modification the same argument holds also in case the two arcs have the same direction.

5 The obstacle problem

Recalling the definition of R_j given in Lemma 4.2, let be defined a radius R such that

$$R < \min\left\{\min_{j\in\mathcal{J}} R_j, \frac{1}{2}\min_{i<j\in N}|c_i - c_j|\right\} \tag{10}$$

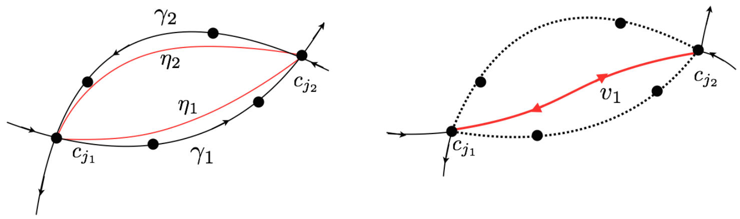


Figure 7: The left figure concerns the situation similar to the one described in Corollary 4.11.

and

$$B_j = B(c_j, R), \quad \forall j \in \mathcal{J}$$

the balls of radius R around the centers where the collisions occur, with the property that whenever it exists $[a,b] \in S$ so that $\bar{u}(t) \in B_j$ for all $t \in [a,b]$, then $[a,b] \cap T_c(\bar{u}) \neq \emptyset$. Note that R is defined small enough so that anytime $\bar{u}$ enters the ball B_j then $\bar{u}$ collides with the center c_j before leaving the ball.

For any $k = 1, \ldots, K$, being K the number of collisions, let be defined

$$a_k = \max\{t < \tau_k : \bar{u}(t) \in B_{j_k}\}, b_k = \min\{t > \tau_k : \bar{u}(t) \in B_{j_k}\}$$

the instants when the path $\bar{u}$ enters and leaves the ball B_{j_k} just before and after the collision time τ_k, and

$$p_k^1 = \bar{u}(a_k), \quad p_k^2 = \bar{u}(b_k) \tag{11}$$

the points where the minimiser enters and leaves B_{j_k}.

Since for Lemma 4.2 the inertia $I_k(\bar{u})(t)$ is strictly convex in (a_k, b_k) it follows that τ_k is the only collision time in the interval $[a_k, b_k]$ and $[a_k, b_k] \cap [a_j, b_j] = \emptyset$ for any $k \neq j$.

Lemma 5.1. *Let $\bar{u}$ be the minimizer for $\mathscr{M}$ in $H_{[\gamma]}$. Assume that $\bar{u}$ has collisions. Then an alternative holds: either $\bar{u}$ is a collision reflection solution, or it exists at least one k so that $p_k^1 \neq p_k^2$.*

Proof. Suppose $\bar{u}$ does not reflect after the collision instant $\tau_k \in T_c$. Arguing by contradiction, assume $p_k^1 = p_k^2$. Then, for Lemma 4.7 $\bar{u}$ is not a minimiser. Otherwise, if a reflection occurs at any collision instants, then $\bar{u}$ is a collision reflection solution according to Definition 1.1.

□

Let us sketch the plan of the proof of the Main theorem, that it will be given in the Section 6: by contradiction assume that $\bar{u}$ has collisions but it is not a collision reflection solution. For Lemma 5.1 there is at least one k so that $p_k^1 \neq p_k^2$ and $\bar{u}$ is not a collision reflection solution within the ball B_k. The absurd will arise by proving that p_k^1 must be equal to p_k^2, that is, anytime a collision occurs, the path $\bar{u}$ has to reflect. Therefore, either the minimizer $\bar{u}$ is collision free or it is a collision reflection solution.

The core of the proof is based on the so called *Obstacle problem*, that allows to analyse the behaviour of the minimisers of a given functional in the neighbourhood of the collisions. It results that if certain hypothesis are satisfied, whenever the minimiser $\bar{u}$ has a collision, it must be a collision reflection solution, at least locally.

In the following we shortly review the technique, mostly for stating the result and fixing the notation, while referring to previous works for the details.

Henceforth we assume that

$$\bar{u} \text{ has } K > 0 \text{ collisions, i.e. } |T_c| = K. \tag{HYP}$$

Without loss of generality, we apply the obstacle problem in the neighbourhood of τ_1. Denote by

$$\widehat{K}_1 := \{v \in H^1([a_1, b_1]; B_{j_1} \setminus c_{j_1}) : v(a_1) = p_1^1, v(b_1) = p_2^1\}$$
$$K_1 := \{v \in H^1([a_1, b_1]; B_{j_1}) : v(a_1) = p_1^1, v(b_1) = p_2^1\}$$

and

$$\widehat{K}_{[\gamma]} := \left\{ v \in \widehat{K}_1 \text{ s.t. the path } \gamma_v(t) := \left\{ \begin{array}{ll} \bar{u}(t) & t \in S \setminus [a_1, b_1] \\ v(t) & t \in [a_1, b_1] \end{array} \right. \text{ belongs to } \widehat{H}_{[\gamma]} \right\}. \tag{12}$$

Moreover let be

$$K_{[\gamma]} := \widehat{K}_{[\gamma]} \cup \{v \in K_1 : \gamma_v \in H_{[\gamma]}\}$$

and

$$\mathscr{M}_1 : K_{[\gamma]} \to \mathbb{R} \cup \{+\infty\}, \quad \mathscr{M}_1(v) = \frac{1}{2} \int_{a_1}^{b_1} |\dot{v}|^2 \, dt \int_{a_1}^{b_1} (V(v) + h) \, dt$$

Since $K_{[\gamma]}$ is weakly closed and $\mathscr{M}_1$ is coercive and weakly lower semicontinuous, the direct method of calculus of variation assures that $\mathscr{M}_1$ admits a minimum. Lemma 3.2 implies that the arc $\bar{u}|_{[a_1,b_1]}$ is a minimizer of $\mathscr{M}_1$.

For $\varepsilon > 0$ let be defined

$$d(\varepsilon) = \min \left\{ \mathscr{M}_1(v) : v \in K_{[\gamma]}, \min_{t \in [a_1, b_1]} |v_1(t) - c_{j_1}| = \varepsilon \right\}.$$

Since the weak convergence in H^1 implies the uniform convergence the set

$$\left\{ v \in K_{[\gamma]}, \quad \min_{t \in [a_1, b_1]} |v(t) - c_{j_1}| = \varepsilon \right\}$$

is weakly closed, hence the function $\varepsilon \mapsto d(\varepsilon)$ is well-posed. Moreover, by definition, $d(0)$ is the minimum of $\mathscr{M}_1$ over those arcs v that collide with c_{j_1}, thus the value $d(0)$ is achieved by $\bar{u}_{[a_1,b_1]}$.

