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On the variational approach to the one and N-centre problem with weak forces

Thesis submitted for the degree of *Dottore di Ricerca*

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Introduction

This thesis is devoted to the study of the solutions of the dynamical system associated to the two-body problem and to the planar N-centre problem. Besides the fundamental role that both these problems play in the Celestial Mechanics, they are viewed with great interest also in other areas of classical and quantum mechanics and theoretical physics: for instance the 2-centre problem is regarded as a model for the motion of a satellite in the neighbourhood of two massive planets, as like as a model for the interactions in a diatomic molecule. Moreover, provided that different types of attraction is considered, the bodies can be replaced by other attractive objects: for instance the interactions between vortices is modelled in the same way of bodies, once the logarithmic potential in place of the gravitational potential is considered, [13].

From the point of view of classical mechanics, the systems of the N-body problem as well as the N-centre problem, are hamiltonian systems where the hamiltonian function is the energy. Unfortunately only two of these systems are completely integrable: the two-body problem and the two-center problem, [4, 5, 6, 14, 27]. However, also in the integrable cases, the collisions play a crucial role in the study of the dynamical systems and represent the major technical obstacle in any work on this type of problem: indeed the collisions are the source of the non completeness of the flow and they are responsible of the chaotic behaviour that the system may present. Our efforts are mainly dedicated to overcame this hurdle and different methods, coming from an hamiltonain point of view, as like as variational approach, are applied.

The work is principally composed by two parts.

In the first part we study the two-body problem when the particles interact with different potentials and we are mostly interested in the case the potential is of logarithmic type. In particular we investigate the connections between the solutions of the two-body problem with α-homogeneous potential functions, i.e. $V(x) = \frac{1}{|x|^\alpha}$, $\alpha \leq 2$, $\alpha \neq 0$ and the one with the logarithmic potential, that can be considered a limiting case for α leading to zero. To this purpose we analize, in chapter 1, the integrability of the system and we derive the equations of motion, while, in chapter 2, starting from a classical theorem of Betrand [3] and by means of a generalization of the Levi-Civita transformation [17], the orbital structure of the collisionless solutions are described. The obtained results, mainly the one concerning the behaviour of the near-collision orbits in presence of a logarithmic potential, suggest the idea to regularize the dynamical system. Then we focus our analysis on solutions leading to collision and, motivated by the discussion in [2], we study the possibility to extend the flow beyond the singularity due to the presence of the collisions: in chapter 3 we deal with some of the regularization techniques known in the literature [17, 16, 19, 20, 22] and we prove that for a large class of central problems the regularitazion of the flow can be attained. In particular, in section 3.5, we prove that for a class of singular potential satisfying a logarithmic type property the collision solutions may be extended following the technique of smoothing of the potential introduced in [2]. Some properties of the extended flow are studied as the degree of continuity of the Poincaré map with respect initial data and the continuity on the Poincaré section.

In the second part of this thesis we concentrate ourselves on the study of the planar N-centre problem, restricted to the gravitational case.

In chapter 4 we deal with the planar two-centre problem: we show the classical results on the integrability of the system and the classification of the bounded orbits [9, 8, 25, 26, 27]. Then, in chapter 5, we join a variational approach and we look for solution of the N-centre problem as critical points of the associated action functional. Again the collisions correspond to the singularities of the flow, thus our contribution is devoted to ensure the solutions to avoid the collision. We show, mainly in section 5.2, how the Levi-Civita transformation and the introduction of suitable sets of paths with certain topological constraints ensure the existence of minimizers for the action free of collisions. The first result we obtain concerns a variational characterisation to the 8-shaped solutions for the two-centre problem, section 5.4.1. Then we extend this result attained for the two-centre problem to systems with a larger number of centres. The most interesting result we obtain, discussed in section 5.4.2, concerns the existence of infinitely many periodic and non-collision solutions with a prescribed topological structure, provided that the centres are placed in symmetric positions with respect an axis.

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Part I

On the singularities of the two body problem

Chapter 1

Introduction

The two-body problem studies the motion of two particles subjected to a force depending only on their mutual distance. We are interested in the study of the two-body problem when the particles, with positions q_1, q_2 interact with a potential V proportional $1/|q_1 - q_2|^{\alpha}$. The motion is governed by Newton's law:

$$m_j \ddot{q}_j = \frac{\partial V}{\partial q_j} \qquad j = 1, 2$$

where m_j indicate the masses of the two bodies.

In studying this problem there is no loss of generality in assuming that the center of mass of the particles is fixed at the origin: $m_1 q_1 + m_2 q_2 = 0$, so the system reduces to that of a single particle of unit mass moving in a plane under the attraction of a central force field with potential $V(x)$ proportional to $1/|x|^{\alpha}$.

The potential functions $V_\alpha(x)$ we will consider are

$$V_\alpha(x) = \frac{1}{\alpha}\frac{1}{|x|^{\alpha}} \qquad \alpha \in \{(-\infty, 2) \setminus 0\}$$

$$V_0(x) = -\log|x| \qquad \alpha = 0$$

Since for $\alpha \neq 0$ the potential functions $V_\alpha(x)$ satisfy the relation $V_\alpha(\lambda x) = |\lambda|^{-\alpha} V_\alpha(x)$, we refer to α as the **Degree of homogeneity** of the potential and we use the notation α-homogeneous potential in order to indicate $V_\alpha(x)$ for $\alpha \neq 0$ and logarithmic potential to indicate $V_0(x)$. Moreover, since for non negative values of α the potential is undefined in $x = 0$, we sometimes call regular the potential $V_\alpha(x)$ with $\alpha < 0$ and singular the potential $V_\alpha(x)$ with $\alpha \geq 0$. We observe that the class of potential functions we have defined includes the classical Kepler potential, for $\alpha = 1$, and the harmonic potential for $\alpha = -2$.

The two body problem with potential function $V_\alpha(x)$ consists in finding a solution $u \in \mathcal{C}^2(T, \mathbb{R}^2)$ associated with the initial value problem

$$\begin{cases} \ddot{u} = \nabla V_\alpha(u) = -\dfrac{u}{|u|^{2+\alpha}} \\ \\ (u(0), \dot{u}(0)) = (q_0, p_0) \end{cases} \tag{1.1}$$

where T is the maximal interval of existence of solution.

We remark that for every $\alpha > -2$ the differential equation of the system (1.1) is singular in the origin, that is the equation is undefined in $|u| = 0$ since the right hand side approaches to $-\infty$ as $|u| \to 0$. On the other hand if $|u| \neq 0$ the force term $\nabla V_\alpha(u)$ is a real analitic vector field then, for every initial data (q_0, p_0) with $q_0 \neq 0$, the classical theorem of existence and uniqueness of solution of ODE, ensures the existence of a unique solution $u(t)$ defined for $t \in [0, t^+]$, where $0 < t^+ \leq \infty$.

Definition 1.1. *If $t^+ < \infty$ then t^+ is said a* **singularity** *of the solution $u(t)$*

Then a singularity of a solution of system (1.1) is a time t^+ beyond which the solution cannot be extended as a smooth function of t.

A collision occurs if the two particles q_1 and q_2 coincides at some moment and this corresponds to $u = 0$, then we say that

Definition 1.2. *A* **Collision** *occurs if $u(t) = 0$ for a certain $t > 0$.*

From this definition follows that, in case $\alpha > -2$, if a solution $u(t)$ has a collision at time $\bar{t}$ then $\bar{t}$ is a singularity of $u(t)$ since the differential equation is undefined for $|u| = 0$. In general singularities of solution don't correspond to singularities of the differential equation, indeed a solution may blow up in finite time. For the two body problem we are taking into account this is not the case: in [19] it's shown that collisions are the only singularities that system (1.1) can develop for $\alpha > 0$, i.e. if $t^* > 0$ is a singularity of $u(t)$, then $u(t) \to 0$ as $t \to t^*$. With similar calculation it's easy to prove that this is true also for $\alpha \geq -2$. Thus

Theorem 1. *For every $\alpha > -2$, if $u(t)$ is a solution of* (1.1) *with a singularity at $\bar{t}$, then $u(t) \to 0$ as $t \to \bar{t}$.*

From a point of view of classical mechanics, the system (1.1) is an Hamiltonian system and admits two integral of motion, i.e. quantities that are constant along solutions of the differential equation: the **Energy integral** E and the **Angular momentum integral** l.