Lemma 5.2. *The function $\varepsilon \mapsto d(\varepsilon)$ is continuous at $\varepsilon = 0$.*

Proof. Look at [38, Lemma17].

□

Given $0 < \varepsilon_1 < \varepsilon_2$, we term

$$K_{[\gamma]}(\varepsilon_1, \varepsilon_2) := \left\{ v \in K_{[\gamma]} : \min_{t \in [a_1, b_1]} |v(t) - c_{j_1}| \in [\varepsilon_1, \varepsilon_2] \right\}.$$

For any $\varepsilon_1 < \varepsilon_2$, $K_{[\gamma]}(\varepsilon_1, \varepsilon_2)$ is a weakly closed subset of $K_{[\gamma]}$, so the restriction of $\mathscr{M}_1$ to $K_{[\gamma]}(\varepsilon_1, \varepsilon_2)$ has a minimum denoted by

$$m(\varepsilon_1, \varepsilon_2) := \min_{v \in K_{[\gamma]}(\varepsilon_1, \varepsilon_2)} \mathscr{M}_1(v).$$

Among all possible minimizers of $\mathscr{M}_1$ in $K_{[\gamma]}(\varepsilon_1, \varepsilon_2)$ we select the following

$$\mathscr{M}(\varepsilon_1, \varepsilon_2) := \left\{ v \in K_{[\gamma]}(\varepsilon_1, \varepsilon_2) : \ \mathscr{M}_1(v) = m(\varepsilon_1, \varepsilon_2) \text{ and } \min_{t \in [a_1, b_1]} |v(t) - c_{j_1}| < \varepsilon_2 \right\}.$$

Lemma 5.3. *Under the hypothesis (HYP), for any $\varepsilon > 0$ there exist $\varepsilon_1 < \varepsilon_2 < \varepsilon$ so that*

$$\mathscr{M}(\varepsilon_1, \varepsilon_2) \neq \emptyset.$$

Proof. Arguing by contradiction, assume that there exists $\bar{\varepsilon}$ such that for any $0 < \varepsilon_1 < \varepsilon_2 < \bar{\varepsilon}$ it holds $\mathscr{M}(\varepsilon_1, \varepsilon_2) = \emptyset$. Then, let us fix ε_2 and ε^*, $0 < \varepsilon_2 < \varepsilon^* < \bar{\varepsilon}$, and define a sequence $\{\varepsilon_{1_n}\}_n$ monotonically decreasing to zero so that $\varepsilon_{1_n} < \varepsilon_2$ for any n. Denote by v_0 the element satisfying $\mathscr{M}_1(v_0) = m(\varepsilon_2, \varepsilon^*)$ and $\{v_n\}_n$ the sequence such that $\mathscr{M}_1(v_n) = m(\varepsilon_{1_n}, \varepsilon^*)$. Since $\mathscr{M}(\varepsilon_2, \varepsilon^*) = \emptyset$ and $\mathscr{M}(\varepsilon_{1_n}, \varepsilon_2) = \emptyset$ for any n, it follows that $\min_{t\in[a_1,b_1]} |v_0(t) - c_{j_1}| = \varepsilon^*$, and, for any n, $\min_{t\in[a_1,b_1]} |v_n(t) - c_{j_1}| = \varepsilon_2$.

It follows that $d(\varepsilon^*) < d(\varepsilon_2)$ and $d(\varepsilon_2) < d(\varepsilon_{1_n})$ for any n. The continuity of $d(\varepsilon)$ at $\varepsilon = 0$ implies that $d(\varepsilon^*) < d(0)$. That contradicts the fact that the minimum of $\mathscr{M}_1$ in $K^1_{[\gamma]}$ has a collision.

□

As a direct consequence of the previous lemma, there exist two sequences $0 < \varepsilon_n < \bar{\varepsilon}_n$ both vanishing, and a sequence $\{v_n\}_n$ such that

$$v_n \in \widehat{K}_{[\gamma]}, \quad \min_{t\in[a_1,b_1]} |v(t) - c_{j_1}| = \varepsilon_n$$

$$\mathscr{M}_1(v_n) = m(\varepsilon_n, \bar{\varepsilon}_n) = d(\varepsilon_n).$$

For any n, denote by

$$T_n = \{t \in [a_1, b_1] : |v_n - c_{j_1}| = \varepsilon_n\}.$$

The next proposition summaries the properties of the sequence $\{v_n\}$. We refer to [38, 34] for the proof.

Proposition 5.4. *For any n*

i) $v_n(t)$ is $\mathscr{C}^1$;

ii) $v_n(t)$ is a $\mathscr{C}^2$ function in any interval $I \subset (a_1, b_1) \setminus T_n$ and solves

$$w_n^2 \ddot{v}_n = \nabla V(v_n(t)), \quad w_n^2 = \frac{\int_{a_1}^{b_1} (V(v_n) + h)\, dt}{\int_{a_1}^{b_1} \frac{1}{2}|\dot{v}_n|^2\, dt};$$

iii) the set T_n consists in a interval, $T_n = [c, d] \subset [a_1, b_1]$;

iv) the inertia $I(t) = |v_n(t) - c_{j_1}|^2$ is strictly convex for $t \in [a_1, c] \cup [d, b_1]$;

v) the energy of the function v_n is constant in $[a_1, b_1]$:

$$\frac{1}{2}|\dot{v}_n|^2 - \frac{V(v_n(t))}{w_n^2} = \frac{h}{w_n^2}, \quad \forall t \in [a_1, b_1]$$

and the sequence $\{w_n\}_n$ is bounded above and uniformly bounded below by a positive constant.

Moreover, writing $v_n(t) = c_{j_1} + \rho_n(t)e^{\imath\theta_n(t)}$, it holds

vi) the function $\theta_n(t)$ is $\mathscr{C}^2$ and strictly monotone in T_n.

The convergence of the sequence $\{v_n\}_n$ is now discussed.