The energy $E(u)$ represents the total energy of the particle whose motion is described by $u(t)$ and it is given by the sum of the kinetic and potential energy while the angular momentum $l(u)$ is given by the cross product between the position and the velocity. Due to the rotational symmetry of the system it is convenient to express the energy and the angular momentum integral in terms of polar coordinates (r, θ) as well as cartesian coordinates.

In the α-homogeneous case the energy is given by

$$E_\alpha(u) = \frac{1}{2}|\dot{u}|^2 - \frac{1}{\alpha}\frac{1}{|u|^\alpha} \qquad E_\alpha(u) = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{\alpha}\frac{1}{r^\alpha}$$

while in the logarithmic case

$$E_0(u) = \frac{1}{2}|\dot{u}|^2 + \log|u| \qquad E_0(u) = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \log r$$

By definition, at every instant t, the angular momentum $l(u)$ of a particle moving on the trajectory $u(t)$ is a vector perpendicular to the plane where $u(t)$ and $\dot{u}(t)$ live, then in general the angular momentum changes in norm and direction. On the other hand, since the central force problem we are studying are necessarily planar, without loss of generality, we denote with l the norm of the angular momentum that, for every $\alpha \in [-1, 2]$, is given by

$$l(u) = |u \wedge \dot{u}| \qquad\qquad l(u) = r^2\dot{\theta}$$

It follows that the energy, at least in non collision instants, can be rewritten in the form

$$E_\alpha(u) = \frac{1}{2}\left(\dot{r}^2 + \frac{l^2}{r^2}\right) - \frac{1}{\alpha}\frac{1}{r^\alpha} \qquad E_0(u) = \frac{1}{2}\left(\dot{r}^2 + \frac{l^2}{r^2}\right) + \log r \tag{1.2}$$

As previously said, the energy and the angular momentum are constant along solutions of equation of motion (1.1)

$$\frac{d}{dt}E(u(t)) = \dot{u}\ddot{u} + \frac{1}{|u|^{\alpha+2}}u\dot{u} = \dot{u}\left(\ddot{u} + \frac{u}{|u|^{\alpha+2}}\right) = 0$$

and

$$\frac{d}{dt}l(u(t)) = \dot{u} \wedge \dot{u} + u \wedge \ddot{u} = -u \wedge \frac{u}{|u|^{\alpha+2}} = 0$$

thus, since we are dealing with systems with two degrees of freedom and there always exist at least two integrals of motions, the two body problem are Liouville integrable.

A different approach to the problem is to consider the system (1.1) following its characterization as a Lagrangian dynamical system. In this formulation the **Lagrangian** function is the sum of the kinetic energy and the potential function V, then in cartesian and polar coordinates, the Lagrangian associated to the two body problem is defined by

$$\begin{array}{lll} L_\alpha(u) = \frac{1}{2}|\dot{u}|^2 + \frac{1}{\alpha}\frac{1}{|u|^\alpha} & L_\alpha(u) = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{\alpha}\frac{1}{r^\alpha} & \alpha \neq 0 \\ L_0(u) = \frac{1}{2}|\dot{u}|^2 - \log|u| & L_0(u) = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \log r & \alpha = 0 \end{array}$$

For the Principle of least action, the solutions of the dynamical system corresponds to stationary points of the action functional

$$\mathcal{A}(u) = \int_T L_\alpha(u)dt$$

defined on a suitable set of paths.

The equation of motion obtained from the functional equation $\frac{\delta\mathcal{A}}{\delta u} = 0$ are the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_i} - \frac{\partial L}{u_i} = 0$$

where u_i are the components of the vector $u(t)$. Applying this formula we achieve the equations of motion of the radial and angular coordinate

$$\ddot{r} - r\dot{\theta}^2 + \frac{1}{r^{\alpha+1}} = 0$$
$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

We observe that the second equation proves that the angular momentum is an integral of motion while we rewrite the first equation in the usual form

$$\ddot{r} - \frac{l^2}{r^3} + \frac{1}{r^{\alpha+1}} = 0 \tag{1.3}$$

Although from a mechanic point of view the two-body problem is solved, since it's completely integrable, explicit solutions of system (1.1) are known only in the case of Kepler and harmonic potential. Our aim is to investigate the shape of solutions in relation with the integrals of motion and with the different degree of the potential. In particular we are mostly interested in understand the behaviour of a solution as it approaches to a collision in order to extend, if it is possible, a solution beyond the singularity. Moreover we will emphasise the different behaviour of solutions in case of regular and singular potential and in which sense the logarithmic potential problem can be considered the limit case of the α-homogeneous problems.

Before starting with the study of these questions, we show that for the class of potential we are taking into account, a collision occurs if and only if the angular momentum is zero.

Theorem 2. *If $\alpha < 2$ then the initial condition (q_0, p_0) leads to collision if and only if the angular momentum $l = q_0 \wedge p_0 = 0$.*

Proof
Suppose $\alpha < 2$ and that the solution $u(t)$ has a collision. From relation (1.2) it follows $r^2E_\alpha - \frac{1}{2}l_\alpha^2 + \frac{1}{\alpha}r^{2-\alpha} \geq 0$ then, since E_α and l_α are constant of motion, as r approaches to zero we have $l_\alpha^2 \leq 0$. Viceversa, if $l = 0$ and the orbit is bounded, the trajectory lies on a line $\theta = const$ and has a collision. □

If $\alpha \geq 2$ the set of initial conditions leading to collision is rather larger than the zero angular momentum ones and includes the set for which the total energy E is negative. See [2] for a description of this kind of solutions.

In order to describe the behaviour of the orbits we give the following definitions.

Definition 1.3. *Let $u(t)$ a collisionless, bounded solution of system (1.1). We call R_-^α and R_+^α respectively the* ***minimal and maximal value of the radial coordinate*** *of the solution.*

Following the terminology adopted in the celestial mechanics to describe the system Sun-Earth we sometimes used the notation pericentre and apocentre to indicate respectively R_-^α and R_+^α.

From relation (1.2) it follows that, given the constants of motion E and l, an orbit exists only for that positive values of radial coordinate satisfying the relation

$$2(E + V_\alpha) - \frac{l^2}{r^2} \geq 0$$

Such values, if exist, belong to an interval whose extremal points are the minimal and maximal values of the radial coordinate R_-^α, R_+^α. It may happens that the interval of solution of the previous relation is not bounded from above: in this case we say that the orbit is unbounded and we define $R_+^\alpha = +\infty$. On the other hand, if a collision occurs, we set $R_-^\alpha = 0$.

Definition 1.4 (Apsidal angle)**.** *Given a solution $u(t)$ of system* (1.1)*, we define the* ***apsidal angle*** *$\Delta_\alpha\theta(u)$ as the angle between the vectors joining two consecutive apocentre and pericentre of the orbit $u(t)$.*

The apsidal angle corresponds to the increment of the angular coordinate $\theta(t)$ of solution $u(t)$ between a minima and the following maxima of $r(t)$. The conservation of the angular momentum and relation (1.2) imply

$$\begin{aligned} \Delta_\alpha\theta(u) &= \int \dot\theta dt = \int \frac{l}{r^2} dt = \int_{R_-^\alpha}^{R_+^\alpha} \frac{l}{r^2 \dot r} dr \\ &= \int_{R_-^\alpha}^{R_+^\alpha} \frac{l}{r^2\sqrt{2(E+V_\alpha) - \frac{l^2}{r^2}}} dr \end{aligned} \tag{1.4}$$

The apsidal angle is the topic on which, more than all, we will concentrate our study in this chapter since it gives us a measure of how much the particle winds around the center of attraction during the motion.

Theorem 3 (Binet)**.** *Let $\varphi(r)$ the force at distance r acting on a particle and directed towards the center of attraction, (r, θ) the polar coordinate of the particle and l the angular momentum. Then it holds*

$$\frac{d^2 z}{d\theta^2} = \frac{\tilde\varphi(z)}{l^2 z^2} - z$$

where $z = \frac{1}{r}$, $\tilde\varphi(z) = \varphi(r)$.