Lemma 5.5. *Up to a subsequence, the sequence $v_n(t)$ uniformly converges to a path $\tilde{u}_1(t) \in K_{[\gamma]}$. The limit $\tilde{u}_1$ has a collision with c_{j_1} and $\mathscr{M}_1(\tilde{u}_1) = d(0)$.*

Proof. Since $\mathscr{M}_1(v_n) = d(\varepsilon_n)$ and $\varepsilon_n \to 0$, from Lemma 5.2, it follows that $\mathscr{M}_1(\varepsilon_n) \to d(0)$. The coercivity of $\mathscr{M}_1$ implies that $\{v_n\}$ is bounded, hence, up to a subsequence still denoted by $\{v_n\}$, it is weakly convergent to a arc $\tilde{u}_1 \in K_{[\gamma]}$. The weak convergence in H^1 implies uniform convergence, thus $\tilde{u}_1$ has to collide with c_{j_1}. □

The last step of the obstacle problem technique consists in the analysis of the behaviour of the minimising sequence v_n and of the limit $\tilde{u}_1$. Again, we do not provide the detailed proof, since it is a small modification of the proof given in [34]. The argument is the following. Denote by Θ_n the total variation for $t \in [a_1, b_1]$ of the angular coordinate θ_n associated to v_n. One shows that $\Theta_n \geq \Delta_n$ for any n and $\Delta_n \to \frac{2\pi}{2-\alpha}$ as $n \to \infty$, see also [36]. It means that for any $\alpha > 1$ the path v_n has to self intersect for n large enough. Therefore, it is enough to prove that all the v_n's are simple to obtain the contradiction. The case $\alpha = 1$ is more delicate. Indeed, even under the hypothesis that v_n are simple, the limit $\tilde{u}_1$ could have the collision. However, if that happens, $\tilde{u}_1$ is proven to be a collision reflection solution. The idea is to combine the minimisation procedure with the Levi Civita regularisation [29]. In the neighbourhood of one of the centres, the potential $V(x)$ is a perturbation of the one-centre keplerian potential and, by means of the classical Levi-Civita change of coordinates, the singular dynamical system is related to a regular one defined on a Riemann surface. As a result, the minimising sequence $\{v_n\}_n$ is associated to a minimising sequence $\{q_n\}_n$ of an equivalent obstacle problem in the regularised system. The weak limit $\tilde{q}$ of the sequence q_n is related to the arc $\tilde{u}_1$ as follows. Assuming that the centre c_{j_1} has been mapped to the origin in the Levi-Civita plane and denoting with s_0 the time when $\tilde{q}(s_0) = 0$, if the limit $\tilde{q}$ is a transmission solution, i.e. if $\tilde{q}(s_0 + s) = -\tilde{q}(s_0 - s)$, then the limit $\tilde{u}_1(t)$ is a collision reflection solution. The path $\tilde{q}$ is a transmission solution if the angle swept by q_n on the obstacle is not larger than π or, equivalently, if the angle Θ_n covered by $v_n(t)$ is not larger than 2π. In the gravitational case, $\alpha = 1$, it is proved that $\Theta_n \leq 2\pi$ for any n, as wanted.

Summarising, the following holds.

Proposition 5.6.

> **Case** $\alpha = 1$. *If for any n the path v_n is simple and the limit $\tilde{u}_1(t)$ has a collision, then $\tilde{u}_1(t)$ is a collision reflection solution within the ball B_1. In particular $p_1^1 = p_2^1$.*
>
> **Case** $1 < \alpha < 2$. *If for any n the path v_n is simple, then the limit $\tilde{u}_1(t)$ is collision-free.*

Proof. The same proof as in [34] and [38]

□

6 Proof of the theorem

We are now in the position to proof the Theorem 1.2. For clarity, let us recall the statement.

Theorem 1.2. *Let $\alpha = 1$, $\mathcal{C} = \{c_1, \dots, c_N\}$, $c_i \in \mathbb{R}^2$, be any set of centres. Then, for any admissible class $[\gamma]$ and any $h > 0$, at least one of the following occurs:*

1. *it exists a non-collision periodic solution $x(s)$ for the N-center problem with energy h and $x(s)$ parametrizes a curve in $[\gamma]$,*

2. *it exists a collision reflection solution.*

Proof. For Remark 2.16 and Theorem 4.1, let $\bar{u}$ be the minimiser for $\mathscr{M}_{[\gamma]}$ in $H_{[\gamma]}$. If $\bar{u}$ has not collisions, then for Theorem 3.3 the path $x(s) = \bar{u}(ws)$ is a classical solution with energy h and $x(s)$ parametrizes a curve in $[\gamma]$, hence 1. holds.

Otherwise, suppose $\bar{u}$ has collisions. We aim at proving that $\bar{u}$ parametrizes a collision reflection solution. Arguing by contradiction, suppose that $\bar{u}$ is not a collision reflection solution. Then, recalling the definitions (10) (11) of R and B_j and of the couple (p_k^1, p_k^2), for Lemma 5.1 it exists at least one k^* so that $p_{k^*}^1 \neq p_{k^*}^2$. Without loss of generality, set $k^* = 1$. We now apply the obstacle problem in the neighbourhood of τ_1 as described in Section 5. Assume for the moment that for any n the arc v_n is simple. This fact will be proved afterwards in Theorem 6.3. According to Proposition 5.6, we conclude that $p_1^1 = p_1^2$, hence the contradiction. Therefore, if $\bar{u}$ has collisions, $\bar{u}$ reflects at any collision instants and the path $x(s) = \bar{u}(ws)$ is a collision reflection solution with energy h of the N-centre problem. □

The proof of Theorem 1.3 is more immediate.

Theorem 1.3. *Let $1 < \alpha < 2$, $\mathcal{C} = \{c_1, \ldots, c_N\}$, $c_i \in \mathbb{R}^2$, be any set of centres. Then, for any admissible class $[\gamma]$ and any $h > 0$ it exists a non-collision periodic solution $x(s)$ for the N-center problem with energy h and $x(s)$ parametrizes a curve in $[\gamma]$.*

Proof. As done in the proof of Theorem 1.2, let $\bar{u}$ be a minimiser of $\mathscr{M}_{[\gamma]}$ on $H_{[\gamma]}$. If $\bar{u}$ is collision free, then $x(s) = \bar{u}(ws)$ provides the periodic solution of (1)-(3). Otherwise, assume by contradiction that $\bar{u}$ has a collision and apply the obstacle problem around one of the collisions. Again, suppose that the paths v_n are simple, then Proposition 5.6 provides the absurd. □

It remains to prove that for any n the arc v_n is simple. From Proposition 5.4 we remind that, for any n, the path $v_n \in \mathscr{C}^1([a_1, b_1]; B_{j_1} \setminus c_{j_1})$, $\min_{t \in [a_1, b_1]} |v_n(t) - c_{j_1}| = \varepsilon_n > 0$ and that $|v_n(t) - c_{j_1}| = \varepsilon_n$ for $t \in [c, d] \subset [a_1, b_1]$. Denote

$$v_n^1 \stackrel{\text{def}}{=} v_n|_{[a_1, c]}, \quad v_n^2 \stackrel{\text{def}}{=} v_n|_{[d, b_1]}.$$

Moreover, define

$$\gamma_n \stackrel{\text{def}}{=} \bar{u}|_{S \setminus [a_1, b_1]} \# v_n. \tag{13}$$

Lemma 6.1.

i) The arcs v_n^1, v_n^2 are simple.

ii) The path v_n does not admit any singular 1-gon neither singular 2-gon.

iii) v_n^1 is nowhere tangent to v_n^2.