Proof

Derive $\frac{1}{r}$ with respect to θ:

$$\frac{d}{d\theta}\frac{1}{r} = \frac{d}{dt}\frac{1}{r}\frac{dt}{d\theta} = -\frac{1}{r^2}\dot r \frac{1}{\dot\theta} = -\frac{\dot r}{l}$$

$$\frac{d^2}{d\theta^2}\frac{1}{r} = \frac{d}{d\theta}\left(-\frac{\dot r}{l}\right) = \frac{d}{dt}\left(-\frac{\dot r}{l}\right)\frac{1}{\dot\theta} = -\frac{\ddot r}{l}\frac{1}{\dot\theta} = -\frac{\ddot r r^2}{l^2}$$

From equation (1.3), generalized to every radial forces $\varphi(r)$, we have

$$\ddot{r} - \frac{l^2}{r^3} + \varphi(r) = 0$$

then

$$\frac{d^2}{d\theta^2}\frac{1}{r} = -\frac{1}{r} + \varphi(r)\frac{r^2}{l^2}$$

□

1.1 Shape of the orbits

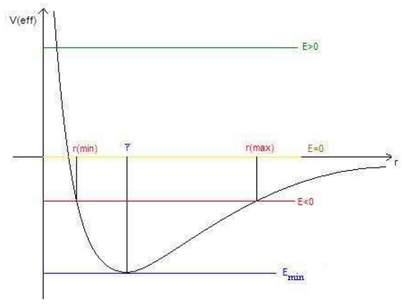
We give now a brief and qualitative description of the shape of the orbits solutions of the two body problem as the degree $\alpha \in [-2, 1]$ and the constant E, l change. We first rewrite the energy formula (1.2)

$$E_\alpha(u) = \frac{1}{2}\dot{r}^2 + V^\alpha_{eff}$$

where $V^\alpha_{eff} = \frac{1}{2}\frac{l^2}{r^2} - V_\alpha$. The energy integral is the sum of a positive part, concerning the radial velocity, plus a term depending only on the radial coordinate and on constant of motion. It turns out that, given the constants of motion E and l, an orbit could exists only for that positive values of radial coordinates satisfying the relation $E \geq V^\alpha_{eff}$. Such values, if exist, belong to an interval whose extremal points, by definition, are the minimal and maximal values of the radial coordinate R^α_-, R^α_+. If $0 < R^\alpha_- \leq R^\alpha_+ < \infty$ the orbit is bounded and the radial coordinate $r(t)$ oscillates between the minimal and the maximal values. In the meantime the angular coordinate $\theta(t)$ increase its value following the relation $\dot{\theta} = \frac{l}{r^2}$. Since, for convenience, the angular momentum is positive defined, the anticlockwise motion is chosen and this direction can not be reversed during the motion. By definition of the apsidal angle, it follows that the quantity $\frac{\Delta_\alpha \theta}{\pi}$ represent the fraction of one revolution that one goes through between successive meeting of the orbit with the apocentre or pericentre. In particular if such fraction is rational then a periodic solution exists, otherwise quasiperiodic.

- **Keplerian case** $\alpha = 1$.

  In the figure above the function V^1_{eff} is drawn for a fixed value of angular momentum l and three different levels of energy are chosen.

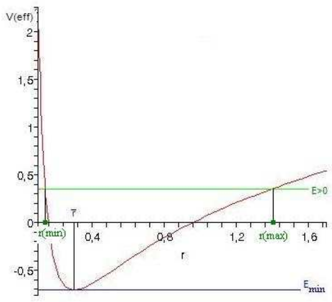


  The minimal of V^1_{eff} is achieved for $\bar{r} = l^2$ and its value is $-\frac{1}{2l^2}$ then there exists orbits only for energy $E \geq -\dfrac{1}{2l^2}$. In particular if $E = -\frac{1}{2l^2}$ the orbit is circular, $r(t) = \bar{r}$, if $-\frac{1}{2l^2} < E < 0$ the orbit is bounded and, as it's well known, the particle describes an ellipse where the center of attraction is one of its foci. Then $\Delta_1\theta = \pi$ and the values R^1_- and R^1_+ are the length of the periapsis and apoapsis of the ellipse. Otherwise $E \geq 0$ corresponds to unbounded orbits, parabolic for $E = 0$ and hyperbolic for $E > 0$.

- **α-homogeneous case** $0 < \alpha < 1$

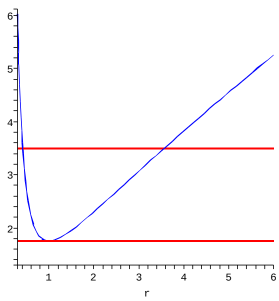
  This case is similar to the previous one: a solution exists for $E \geq -\dfrac{2-\alpha}{2\alpha} l^{-\frac{2\alpha}{2-\alpha}}$ and the orbits are bounded if $E < 0$ and unbounded if $E \geq 0$. For bounded orbits, the values of R^α_-, R^α_+ are the positive solutions of equation $-\dfrac{1}{2}\dfrac{l^2}{r^2} + \dfrac{1}{\alpha}\dfrac{1}{|r|^\alpha} + E = 0$. The difference with respect to the Keplerian case is that the shape of the orbits is no longer an ellipse, but, as we will show in the following sections, it changes with α and with E and l.

- **Logarithmic-potential case** $\alpha = 0$



In this case the function $V^0_{eff} = \frac{1}{2}\frac{l^2}{r^2} + \log r$ attains its minimal value $\frac{1}{2} + \log l$ for $\bar{r} = l$ and, for every $l > 0$, $\lim_{r\to 0} V^0_{eff} = \lim_{r\to\infty} V^0_{eff} = +\infty$. From this considerations it descends that there exist orbits for energy value $E \geq \frac{1}{2} + \log(l)$ and that all the orbits are bounded. The maximal and minimal values of the radial coordinate are solution of $-\dfrac{1}{2}\dfrac{l^2}{r^2} - \log(r) + E = 0$.

- **Regular-potential** $-2 < \alpha < 0$



Since $\alpha < 0$, the potential function V_α is negative, thus the energy E of a solution has to be positive. More precisely, once the value of angular momentum has been chosen, the energy has to satisfy $E \geq -\frac{2-\alpha}{2\alpha} l^{-\frac{2\alpha}{2-\alpha}}$. All the orbits are bounded and R^α_-, R^α_+ are solutions of equation $-\dfrac{1}{2}\dfrac{l^2}{r^2} + \dfrac{1}{\alpha}\dfrac{1}{|r|^\alpha} + E = 0$.

- **Harmonic potential** $\alpha = -2$
  As in the previous case, solutions exist provide the constants of motion satisfy $E \geq l$. All the solutions of the Harmonic, also called elastic, problem are explicitly known and are given as a combination of trigonometric functions. All the orbits lie on ellipses centered in the origin whose semiaxis have length equal to R^α_-, R^α_+. Clearly, for every solution $u(t)$, $\Delta_{-2}\theta(u) = \frac{\pi}{2}$.

We have said something about $\Delta_\alpha\theta$ only in the case $\alpha = 1, \alpha = -2$ because, as we will see later, the Keplerian and harmonic are the only attractive law that impose the orbits to be closed and for which the apsidal angle does not depend on the value of energy and angular momentum.

1.2 A generalization of the Levi-Civita transformation

A solution of the two body problem, at least for $\alpha > -2$, ends to exists when a collision occurs since the Newton's equation that describes the motion is singular in the origin. A regularization theory is a method to transform singular differential equations into regular ones, providing a mathematical tool to analyse motion leading to collisions. In this framework the classical Levi-Civita transformation, first introduced in [17], is a useful method to regularise the planar Kepler problem: by a suitable time-space variable changing the singularity of the Newton's equation is removed and the Kepler problem is equivalent to the harmonic equation. Essentially the Levi-Civita regularization is based on three steps: a change of coordinate $u = w^2$ in the complexified space, the introduction of a frictitious time in order to overcome the fact that the velocity become infinity at the singularity and, finally, the use of the conservation of energy.

A generalization of the Levi-Civita transformation in the three dimensional space is known as Kustaanheimo-Stiefel transformation [16, 20].