Proof. For point *iv*) in Proposition 5.4, the two arcs are simple. This excludes the possibility for v_n to perform singular 1-gon. Also, following the same arguments as in the proof of Proposition 4.6, the two arcs v_n^1 and v_n^2 don't have any tangencies, neither they can intersect each other so to create singular 2-gon. □

Hence, the possible self intersections of v_n are due to transversal intersections of v_n^1 with v_n^2 or to multiple covering of part of the obstacle (tangential self intersection). The remaining of this section aims at proving that both the transversal and tangential intersections can not occur. The argument is based on the fact that the path γ_n given in (13) is in the closure of $\widehat{H}_{[\gamma]}$ and $[\gamma]$ is an admissible class.

First, let us construct a path γ_n^*, free of collisions, without self tangencies, so that $\gamma_n^* \in [\gamma]$ and γ_n^* visits the balls B_j in the same order as γ_n.

If τ_1 is the only collision time for $\bar{u}$ then $\gamma_n(t) \in \mathbb{R}^2_{\mathcal{C}}$ and $\gamma_n \in [\gamma]$. In that case, define $\gamma_n^* = \gamma_n$. Otherwise, let $\{\tau_2, \dots, \tau_K\}$ and $\{c_{j_2}, \dots, c_{j_K}\}$ be respectively the collision instants for $\gamma_n(t)$ and the centres where the collisions occur. Since $H_{[\gamma]}$ is the weak H^1 closure of $\widehat{H}_{[\gamma]}$, for any $0 < \rho < R$ small enough, there exists a collection of $K-1$ arcs $\{\hat{u}_i\}_{i=2}^K$ so that

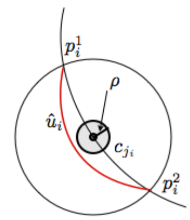
i) $\hat{u}_i : [a_i, b_i] \to B_{j_i} \setminus B_{j_i}(\rho)$

ii) $\hat{u}_i(a_i) = p_i^1, \hat{u}_i(b_i) = p_i^2$

iii) the path

$$\hat{\gamma}_n \stackrel{\text{def}}{=} \begin{cases} v_n(t) & t \in [a_1, b_1] \\ \hat{u}_i(t) & t \in [a_i, b_i], \quad i = 2, \dots, K \\ \bar{u}(t) & t \in S \setminus \bigcup_{i=1}^K [a_i, b_i] \end{cases}$$

belongs to $\widehat{H}_{[\gamma]}$.



Clearly the choice of the arc $\hat{u}_i$ is not unique. Following [23], the arcs $\hat{u}_i$ can be selected so that

iv) for any i the arc $\hat{u}_i$ does not have self tangencies and $\hat{u}_i$ minimises the number of self intersections in its homotopy class (with respect to the relative homotopy),

v) for any $i \neq k$ so that $c_{j_i} = c_{j_k}$, if any, the arcs $\hat{u}_i$ and $\hat{u}_k$ do not intersect tangentially and they minimise the number of intersections.

vi) if $c_{j_i} = c_{j_1}$ for some $i \in \{2, \dots, K\}$, then $\hat{u}_i$ is nowhere tangent to v_n and it minimises the number of intersections with v_n.

Since $\hat{\gamma}_n$ is equal to $\bar{u}$ outside $\bigcup_j B_j$, $\hat{\gamma}_n$ inherits the self tangencies from $\bar{u}$. As described in Proposition 4.6, Lemma 4.10 and Corollary 4.11, there may exists a set of couples of indexes $\mathcal{I}_{tg} \subset \{(i,j) \in \{1, 2, \dots, K\}^2\}$, so that $\hat{\gamma}_n|_{[b_i, a_{i+1}]}$ is tangent to $\hat{\gamma}_n|_{[b_j, a_{j+1}]}$ for any $(i,j) \in \mathcal{I}_{tg}$.

If $\mathcal{I}_{tg} \neq \emptyset$, say $\mathcal{I}_{tg} = \{(i_1, j_1), \dots, (i_m, j_m)\}$, for $\epsilon > 0$ small enough define m homotopies relative to the boundaries,

$$\Phi_k(s,t) : [0,1] \times [b_{i_k} - \epsilon, a_{i_k+1} + \epsilon] \cup [b_{j_k} - \epsilon, a_{j_k+1} + \epsilon]$$

to be applied to $\hat{\gamma}$ in sequence, so that the tangent arcs are moved to a different configuration without tangencies and with the minimal number of transversal intersections, see Figure 8. Indeed, the tangent arcs can be homotoped to two arcs with at most one intersection [2]. Also, the homotopies Φ_k can be defined so that the number of intersections between the two arcs involved in the homotopy with the remaining part of the curve $\hat{\gamma}_n$ does not change.

Denote by $\gamma_n^*(t)$ the path resulting after applying all the homotopies Φ_k. By construction, γ_n^* satisfies the following properties:

Lemma 6.2.

- $\gamma_n^*(t) \in [\gamma]$,
- $\gamma_n^*|_{[a_1, b_1]} = v_n$,
- $\gamma_n^*(t)$ *visits the same balls* B_j *in the same order as* γ_n.
- $\gamma_n^*(t)$ *has no tangential self intersection in* $S \setminus [a_1, b_1]$.

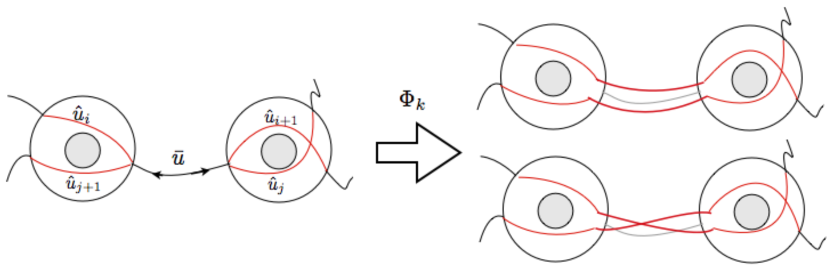


Figure 8: The self tangency is removed by homotopy. It results two arcs that intersect at most once.

Theorem 6.3. *For any n, the arc v_n is simple.*

Proof. Suppose by contradiction that for some n the path $v_n(t)$ is not simple, i.e. t it exists at least one couple $t^* < t_*$ such that $v_n(t^*) = v_n(t_*)$.

For the moment assume that v_n does not have any self intersection in the interval $[c, d]$ (later on we will be back to this eventuality). Hence, the self-intersections of v_n are due to intersections of v_n^1 with v_n^2. Denote by

$$a_1 < t_1^1 < \cdots < t_m^1 < c, \quad \text{and} \quad d < t_m^2 < \cdots < t_1^2 < b_1 \tag{14}$$

the instants of transversal intersections and

$$I_i^{tr} = v_n(t_i^1) = v_n(t_i^2), \quad i = 1, \ldots, m$$

the points where they occur. For Proposition 5.4 the angular velocity of v_n^1, v_n^2 is constant and v_n^1, v_n^2 have finite length. Thus, the number of intersections must be finite.