Roughly speaking the Levi-Civita transformation works well since the singular Kepler problem is trasformed into the regular Harmonic problem, that is the α-homogeneous two body problem with $\alpha = 1$ is linked to the problem with $\alpha = -2$. We show that this procedure can be generalized to problems with degree of homogeneity $\alpha \in (0, 2]$; the two body problem with singular potential V_α, $\alpha \in (0, 2]$ are, in same sense, equivalent to the ones with a regular potential $V_{-\beta}$, $\beta \in [0, \infty)$. See also [19, 12]

Theorem 4 (Levi-Civita)**.** *Let $\alpha \in (0, 2]$ and $\beta = \frac{2\alpha}{2-\alpha}$. The systems*

1. $\ddot{u} = -\dfrac{u}{|u|^{2+\alpha}}$
2. $w'' = -K\dfrac{w}{|w|^{2-\beta}}$

are equivalent in the sense that if $u(t)$ is a solution of equation 1 with apsidal angle $\Delta_\alpha\theta$, then there exists a solution $w(s)$ of system 2 with apsidal angle

$$\Delta_{-\beta}\theta(w) = \frac{2-\alpha}{2}\Delta_\alpha\theta(u) \tag{1.5}$$

The constant K depends on the energy E of $u(t)$ and on α.

Proof
Let $\alpha \in (0, 2]$ and $x(t)$ be a solution of system *1* with energy E.

We define the following change of coordinate as a generalization of the Levi-Civita transformation: consider $x(t)$ in complex form, $x(t) = |x| \exp(i\varphi)$, and define $w = |w| \exp(i\vartheta)$ such that $x = w^\gamma$. This means $|x| = |w|^\gamma$, $\varphi = \gamma\vartheta$. It follows

$$|\dot{x}|^2 = \gamma^2|w|^{2(\gamma-1)}|\dot{w}|^2$$

In the new coordinate w the Lagrangian and the energy integral E (1.2) have the form

$$L(w, \dot{w}) = \frac{1}{2}\gamma^2|w|^{2(\gamma-1)}|\dot{w}|^2 + \frac{1}{\alpha}\frac{1}{|w|^{\alpha\gamma}}$$

$$E = \frac{1}{2}\gamma^2|w|^{2(\gamma-1)}|\dot{w}|^2 - \frac{1}{\alpha}\frac{1}{|w|^{\alpha\gamma}}$$

From the Lagrangian we derive the Euler-Lagrange equations for the coordinate $w(t)$

$$\frac{d}{dt}(\gamma^2|w|^{2(\gamma-1)}\dot{w}) - \gamma^2(\gamma-1)|w|^{2(\gamma-2)}w|\dot{w}|^2 + \gamma\frac{w}{|w|^{\alpha\gamma+2}} = 0$$

We observe that the velocity $|\dot{w}|$ is still infinite at collision, indeed

$$|\dot{w}(t)| = \sqrt{c\frac{E}{|w|^{2(\gamma-1)}} + \frac{k}{|w|^{\alpha\gamma+2(\gamma-1)}}}$$

then, in order to control the increase of speed at the singularity, we introduce the frictitious time s by the time rescale

$$\frac{d}{dt} = \frac{1}{|w|^{2(\gamma-1)}}\frac{d}{ds}$$

Denoting by $w' = \frac{dw}{ds}$ it follows

$$\frac{d}{dt}(\gamma^2|w|^{2(\gamma-1)}\dot{w}) = \frac{1}{|w|^{2(\gamma-1)}}\frac{d}{ds}\left(\gamma^2|w|^{2(\gamma-1)}\dot{w}\right) = \frac{1}{|w|^{2(\gamma-1)}}\frac{d}{ds}(\gamma^2 w') = \frac{\gamma^2}{|w|^{2(\gamma-1)}}w''$$

and

$$|w|^{2(\gamma-2)}w|\dot{w}|^2 = \frac{|w|^{2(\gamma-2)}w}{|w|^{4(\gamma-1)}}|w'|^2 = \frac{w}{|w|^{2\gamma}}|w'|^2$$

Substituting these relations in the above Euler-Lagrange equation, it descends that $w(s)$ solves

$$\frac{\gamma^2}{|w|^{2(\gamma-1)}}w'' - \gamma^2(\gamma-1)\frac{w}{|w|^{2\gamma}}|w'|^2 + \frac{\gamma w}{|w|^{\alpha\gamma+2}} = 0 \tag{1.6}$$

In the new space-time variable (w,s) the energy integral is written as

$$E = \frac{1}{2}\gamma^2|w|^{2(\gamma-1)}\frac{|w'|^2}{|w|^{4(\gamma-1)}} - \frac{1}{\alpha}\frac{1}{|w|^{\alpha\gamma}}$$

therefore

$$\frac{|w'|^2}{|w|^{2\gamma}} = \frac{2}{\gamma^2}\left(E + \frac{1}{\alpha}\frac{1}{|w|^{\alpha\gamma}}\right)$$

After replacing in (1.6) we obtain

$$w'' = E\frac{2(\gamma-1)}{\gamma^2}w|w|^{2\gamma-4} + \frac{(2-\alpha)\gamma-2}{\alpha\gamma}w|w|^{2(\gamma-1)-2-\alpha\gamma}$$

Choose

$$\gamma = \frac{2}{2-\alpha}$$

so that

$$w'' = \frac{(2-\alpha)\alpha}{2}E\frac{w}{|w|^{2-\frac{2\alpha}{2-\alpha}}} = -K\frac{w}{|w|^{2-\beta}}$$

□

The same result, obtained with a different approach, is described in [12].

We remark that in the case of Kepler potential, $\alpha = 1$ the transformation $u = w^\gamma$ is the same adopted by the classical Levi Civita regularization for the Kepler problem, $u = w^2$. In this sense the transformation $u = w^\gamma$, $\gamma = \frac{2}{2-\alpha}$ can be considered as a generalization of the Levi Civita one. On the other hand it would be improper to define this theory an extension of the Levi-Civita regularization, since for $\alpha \in (0,1)$ the corresponding β belongs to $(0,2)$ and the system *2* is singular anyway. The theorem is useful because a relation between the apsidal angle of solutions of different α-homogeneous problem is given. More precisely, the α-homogeneous problem with $\alpha \in (0,1)$ are associated to the ones with $\alpha \in (-2,0)$ and the problems with degree $\alpha \in [1,2)$ are linked to the ones with $\alpha \in (-\infty,-2]$:

$$\begin{array}{rcl} \text{singular potential} & & \text{regular potential} \\ (0,1) & \longleftarrow - - - - - - - - \longrightarrow & (-2,0) \\ 1 & \longleftarrow - \textit{ Levi Civita } - \longrightarrow & -2 \\ (1,2) & \longleftarrow - - - - - - - - \longrightarrow & (-\infty,-2) \end{array}$$

Following this scheme and relation (1.5), informations obtained about the apsidal angle of a solution of the α-homogeneous problem for certain values of α can be easily extended to a bigger set of degree of homogeneity. The behalf of this approach is that sometimes could be easier to infer results for regular potential, instead of singular potential.

Chapter 2

The behaviour of the apsidal angle

In this chapter we study the apsidal angle $\Delta_\alpha\theta(u)$ for solutions of system (1.1) with $\alpha \in (-\infty, 2]$. Looking at the definition (1.4), one can realize that in general the apsidal angle of a solution of the α-homogeneous problem depends on the constants of motion E and l of the solution itself and that the difficulties arise from the fact that integrand in formula (1.4) is singular at the endpoints and the endpoints themselves are implicitly defined. We have mentioned in §1.1 that the value of the apsidal angle is well known in the case $\alpha = 1, -2$ and it is constant with respect to the energy and angular momentum of the orbits, but this semplification holds only in these classical cases. Otherwise, for different value of degree α, the apsidal angle of a solution strictly depends on the constants of motion: we would like to study this dependence and to this aim, far from obtain a complete description of such phenomena, we will prove an over and upper bound for the apsidal angle and its asymptotical limit for near-circular and near-radial orbits. The theorem 4 and the accomplished remarks, suggest to split the discussion in two section: we first study the apsidal angle in the case of α-homogeneous potential, §2.1, then in case of logarithmic potential §2.2. Some of these results, obtained from different approach, could be found in the paper of Touma, Tremaine [24] where the case $\alpha \in [-2, 1]$ is considered.

2.1 The apsidal angle for α-homogeneous potential problem

In the following we suppose $\alpha \in (-\infty, 2]$, $\alpha \neq 0$. The apsidal angle of a solution $u(t)$ is a function of both the constants of motion E and l, but, as we will show in the following theorem, the homogeneity of the potential implies that the apsidal angle is energy-scale free. This means that we can choose a value of energy and study the apsidal angle as function of angular momentum l.