Consider the path γ_n^* previously introduced. For Lemma 6.2 γ_n^* belongs to an admissible class. Note that γ_n^* has an innermost loop $\gamma_n^*|_{[t_1^m, t_2^m]}$ enclosing only one centre. Therefore, by definition of admissible class, γ_n^* is not taut. For Theorem 2.12 γ_n^* has a singular 1-gon or singular 2-gon, denoted by Σ. Since for Lemma 6.1 and Lemma 6.2 the arcs v_n^1 and v_n^2 do not produce singular 1-gon neither 2-gon and γ_n^* has no self tangencies, it follows that the point $p = v_n(t_1^1) = v_n(t_1^2)$ is necessarily one of the end-points of the singular object Σ whose boundary lies partially inside B_{j_1} and partially outside B_{j_1}. We show that whatever singular object Σ is (1-gon or 2-gon), a contradiction arises.

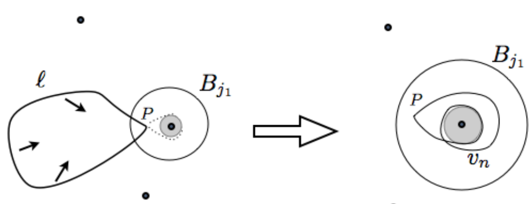


Figure 9: If Σ is a singular 1-gon, by shrinking the loop ℓ it results a path still belonging to $[\gamma]$. The new path is taut and contains an innermost loop enclosing only one center.

Let us first consider the case that Σ is a singular 1-gon and denote by $\ell = \gamma_n^*|_{[t_1^2, t_1^1]}$ the null homotopic loop that bounds Σ, as showed in Figure 9. Shrinking ℓ to a point, we have that $v_n|_{[t_1^1,t_1^2]} \in [\gamma]$ and $v_n|_{[t_1^1,t_1^2]}$ is taut, since no singular 1-gon or 2-gon lies in B_{j_1}. That contradicts the definition of admissible class, since $v_n|_{[t_1^1,t_1^2]}$ has an innermost loop containing only one center.

Suppose now that Σ is a singular 2-gon. Denote by q the second end-point (the first one is p) and α, β the edges of Σ. For Lemma 6.1, at least one of the edges is not completely within the ball B_{j_1}. Without loss of generality, the case when one of the edges completely lies in the ball is depicted in figure 10 a).

Looking at the definition of $\hat{\gamma}_n$, we see α as the composition of three arcs: $\alpha = \alpha_1 \# \alpha_2 \# \alpha_3$: the first is a portion of the arc v_n from the point p up to the point p_1^2 on the boundary of B_{j_1}, the second is made by part of $\bar{u}$ and eventually some $\hat{u}_i$ up to the point $p_1^{j_*}$ where α is back on B_{j_1} and, finally, α_3 is part of $\hat{u}_{j_*}$. Consider the arc $\ell = \bar{u}|_{[\tau_1,b_1]} \# \alpha_2 \# \bar{u}|_{[a_{j_*},\tau_{j_*}]}$. The arc $\ell \in \widehat{H}_{c_{j_1},c_{j_1}}$, it is homotopic to the point c_{j_1}, hence the loop $\bar{u}|_{S\setminus[\tau_1,\tau_{j_*}]} \# \ell$ belongs to $H_{[\gamma]}$. That contradicts Lemma 4.7 or the associated Corollary 4.8.

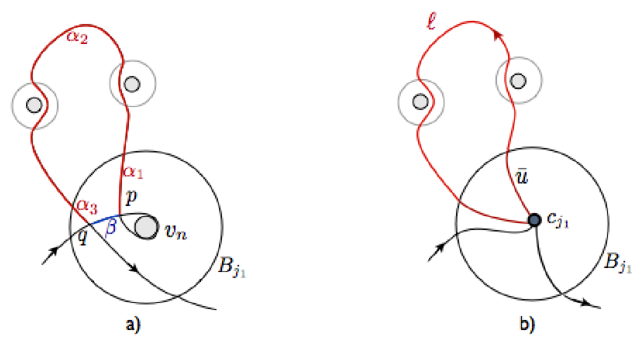


Figure 10: It concerns the situation when one edge of the singular 2-gon is completely inside the ball B_{j_1}.

Consider now the case when none of the edges α, β lies completely in the ball B_{j_1}. Depending whether q belongs to B_{j_1} (case 1), to $B_{j_k}, k \neq 1$ (case 2) or outside any of the balls B_j (case 3), the situation is depicted in the Figure 11. Arguing as before, the same procedure leads to a configuration that contradicts Lemma 4.10 and related Corollary 4.11 in case 1 and case 2, and Lemma 4.9 in case 3.

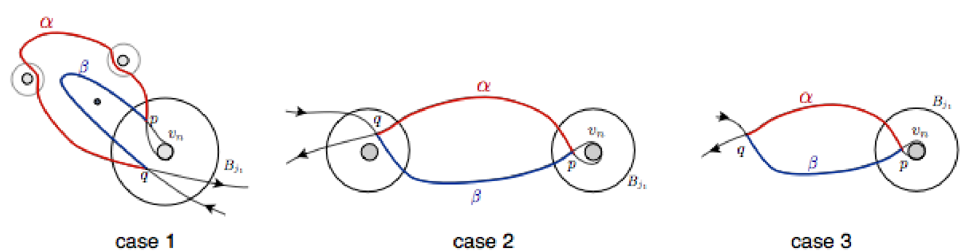


Figure 11: The three figures depict the situations when both the edges of the singular 2-gon exit the ball B_{j_1}.

It remains to consider the case when v_n has tangential self intersections on the obstacle. The idea is to replace v_n with an homotopically equivalent path ν_n without self tangencies and equal to $v_n(t)$ for those t where $v_n(t)$ is far away from the obstacle. The path ν_n has at least one self intersection and it satisfies the properties stated in Lemma 6.1. Then, the same procedure as before can be applied to ν_n, proving that ν_n can not have self intersections, hence the original v_n must be simple.

Let us show how to construct ν_n. Remember from Proposition 5.4 that the angular coordinate $\theta(t)$ is monotone for $t \in [c,d]$. As depicted in Figure 12, define $\tilde{R} = |I_m^{tr} - c_{j_1}|$ the distance between the centre c_{j_1} and the traversal self intersection point of v_n closest to the obstacle. (In case v_n does not have transversal self intersections, consider $t_1^m = a_1, t_2^m = b_1$ and $\tilde{R} = R$).

For a choice of $\tilde{\varepsilon}_n \in (\varepsilon_n, \tilde{R})$, being ε_n the radius of the obstacle, define

$$\tilde{c} = \max\{t < c : |v_n(t) - c_{j_1}| = \tilde{\varepsilon}_n\} \quad \tilde{d} = \min\{t > d : |v_n(t) - c_{j_1}| = \tilde{\varepsilon}_n\}$$

and denote A the annulus $A = B_{c_{j_1}}(\tilde{\varepsilon}_n) \setminus B_{c_{j_1}}(\varepsilon_n)$. Consider an homotopy $\Phi : [0,1] \times [\tilde{c}, \tilde{d}] \to A$ that fixes the end points and moves the arc $v_n|_{[\tilde{c},\tilde{d}]}$ to the smooth arc $\tilde{v}$, free of self tangencies and with the minimal number of self intersections. The monotonicity of $\theta(t)$ implies that v_n turns on the obstacle for more than 2π. This, together with the fact that $v_n|_{[\tilde{c},c)}, v_n|_{(d,\tilde{d}]}$ are simple and disjoint, implies that $\tilde{v}$ has at least one double point. Indeed, $\tilde{v}$ has as many double points as the number of times v_n turns around the obstacle.

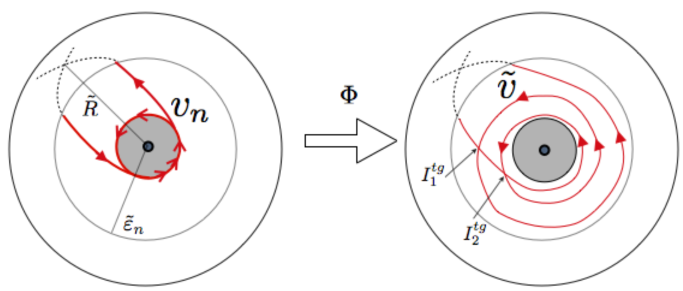


Figure 12: Relaxation of the self tangency. By means of an homotpy Φ the self tangent arc is moved to an arc with transversal self-intersections only.

Denote by $I_1^{tg}, \ldots I_s^{tg}$ such double points, labelled from the outermost to the innermost, see Figure 12. Since $\tilde{v}$ minimises the number of self intersections, $\tilde{v}$ does not admit singular 1-gon neither singular 2-gon, [23]. However, there could be a singular 2-gon whose end-points are given by I_m^{tr} and I_1^{tg}. If that happens, the original path v_n would behave as depicted in Figure 13 and the curve w_n drawn in Figure 13 would be a minimiser as like as v_n without being $\mathcal{C}^1$, providing a contradiction.

Replace $v_n|_{[\tilde{c},\tilde{d}]}$ by $\tilde{v}$ and define $\nu_n = v_n|_{[a_1,\tilde{c}]} \# \tilde{v} \# v_n|_{[\tilde{d},b_1]}$.

□

7 Admissible classes as sequence of symbols

In the previous section we proved the existence of a periodic solution for the N center problem in any admissible class $[\gamma]$. According to the Definition 2.15, to test whether a

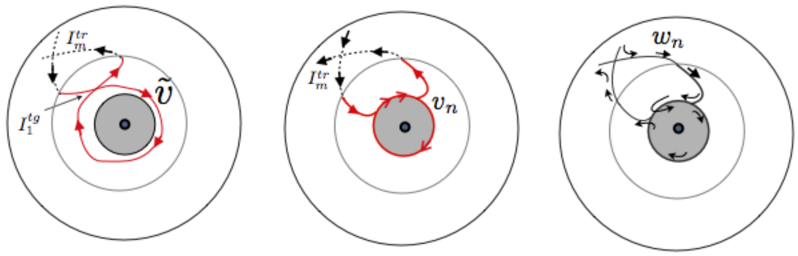


Figure 13: Construction of the path w_n. The figure depicts the situation when v_n turns around the obstacle only once. The same construction provides the contradiction even in case v_n turns around the obstacle many times.

given class is admissible one has to count the number of centers in any innermost loop of a taut representative of the homotopy class. Such a test could be time consuming and tedious, however it is straightforward thanks the Theorem 2.12.

Nevertheless, a natural question arises whether it is possible to prescribe sufficient conditions to a loop γ so that its homotopy class $[\gamma]$ results to be admissible. That is exactly the content of this section: given the centers in any position on the plane we propose a method to draw infinitely many closed curves each of them belonging to a different admissible class. The idea is to consider a simple polygonal having the centers as vertices and to construct different closed curves whose sequence of consecutive crossings with the edges of the polygonal is prescribed in advance. Certain conditions on the sequence of the consecutive crossings will imply the co-existence of at least two centers in any innermost loop of a taut representative, hence the admissibility of the homotopy class.

Let us introduce the technique in the case N centers are placed at vertices of a regular N-gon, then we will show how to extend to the general case.

Let $\mathcal{P}$ be a regular N-gon of unit radius and centered in the origin of $\mathbb{R}^2$. Let $c_1, c_2, \ldots, c_N$ be N-centers placed at the vertices of $\mathcal{P}$ and labelled according with the counterclockwise orientation, i.e. $c_i = (\cos(\frac{2\pi}{N}(i-1)), \sin(\frac{2\pi}{N}(i-1)))$. Denote by

$$\mathscr{L} = \{l_1, l_2, \ldots, l_N\}$$

the set of the edges of $\mathcal{P}$, where $l_i = \{\lambda c_i + (1-\lambda)c_{i+1}, \lambda \in [0,1]\}$ is the edge joining c_i and c_{i+1}, for $i = 1, \ldots, N$. To simplify the exposition we will always assume the cyclic action of the labels on a set of N elements, so that for instance $c_{N+1} = c_1$. Denote $\partial\mathcal{P} = \bigcup l_i$. On $\mathscr{L}$ we consider the Lee distance

$$d(l_i, l_j) = \min(|i-j|, N - |i-j|).$$

Clearly $d(l_i, l_j) = 0 \Leftrightarrow i = j$ and $d(l_i, l_j) = 1$ if and only if the edges l_i and l_j are adjacent. Moreover $\max_{1 \leq i,j \leq N} d(l_i, l_j) = \lfloor \frac{N}{2} \rfloor$, thus $d(l_i, l_j) \geq 2$ for some i, j if and only if $N \geq 4$.

Thinking at l_i as *letters*, we introduce the *words* with alphabet $\mathscr{L}$ as follows:

Definition 7.1. We define a word W with alphabet $\mathscr{L}$ any possible string

$$W = w_1 w_2 \ldots w_M$$

where $w_k \in \mathscr{L}$ for any $1 \leq k \leq M$ and $M < \infty$. M is said the length of the word and we denote $M = |W|$.