Theorem 5 (Scaling invariance)**.** *For fixed $\alpha \in (-\infty, 2)$, $\alpha \neq 0$, let $u(t)$ be a solution of the α-homogeneous 2-body problem* (1.1) *whose energy, angular momentum and apsidal angle are denoted by $E, l, \Delta_\alpha(u)$ respectively.*

Then, for every $\lambda > 0$, the rescaled path

$$v(t) = \lambda u\left(\lambda^{-\frac{\alpha+2}{2}} t\right) \tag{2.1}$$

is again a solution of the α-homogeneous problem and the corresponding energy E_v, angular momentum l_v, apsidal angle $\Delta_\alpha(v)$ are mapped under the scaling (2.1) *by*

$$E_v = \lambda^{-\alpha} E \qquad l_v = \lambda^{\frac{2-\alpha}{2}} l \tag{2.2}$$
$$\Delta_\alpha(v) = \Delta_\alpha(u)$$

Proof

Denoting with $\bar{t} = \lambda^{-\frac{\alpha+2}{2}} t$, from straightforward calculations, we glean

$$\ddot{v}(t) = \lambda^{-(\alpha+1)} \ddot{u}(\bar{t}) = -\frac{\lambda^{-(\alpha+1)}}{\lambda} \frac{v(t)}{|\frac{v(t)}{\lambda}|^{2+\alpha}} = -\frac{v(t)}{|v(t)|^{2+\alpha}}$$

thus $v(t)$ is solution of system (1.1).

The following relation between the velocities of u and v

$$\dot{v}(t) = \lambda \lambda^{-\frac{\alpha}{2}-1} \dot{u}(\bar{t}) = \lambda^{-\frac{\alpha}{2}} \dot{u}(\bar{t})$$

implies

$$E_v = \frac{1}{2}|\dot{v}(t)|^2 - \frac{1}{\alpha}\frac{1}{|v(t)|^\alpha} = \frac{1}{2}\lambda^{-\alpha}|\dot{u}(\bar{t})|^2 - \frac{1}{\alpha}\lambda^{-\alpha}\frac{1}{|u(\bar{t})|^\alpha} = \lambda^{-\alpha} E$$

and

$$l_v = v(t) \wedge \dot{v}(t) = \lambda u(\bar{t}) \wedge \lambda^{-\frac{\alpha}{2}} \dot{u}(\bar{t}) = \lambda^{\frac{2-\alpha}{2}} l$$

then (2.2) are shown.

In order to prove that the apsidal angle does not change under the scaling (2.1) we first have to calculate the values of the pericentre $\tilde{r}_-$ and apocentre $\tilde{r}_+$ of the orbit $v(t)$.

By definition $\tilde{r}_-$ and $\tilde{r}_+$ are the positive solutions of equation $2E_v + \frac{2}{\alpha}\frac{1}{r^\alpha} - \frac{l_v^2}{r^2} = 0$. For the relations (2.2) $\tilde{r}_\pm$ are solutions of $\lambda^{-\alpha}(2E + \frac{2}{\alpha}\frac{\lambda^\alpha}{r^\alpha} - \frac{\lambda^2}{r^2} l^2) = 0$, but for definition of R_-^α and R_+^α we infer that

$$\tilde{r}_\pm = \lambda R_\pm^\alpha$$

where $R_\pm^\alpha$ indicate the minimal and maximal values of radial coordinate of solution $u(t)$.

Thus

$$\begin{aligned}\Delta_\alpha(v) &= \int_{\tilde{r}_-}^{\tilde{r}_+} \frac{l_v}{s\sqrt{2E_v + \frac{2}{\alpha}\frac{1}{s^\alpha} - \frac{l_v^2}{s^2}}} ds = \int_{\tilde{r}_-}^{\tilde{r}_+} \frac{\lambda^2 l}{s\sqrt{2E + \frac{2}{\alpha}\frac{\lambda^\alpha}{s^\alpha} - \lambda^2 \frac{l^2}{s^2}}} ds \\ &= \int_{R_-^\alpha}^{R_+^\alpha} \frac{l}{r\sqrt{2E + \frac{2}{\alpha}\frac{1}{r^\alpha} - \frac{l^2}{r^2}}} dr = \Delta_\alpha(u)\end{aligned}$$

where again the relations (2.2) are used and the variable changing $s = \lambda r$ is performed. □

As a consequence of the scaling symmetries it follows that for every point P_0 in the energy-angular momentum plane there exists a line passing through P_0 where the apsidal angle $\Delta_\alpha \theta(E, l)$ is constant. More precisely the level set passing through $P_0 = (E_0, l_0)$ is given by

$$E = K l^{\frac{2\alpha}{2-\alpha}}$$

where $K = E_0 l_0^{-\frac{2\alpha}{2-\alpha}}$.

Remark 2.1. *Though the apsidal angle is well defined also for unbounded solutions, we are mostly interested on its behaviour in case of bounded orbits. To this purpose we assume that if $u(t)$ is a solution of the α-homogeneous problem with $\alpha \in (0, 2)$ then the energy of u is negative.*

In case $\alpha \in (-\infty, 0)$ no assumption on the value of the energy is required, since a solution exists only if the energy is positive and all the solutions are bounded.

Moreover for every $\alpha \in (-\infty, 2)$, $\alpha \neq 0$, given a value of energy E, we denote with l_{max} the quantity

$$l_{max} = \left(-\frac{2\alpha}{2-\alpha}E\right)^{-\frac{2-\alpha}{2\alpha}}$$

Clearly l_{max} is the maximal value of the angular momentum that a bounded solution of the α-homogeneous problem with energy E can support and the solution with energy E and angular momentum l_{max} corresponds to the circular orbit. Therefore, once one fixes the value of energy according to remark 2.1, a solution of system (1.1) exists for every value of angular momentum in the range $[0, l_{max}]$ and we denote near-radial the orbits with angular momentum approaching to zero and near-circular if the angular momentum is close to l_{max}.

We start the study of the apsidal angle recalling the classical Bertrand Theorem [3].

Theorem 6 (Bertrand's Theorem)**.** *The only two central force laws that give rise to closed orbits independent by of the choice of initial conditions are linear and inverse square.*

The Bertrand's theorem states that the apsidal angle of a solution of the elastic and Kepler problem is rationally proportional to π and it does not depend on the values of energy and angular momentum. Furthermore it states that all the other laws of attraction allow the orbits to be closed, but only the Kepler and elastic one impose it.

We don't prove the Bertrand's theorem, see [3], but we follow the Bertrand argument in the proof of the next theorem. Some of the following results are contained also in [2, 24].

Theorem 7. *For every $\alpha \in (-\infty, 2), \alpha \neq 0$, a value of energy E is chosen according to remark 2.1. For every $l \in (0, l_{max})$ let $u_l(t)$ be a solutions of system (1.1) with energy E and angular momentum equal to l and denote with $\Delta^l_\alpha\theta(u)$ the apsidal angle of solution $u_l(t)$.*
Then $\forall l \in (0, l_{max})$ it holds

$$\begin{aligned}
\pi &\leq \Delta^l_\alpha\theta \leq \frac{\pi}{2-\alpha} && \alpha \in (1,2)\\
\frac{\pi}{2-\alpha} &\leq \Delta^l_\alpha\theta \leq \pi && \alpha \in (0,1)\\
\frac{\pi}{2} &\leq \Delta^l_\alpha\theta \leq \frac{2\pi}{2-\alpha} && \alpha \in (-2,0)\\
\frac{2\pi}{2-\alpha} &\leq \Delta^l_\alpha\theta \leq \frac{\pi}{2} && \alpha \in (-\infty,-2)
\end{aligned} \tag{2.3}$$

Moreover for every $\alpha \in (-\infty, 2)$, $\alpha \neq 0$

$$\lim_{l\to l_{max}} \Delta^l_\alpha\theta = \frac{\pi}{\sqrt{2-\alpha}}$$

and

$$\begin{aligned}
&\lim_{l\to 0} \Delta^l_\alpha\theta = \frac{\pi}{2-\alpha} && \text{for } \alpha \in (0,2)\\
&\lim_{l\to 0} \Delta^l_\alpha\theta = \frac{\pi}{2} && \text{for } \alpha \in (-\infty,0)
\end{aligned}$$