In the next Definition we fix further notations and we introduce some useful operations.

Definition 7.2.

i) Denote by $\mathcal{W} = \{W : |W| < \infty\}$ the set of all possible words of finite length with alphabet $\mathcal{L}$ and by $\mathcal{W}_e = \{W \in \mathcal{W} : |W| = 2n, n \in \mathbb{N}\}$ the set of words of even length;

ii) Given two words $W^1 = w_1^1 w_1^2 \ldots w_{k_1}^1$ and $W^2 = w_1^2 w_2^2 \ldots w_{k_2}^2$ we define the composition of W^1 and W^2 as the word $W = W^1 W^2 = w_1^1 w_1^2 \ldots w_{k_1}^1 w_1^2 w_2^2 \ldots w_{k_2}^2$.

iii) A word W is said a power of W^1 if there exists $n > 1$ so that $W = W^1 W^1 \ldots W^1$ n-times.

iv) A word W of length M is said irreducible if $w_i \neq w_{i+1}$, for $i = 1, \ldots, M-1$ and $w_M \neq w_1$.

From now on we consider only irreducible words. Given a word W, we associate to W a path $\gamma(t)$ in the punctured plane so that the sequence of consecutive crossings of $\gamma(t)$ with the edges of $\mathcal{P}$ is exactly the one prescribed by W.

Definition 7.3. For any given word $W \in \mathcal{W}_e$, define γ_W any closed path with the following properties:

i) $\gamma_W \in \mathscr{C}^1([0,1], \mathbb{R}^2_{\mathcal{C}})$, $\gamma_W(0) = \gamma_W(1)$;

ii) γ_W is nowhere tangent to $\partial\mathcal{P}$ and $|\gamma_W([0,1]) \cap \partial\mathcal{P}| = |W|$;

iii) $\gamma_W(t_i) \in w_i$ for any $1 \leq i \leq |W|$, where $\{0 \leq t_1 < t_2 < \cdots < t_{|W|} < 1\} = \gamma_W^{-1}(\gamma_W \cap \partial\mathcal{P})$.

Since we allow the path $\gamma_W(t)$ to only have transversal intersections with $\partial\mathcal{P}$ it turns out that $\gamma_W(t)$ has an even number of crossings with $\partial\mathcal{P}$. That is the reason why we consider only words of even length.

The above definition is well posed in the sense that for any given W of even length it is always possible to define a path with properties $i), ii), iii)$. Clearly the path γ_W is not unique, since for instance small perturbations produce paths with the same sequence of consecutive crossings with the edges of the polygon $\mathcal{P}$. Even more, the same word W could produce paths γ_W belonging to different homotopy classes: indeed any time the path γ_W leaves $\mathcal{P}$ crossing the edge w_i, there are two different directions it can follow to reach the next edge w_{i+1}, namely the clockwise or counterclockwise direction and even it can wind around the polygon $\mathcal{P}$ many times before entering. In general, different choice of the directions produce non homotopic curves, see Fig.14. Nevertheless, the following result holds.

Theorem 7.4. *Let $N \geq 4$ centers be placed at the vertices of a regular N-gon and let W be any irreducible word of even length such that*

$$d(w_i, w_{i+1}) \geq 2, \quad \forall\, 1 \leq i \leq |W|. \tag{15}$$

Let $\gamma_W(t)$ be any path associated to W according with Definition 7.3. Then, the homotopy class $[\gamma_W]$ is admissible.

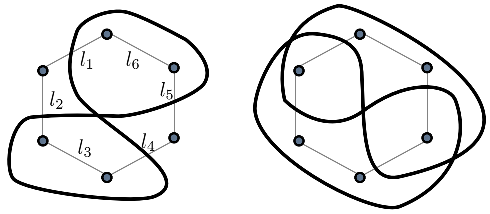


Figure 14: Both the curves are a realisation of the word $W = l_1 l_4 l_2 l_6$.

Proof. First note that the irreducibility of W implies that γ_W minimizes the number of possible intersections with $\partial\mathcal{P}$ in its homotopy class. Indeed suppose that there exists an homotopy ϕ_s that moves γ_W to $\phi_1(\gamma_W)$ with a less number of crossings with $\partial\mathcal{P}$. Since the number of intersections between two closed curves only changes in correspondence of tangential crossings, the change of number of crossings between $\phi_s(\gamma_W)$ and $\partial\mathcal{P}$ occurs only at instants s_i when $\phi_{s_i}(\gamma_W)$ is tangent to one of the edges l_j. At the first of these instants, say s_1, suppose that $\phi_{s_1}(\gamma_W)$ is tangent to l_j. Then the path $\gamma_W = \phi_0(\gamma_W)$ has two consecutive crossing with l_j. That contradicts the irreducibility of the word W.

The definition of admissibility of the class $[\gamma_W]$ relies on some conditions to be satisfied by a taut path $\hat{\gamma} \in [\gamma_W]$. The constructed curve γ_W does not need to be taut, however there exists a taut path $\hat{\gamma}$ with the same sequence of crossings of $\partial\mathcal{P}$ as γ_W. As before let ϕ_s be the homotopy that moves γ_W to $\hat{\gamma}$: at any s the number of crossing of $\phi_s(\gamma_W)$ with $\partial\mathcal{P}$ can not be less then the number of crossing of γ_W and it can increase only at instants of tangencies with one of the l_j. Such tangencies could be avoided, since they produce null-homotopyc 2-gons with the perimeter of $\mathcal{P}$.

Suppose first that $\hat{\gamma}$ is not simple and let $\hat{\gamma}(\alpha)$ be an innermost loop. For Lemma (2.6) $\hat{\gamma}(\alpha)$ is a simple closed curve and an alternative occurs: $\hat{\gamma}(\alpha)$ does not intersect $\partial\mathcal{P}$ or it has an even number of crossings corresponding to a part of the word W. In the former case all the centers lie in the region bounded by $\hat{\gamma}(\alpha)$ (it can not be the opposite because $\hat{\gamma}(\alpha)$ is not null homotopic) so condition A1 in the Definition 2.15 holds. In the latter case the set $\mathcal{C}$ is split in two subsets $\mathcal{C}_{int}$ and $\mathcal{C}_{ext}$ of those centers that respectively lie in the bounded and unbounded region of the plane separated by $\hat{\gamma}(\alpha)$. Condition (15) assures that both $\mathcal{C}_{int}$ and $\mathcal{C}_{ext}$ contain at least two centers, thus condition A1 is still satisfied.

In case $\hat{\gamma}$ is simple the innermost loop is given by the curve itself and the same argument as before applies.