Proof

Let $\varphi(r) = \frac{1}{r^{\alpha+1}}$ the force at distance r acting on the particle and directed to the center of attraction, r and θ the polar coordinate of the moving point. Set $z = \frac{1}{r}$ and $\psi(z) = r^2\varphi(r)$. For theorem (3) the function $z(\theta)$ satisfies the equation

$$\frac{d^2z}{d\theta^2} + z - \frac{1}{l^2}\psi(z) = 0.$$

Multiply the last equation by $2dz$, then, by integration we obtain

$$\int 2\frac{d^2z}{d\theta^2}dz + z^2 - \frac{1}{l^2}w(z) = 0 \tag{2.4}$$

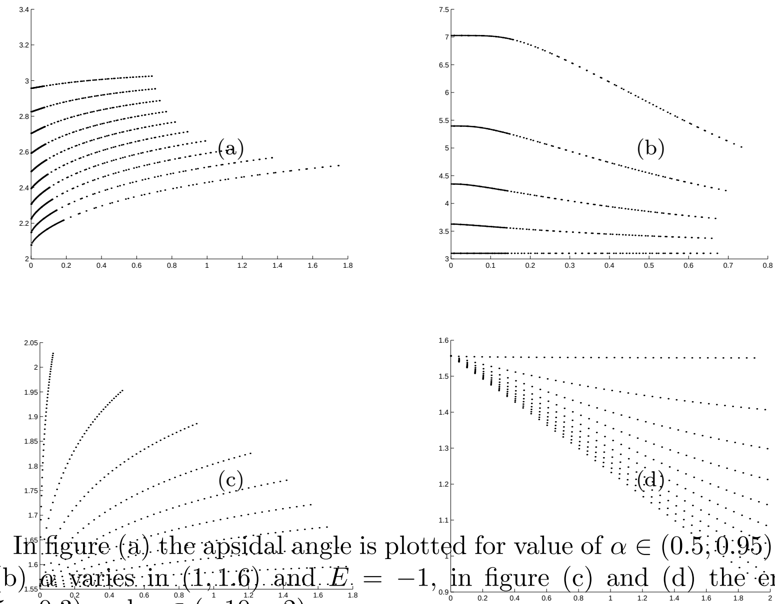


Figure 2.1: In figure (a) the apsidal angle is plotted for value of $\alpha \in (0.5, 0.95)$ and energy $E = -1$. In figure (b) α varies in $(1, 1.6)$ and $E = -1$, in figure (c) and (d) the energy is $E = 2$ and $\alpha \in (-1.95, -0.3)$ and $\alpha \in (-10, -2)$

where the function $w(z)$ is given by

$$w(z) = 2 \int \psi(z) dz$$

Since $\displaystyle\int 2\frac{d^2z}{d\theta^2}dz = \int 2\frac{d^2z}{d\theta^2}\frac{dz}{d\theta}d\theta = \left(\frac{dz}{d\theta}\right)^2 - h$, where h is an arbitrary constant of integration, from (2.4) is follows

$$\left(\frac{dz}{d\theta}\right)^2 + z^2 - \frac{1}{l^2}w(z) - h = 0 \tag{2.5}$$

We infer

$$d\theta = \pm \frac{dz}{\sqrt{h + \frac{1}{l^2}w(z) - z^2}}$$

thus the apsidal angle of a solution $u(t)$ describing the motion of a particle subjected to a central force $\varphi(r)$ is expressed by

$$\Delta^l_\alpha \theta(u) = \int_a^b \frac{dz}{\sqrt{h + \frac{1}{l^2}w(z) - z^2}} \tag{2.6}$$

where $a = \frac{1}{R^\alpha_+}$ and $b = \frac{1}{R^\alpha_-}$ denote the inverse of the pericentre and apocentre distance.

We remind that the time derivative of the radial coordinate is zero in $R^\alpha_\pm$ and this implies $\frac{dz}{d\theta} = 0$ for $z = a, b$. This consideration and formula (2.5) allow us to express the constants h and l^2 in terms of a, b. Indeed

$$h + \frac{1}{l^2}w(a) - a^2 = 0$$

$$h + \frac{1}{l^2}w(b) - b^2 = 0$$

then, as function of a and b, the constant h and l^2 have the form

$$\begin{aligned} l^2 &= \frac{w(b)-w(a)}{b^2-a^2} \\ h &= \frac{a^2 w(b) - b^2 w(a)}{w(b)-w(a)} \end{aligned}$$

It can be shown that the constant h is connected to the energy E by the relation $h = \frac{2E}{l^2}$. Substituting the previous relations in (2.6) we obtain

$$\Delta^l_\alpha \theta(u) = \int_a^b \frac{\sqrt{w(b)-w(a)}}{\sqrt{a^2 w(b) - b^2 w(a) + (b^2-a^2)w(z) - z^2(w(b)-w(a))}} dz$$

that is equivalent to

$$\Delta^l_\alpha \theta(u) = \int_a^b \frac{dz}{\sqrt{a^2 - z^2 + (b^2-a^2)\frac{w(z)-w(a)}{w(b)-w(a)}}} \tag{2.7}$$

The last formula is true for every kind of central problem in which the attracting force depends only on the distance of the particle from the centre of attraction.

We are interested on the study of the apsidal angle in case of α-homogeneous potential problem: the attracting force is $\varphi(r) = \frac{1}{r^{\alpha+1}}$,therefore $\psi(z) = z^{\alpha-1}$ and $w(z) = \frac{2}{\alpha} z^\alpha$. Using formula (2.7) we can easily obtain the classical and well know result concerning the apsidal angle for a solution of Kepler and elastic problem; in the first case $\alpha = 1$, $w(z) = 2z$ and

$$\begin{aligned} \Delta^l_1 \theta(u) &= \int_a^b \frac{dz}{\sqrt{a^2 - z^2 + (b^2-a^2)\frac{z-a}{b-a}}} \\ &= \int_a^b \frac{dz}{\sqrt{(z-a)(b-z)}} \\ &= \pi \end{aligned}$$

for any choice of $a < b$.

In the same way, for the harmonic potential one has $\alpha = -2$, $w(z) = -z^{-2}$, then

$$\begin{aligned} \Delta^l_{-2} \theta(u) &= \int_a^b \frac{dz}{\sqrt{a^2 - z^2 + (b^2-a^2)\frac{(z^2-a^2)b^2}{(b^2-a^2)z^2}}} \\ &= \int_a^b \frac{z}{\sqrt{(z^2-a^2)(b^2-z^2)}} dz \\ &= \frac{1}{2}\int_{a^2}^{b^2} \frac{dt}{\sqrt{(t-a^2)(b^2-t)}} = \frac{\pi}{2} \end{aligned} \tag{2.8}$$

again for every choice of $a < b$.

The previous calculation showed that the apsidal angle of a solution of the Kepler and harmonic problem is independent on the value of the endpoints a and b, i.e. it's independent on the values of energy and angular momentum of the solution. Bertrand showed that, among all the homogeneous potentials, only these two potentials admits this property.

In order to study the apsidal angle of a solution of a generic α-homogeneous problem, we set $w_\alpha(z) = \frac{2}{\alpha} z^\alpha$ and

$$F_\alpha := \frac{w_\alpha(z) - w_\alpha(a)}{w_\alpha(b) - w_\alpha(a)}$$

The relation (2.3) are equivalent to the following

$$
\begin{array}{llll}
\Delta^l_\alpha\theta > \dfrac{\pi}{2-\alpha} & \textit{for } \alpha \in (0,1) & \Delta^l_\alpha\theta < \dfrac{\pi}{2-\alpha} & \textit{for } \alpha \in (1,2) \\
\Delta^l_\alpha\theta > \dfrac{\pi}{2} & \textit{for } \alpha \in (-2,0) & \Delta^l_\alpha\theta < \dfrac{\pi}{2} & \textit{for } \alpha \in (-\infty,-2)
\end{array}
\tag{2.9}
$$

and

$$
\begin{array}{llll}
\Delta^l_\alpha\theta < \pi & \textit{for } \alpha \in (0,1) & \Delta^l_\alpha\theta > \pi & \textit{for } \alpha \in (1,2) \\
\Delta^l_\alpha\theta < \dfrac{2\pi}{2-\alpha} & \textit{for } \alpha \in (-2,0) & \Delta^l_\alpha\theta > \dfrac{2\pi}{2-\alpha} & \textit{for } \alpha \in (-\infty,-2)
\end{array}
\tag{2.10}
$$

Suppose for a moment $\alpha \in (-\infty, 0)$: since the functions $w_\alpha(z) = \frac{2}{\alpha} z^\alpha$ are negative and monotonically increasing, the functions $F_\alpha(z)$ are positive for every choice of $0 < a < b$ and for every $a \le z \le b$. In order to prove relations (2.9) we compare the functions $F_\alpha(z)$ and $F_{-2}(z)$.