□

The above construction of the symbolic dynamic is based on the fact that the edges $\{l_i\}_i$ do not intersect each other so it is possible to label them according to an orientation and to define the suitable distance. However the same theory applies anytime it is possible to define a simple polygonal joining the centers (i.e. a union of non-intersecting, line segments that are joined pair-wise to form a closed path), no matter if it is a regular N-gon. Furthermore the edges $\{l_i\}_i$ need not to be segments, they could be smooth arcs joining two of the centers.

This remark allows us to extend the construction of the symbolic dynamic to the general case, whatever is the location of the N centers. It is enough to draw a simple polygonal (even curvilinear) where each edge joins two of the centers.

Suppose first that all the N centers are placed along a straight line r, as in Figure 15 a): then we can label them according to the direction of the line and we can define the curvilinear polygonal $\mathcal{P}$ with edges $l_i = c_{i+1} - c_i$ for $i = 1, \ldots, N-1$ and the edge l_N as any arc joining c_1 and c_N without crossing r.

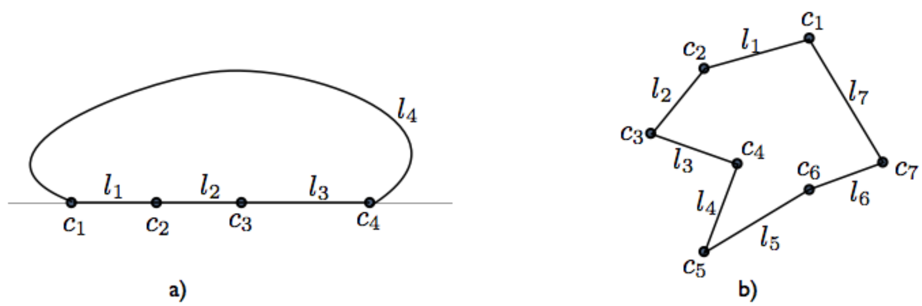


Figure 15: Labelling of the centres and construction of the curvilinear polygonal in case the centres are on the same line (a) or in general position (b).

Suppose now that the N centers are not placed on the same straight line so that the interior of the convex hull $H_{\mathcal{C}} = \left\{ \sum_{i=1}^{N} \alpha_i c_i \middle| c_i \in \mathcal{C}, \, \alpha_i \in \mathbb{R}^+ \cup \{0\}, \, \sum_{i=1}^{N} \alpha_i = 1 \right\}$ is not empty, Figure 15 b). Choose a point p in the interior of $H_{\mathcal{C}}$ so that there are no two centers aligned with p. (Such point p always exists: indeed let $\mathcal{S}$ be the union of all possible straight lines passing through two of the centers and denote $\mathcal{O} = \mathring{H}_{\mathcal{C}} \setminus \mathcal{S}$, $\mathring{H}_C$ denoting the interior of $H_{\mathcal{C}}$. Any point of $\mathcal{O}$ satisfies the requirements for p and the set $\mathcal{O}$ is not empty. Indeed it is of full measure in $\mathring{H}_{\mathcal{C}}$ being $\mathcal{S}$ a finite union of one dimensional differentiable objects.) Then, consider R large enough so that all the centers $\{c_j\}_j$ lie in the ball $B_R(p)$ centered at p and with radius R, and define the map

$$\begin{aligned} \varphi : \mathbb{R}^2 \setminus p &\to \partial B_R(p) \\ x &\mapsto p + R\frac{x-p}{|x-p|}. \end{aligned} \tag{16}$$

The set $\varphi(\mathcal{C})$ consists of N points on the circle $\partial B_R(p)$ so they can be labelled according with an orientation. Denote by $\{\tilde{c}_j\}_j$ such points and construct the polygonal $\tilde{\mathcal{P}}$ with $\{\tilde{c}_j\}_j$ as vertices and $\tilde{l}_j = \tilde{c}_{j+1} - \tilde{c}_j$, $j = 1, \ldots, N$ as edges. It turns out that p is within the convex hull of $\varphi(\mathcal{C})$ (indeed if $p = \sum_i \alpha_i c_i$, $\sum \alpha_i = 1$, it follows $p = \sum_i \beta_i \varphi(c_i)$ where $\beta_i = \alpha_i |c_i - p| / \sum_j \alpha_j |c_j - p|$) and in particular it is inside the polygonal $\tilde{\mathcal{P}}$. Thus, denoting by (ρ, θ) a system of polar coordinates centered at p, it follows that any edge $\tilde{l}_j$ stays in a strip $(0, R) \times \Theta_j$ so that $\mathring{\Theta}_j \cap \mathring{\Theta}_k = \emptyset$. Hence, labelling the centers $\{c_j\}_j$ so that $\varphi(c_j) = \tilde{c}_j$, the polygonal $\mathcal{P}$ with edges $l_j = c_{j+1} - c_j$ results to be simple because each l_j lives in the strip $(0, R) \times \Theta_j$.

For instance, let $\mathcal{C} = \{c_1, \ldots, c_7\}$ be the set of seven centres as in figure 15 b). The all the following words

$$W_1 = l_1 l_3 l_7 l_4 l_2 l_6, \quad W_2 = l_2 l_5 l_3 l_7, \quad W_3 = l_2 l_6 l_2 l_7 l_5 l_1 l_6 l_3 l_5 l_7$$

$$W_4 = l_4 l_2 l_7 l_5 l_3 l_6 l_1 l_4 l_1 \quad W_5 = l_3 l_6 l_3 l_7 l_5 l_1 l_5 l_1$$

generate a curve γ_W so that $[\gamma_W]$ is admissible. Figure 16 depicts the loops γ_{W_1}, γ_{W_2} and γ_{W_3}.

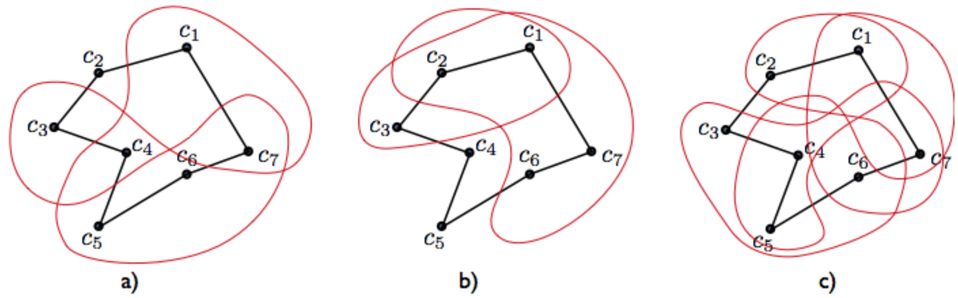


Figure 16: The three curves represent γ_W in case W is W_1, W_2, W_3 respectively.

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Disclosure of potential conflict of interest

The author declares that he has no conflict of interest.

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