We define

$$H_\alpha(z) := \frac{F_\alpha}{F_{-2}}(z) = \frac{w_{-2}(b) - w_{-2}(a)}{w_\alpha(b) - w_\alpha(a)} \frac{w_\alpha(z) - w_\alpha(a)}{w_{-2}(z) - w_{-2}(a)}$$

The aim is to prove that, for every $z \in [a, b]$, the quotient $H_\alpha(z) \le 1$ for certain values of $\alpha \in (-\infty, 0)$ and $H_\alpha(z) > 1$ otherwise.

It holds $H(z) > 0$, $H(b) = 1$, then it enough to study the sign of

$$H'_\alpha(z) = \left(\frac{w_{-2}(b) - w_{-2}(a)}{w_\alpha(b) - w_\alpha(a)}\right) \frac{w'_\alpha(z)\big(w_{-2}(z) - w_{-2}(a)\big) - w'_{-2}(z)\big(w_\alpha(z) - w_\alpha(a)\big)}{\big(w_{-2}(z) - w_{-2}(a)\big)^2}$$

With easy calculations we obtain

$$
\begin{aligned}
H'_\alpha(z) &= C\frac{2a^2}{(z^2-a^2)^2}\frac{1}{\alpha}\big(\alpha z^{\alpha+3} - (2+\alpha)a^2 z^{\alpha+1} + 2a^{2+\alpha} z\big) \\
&= C\frac{2a^2}{(z^2-a^2)^2} z^{\alpha+1} \frac{\big(\alpha z^2 - (2+\alpha)a^2 + 2a^{2+\alpha} z^{-\alpha}\big)}{\alpha} \\
&= C\frac{2a^2}{(z^2-a^2)^2} z^{\alpha+1} N(z), \qquad C > 0
\end{aligned}
$$

where $N(a) = 0$ and

$$N'(z) = 2z - 2a^{2+\alpha} z^{-1-\alpha} = 2z\left(1 - \left(\frac{a}{z}\right)^{2+\alpha}\right)$$

The sign of the last derivative depends on the value of $\alpha + 2$: if $\alpha \in (-2, 0)$ we have $N'(z) > 0$, then $H_\alpha(z)$ is increasing for $z \in [a, b]$. Otherwise if $\alpha \in (-\infty, -2)$ the function $H_\alpha(z)$ is decreasing. Thus we conclude that in case $\alpha \in (-2, 0)$, $0 \le H_\alpha(z) \le 1$ and

$$0 \le F_\alpha(z) \le F_{-2}(z) \tag{2.11}$$

On the other hand, if $\alpha \in (-\infty, -2)$, $H(z) \ge 1$ and

$$0 \le F_{-2}(z) \le F_\alpha(z) \tag{2.12}$$

Since formula (2.7) can be rewritten in the form

$$\Delta^l_\alpha\theta(u) = \int_a^b \frac{dz}{\sqrt{a^2 - z^2 + (b^2 - a^2)F_\alpha(z)}}$$

(2.11) and (2.12) yield

$$\begin{aligned}\Delta^l_\alpha\theta(u) \;&>\; \Delta^l_{-2}\theta(u) = \frac{\pi}{2} \quad \textit{for } \alpha \in (-2, 0)\\ \Delta^l_\alpha\theta(u) \;&<\; \Delta^l_{-2}\theta(u) = \frac{\pi}{2} \quad \textit{for } \alpha \in (-\infty, -2)\end{aligned}$$

The remaining relations in (2.9) concerning $\alpha \in (0,2)$ comes from the case $\alpha \in (-\infty, 0)$ applying theorem (4). Indeed to every solution $u(t)$ of the α-homogeneous problem with $\alpha \in (1,2)$ it corresponds a solution $w(t)$ for the α-homogeneous problem with $\bar{\alpha} \in (-\infty, -2)$ such that $\Delta_\alpha\theta(u) = \frac{2}{2-\alpha}\Delta_{\bar{\alpha}}(w)$; then $\Delta_\alpha\theta(u) < \frac{2}{2-\alpha}\frac{\pi}{2} = \frac{\pi}{2-\alpha}$. In the same way the result concerning $\alpha \in (0,1)$ is obtained considering the duality with the set $\bar{\alpha} \in (-2, 0)$.

In the sequel, keeping in mind that the functions $w_\alpha(z)$ depend on α, we will omit to label $w(z)$ any time we write it.

In order to prove relation (2.10) we rewrite $\Delta^l_\alpha\theta(u)$ as follow:

$$\begin{aligned}\Delta^l_\alpha\theta(u) &= \int_a^b \frac{dz}{\sqrt{(z-a)\left((b+a)\left[\frac{(b-a)\big(w(z)-w(a)\big)}{\big(w(b)-w(a)\big)(z-a)}\right] - (z+a)\right)}}\\ &= \int_a^b \frac{dz}{\sqrt{(z-a)}\sqrt{(b+a)\left[\frac{(b-a)\big(w(z)-w(a)\big)}{\big(w(b)-w(a)\big)(z-a)} - 1\right] + (b-z)}}\\ &= \int_a^b \frac{1}{\sqrt{z-a}}\frac{1}{\sqrt{b-z}}\frac{1}{\sqrt{1+\varepsilon(z)}}dz \qquad (2.13)\end{aligned}$$

where

$$\varepsilon(z) = \frac{b+a}{b-z}\left[\frac{\frac{w(z)-w(a)}{z-a}}{\frac{w(b)-w(a)}{b-a}} - 1\right] \qquad (2.14)$$

For every $\alpha \in (0,1)$, since $w(z)$ is a concave and increasing function

$$\frac{\frac{w(z)-w(a)}{z-a}}{\frac{w(b)-w(a)}{b-a}} > 1 \quad \forall a \le z \le b$$

and

$$\varepsilon(z) > 0 \quad \forall z \in [a,b]\ . \qquad (2.15)$$

Otherwise if $\alpha \in (1,2)$, $w(z)$ is a convex and increasing function then

$$\frac{\frac{w(z)-w(a)}{z-a}}{\frac{w(b)-w(a)}{b-a}} < 1 \quad \forall a \le z \le b$$

and

$$\varepsilon(z) < 0 \quad \forall z \in [a,b]\ . \qquad (2.16)$$

Moreover, in the last case $\alpha \in (1,2)$, we show that

$$\varepsilon(z) > -1 \qquad (2.17)$$

for every $z \in [a,b]$. Indeed we define $E(z) = \varepsilon(z) + 1$ thus, using $w(z) = \frac{2}{\alpha}z^\alpha$,

$$E(z) = \frac{b+a}{b-z}\left[\frac{\frac{z^\alpha - a^\alpha}{z-a}}{\frac{b^\alpha - a^\alpha}{b-a}} - 1\right] + 1 = \frac{b+a}{b-z}\left(\frac{z^\alpha - a^\alpha}{z-a}\right)\left(\frac{b-a}{b^\alpha - a^\alpha}\right) - \frac{z+a}{b-z}$$

Collecting the positive term $(z+a)\dfrac{(b^2-a^2)}{b^\alpha-a^\alpha}$, the sign of $E(z)$ is the same of the following function

$$\tilde{E}(z) = \frac{1}{b-z}\left[\frac{z^\alpha-a^\alpha}{z^2-a^2} - \frac{b^\alpha-a^\alpha}{b^2-a^2}\right] = \frac{g(z)-g(b)}{b-z} = -g'(\xi)$$

where $\xi \in [z,b]$ and the last equality comes from the Lagrange's theorem is applied to the function $g(z) = \dfrac{z^\alpha-a^\alpha}{z^2-a^2} = a^{\alpha-2}\dfrac{\left(\frac{z}{a}\right)^\alpha-1}{\left(\frac{z}{a}\right)^2-1}$. Since for $\alpha\in(0,2)$ the function $\tilde{g}(t)=\frac{t^\alpha-1}{t^2-1}$ is strictly decreasing for $t>0$, it descends that $\tilde{E}(z)>0$ and $\varepsilon(z)$ strictly bigger than -1.

From the estimates (2.15),(2.16) it follows that for every solution $u(t)$ of the α-homogeneous problem

$$\Delta^l_\alpha\theta(u) < \int_a^b \frac{1}{\sqrt{z-a}}\frac{1}{\sqrt{b-z}} = \pi \qquad \alpha\in(0,1)$$

$$\Delta^l_\alpha\theta(u) > \int_a^b \frac{1}{\sqrt{z-a}}\frac{1}{\sqrt{b-z}} = \pi \qquad \alpha\in(1,2)$$

and applying again the duality between positive and negative value of α we obtain the result (2.10) also for $\alpha\in(-\infty,0)$. More precisely the set $\alpha\in(-2,0)$ is linked to the set $\bar\alpha\in(0,1)$ by duality $\bar\alpha=-\frac{2\alpha}{2-\alpha}$ and

$$\Delta^l_\alpha\theta(u) = \frac{2-\bar\alpha}{2}\Delta^l_{\bar\alpha}\theta(w) = \frac{2}{2-\alpha}\Delta^l_{\bar\alpha}\theta(w) < \frac{2}{2-\alpha}\pi$$

Similarly the set $\alpha\in(-\infty,-2)$ is linked to the set $\bar\alpha\in(1,2)$ and relation (1.5) yields

$$\Delta^l_\alpha\theta(u) > \frac{2}{2-\alpha}\pi$$

To conclude the proof of the theorem it remains to calculate the limit of the apsidal angle when the angular momentum tends to zero and to l_{max}.

We suppose $\alpha\in(0,2)$ and we change the variable in integral (2.13) in order to obtain a fixed end-points integral. Let $s=\frac{z-a}{b-a}$ and $L=b-a$,

$$\Delta^l_\alpha\theta(u) = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{1+\varepsilon(Ls+a)}}ds \tag{2.18}$$

$$\varepsilon(Ls+a) = \frac{(L+2a)}{L(1-s)}\left[\frac{w(Ls+a)-w(a)}{s\big(w(L+a)-w(a)\big)}-1\right]$$

In this setting the parameter L brings in itself the dependence on the angular momentum l; indeed L is decreasing with respect to l, since, by definition, $L=b-a=\frac{1}{R_-^\alpha}-\frac{1}{R_+^\alpha}$ and R_-^α, R_+^α are respectively increasing and decreasing with respect to l. Moreover $\lim_{l\to0}L=\infty$ and $\lim_{l\to l_{max}}L=0$.

In the α-homogeneous potential case we have $w(z)=\frac{2}{\alpha}z^\alpha$ thus

$$\varepsilon(Ls+a) = \frac{L+2a}{L(1-s)}\left[\frac{(Ls+a)^\alpha-a^\alpha}{s\big((L+a)^\alpha-a^\alpha\big)}-1\right]$$

Since for every $\alpha\in(0,2)$ the measure of the interval $[0,1]$ given by $\displaystyle\int_0^1\frac{ds}{\sqrt{s(1-s)}}$ is finite and, for (2.15)(2.17), the term $\dfrac{1}{\sqrt{1+\varepsilon(Ls+a)}}$ is uniformly bounded for $s\in(0,1)$ and $L\in(0,+\infty)$, it follows that we can pass to the limit for $L \to 0$ and $L \to \infty$ under the integral sign in (2.18). Since

$$\lim_{L\to 0} \varepsilon(Ls + a) = 1 - \alpha$$

$$\lim_{L\to\infty} \varepsilon(Ls + a) = \frac{s - s^\alpha}{s^2 - s}$$

we conclude

$$\Delta_\alpha^{l_{max}}\theta = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{2-\alpha}}ds = \frac{\pi}{\sqrt{2-\alpha}}$$

$$\Delta_\alpha^0\theta = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{1+\frac{s-s^\alpha}{s^2-s}}}ds = \frac{\pi}{2-\alpha}$$

The result for $\alpha \in (-\infty, 0)$ follows from the duality stated in theorem 4. □

Lemma 2.1. *Let R_+^α and R_-^α be fixed. Then the quantity $\Delta_\alpha^l\theta(u)$ is increasing with respect to α for $\alpha \in (-\infty, 2)$, $\alpha \neq 0$.*

Proof
Write $\Delta_\alpha^l\theta(u)$ as in (2.7)

$$\Delta_\alpha^l\theta(u) = \int_a^b \frac{dz}{\sqrt{a^2 - z^2 + (b^2 - a^2)\frac{w(z)-w(a)}{w(b)-w(a)}}}$$

We want to show that for every choice of $b > a > 0$ and for every fixed $z \in [a, b]$

$$\frac{d}{d\alpha}\frac{w(z) - w(a)}{w(b) - w(a)} < 0 \quad \forall\alpha \in (-\infty, 2), \alpha \neq 0$$

Since $w(z) = \frac{2}{\alpha}z^\alpha$,

$$\frac{w(z) - w(a)}{w(b) - w(a)} = \frac{z^\alpha - a^\alpha}{b^\alpha - a^\alpha} = \frac{\left(\frac{z}{a}\right)^\alpha - 1}{\left(\frac{b}{a}\right)^\alpha - 1}$$

then it's enough to prove that for every $1 < t < s$

$$\frac{d}{d\alpha}\left(\frac{t^\alpha - 1}{s^\alpha - 1}\right) < 0 \tag{2.19}$$

Derivating it follows that the previous relation holds if and only if

$$(t^\alpha \log(t))(s^\alpha - 1) - (t^\alpha - 1)(s^\alpha \log(s)) < 0$$

that, for $\alpha \neq 0$, is equivalent to

$$\frac{t^\alpha \log(t)}{t^\alpha - 1} < \frac{s^\alpha \log(s)}{s^\alpha - 1} \qquad \forall\ 1 < t < s$$

or $\frac{d}{dx}\left(\frac{x^\alpha \log(x)}{x^\alpha - 1}\right) > 0$ for every $x > 1$.

Since $x^\alpha > 1 + \alpha\log(x)$ for every positive x and real $\alpha \neq 0$,

$$\frac{d}{dx}\left(\frac{x^\alpha \log(x)}{x^\alpha - 1}\right) = \frac{x^\alpha - 1 - \alpha\log x}{(x^\alpha - 1)^2} > 0$$

□

Conjecture *For every value of α the apsidal angle is a monotone with respect to l. In particular $\Delta_\alpha\theta(u)$ is increasing for $-2 < \alpha < 1$ and decreasing otherwise.*

It's numerical evident that the conjecture is true, but, to the best of my knowledge, that statement is not even proved. We remark that the conjecture implies that the upperbound for $\Delta_\alpha\theta(u)$, $\alpha \in (-2, 1)$ and the lowerbound for $\Delta_\alpha\theta(u)$, $\alpha \in (-\infty, -2)$ and $\alpha \in (1, 2)$ proved in theorem 7 are not optimal.

2.2 The apsidal angle for logarithmic potential

In this section we deal about the logarithmic potential problem, in particular we will give some consideration about the apsidal angle of a solution of system (1.1) in case $\alpha = 0$.

Let R_-, R_+ the minimal and maximal values of the radial coordinate and $\Delta^l\theta(u)$ the apsidal angle of the solution $u(t)$. We recall that R_-, R_+ are solutions of the equation

$$E - \frac{1}{2}\frac{l^2}{r^2} - \log r = 0 \tag{2.20}$$

and

$$\Delta^l\theta = \int_{R_-}^{R_+} \frac{l}{r\sqrt{2r^2E - 2r^2\log r - l^2}}dr \tag{2.21}$$

We also remind that, once one fixes the value of the angular momentum l, a solution exists provide the energy E satisfies $E \geq \frac{1}{2} + \log(l)$ and all the orbits are bounded. Conversely, if the value of energy E is fixed, there exist solution only for values of angular momentum in the set $l \in [0, e^{E-\frac{1}{2}}]$. For theorem 1 the zero angular momentum solution corresponds to the collision orbit, while the solution that admits the maximal value of angular momentum, $l_{max} = e^{E-\frac{1}{2}}$, corresponds to the circular orbit.

In the next theorem we prove the limit behaviour of the apsidal angle as $l \to 0$ and $l \to l_{max}$, together with a lowerbound for every values of angular momentum.

Theorem 8. *Let $u_l(t)$ a solution of system (1.1) with $\alpha = 0$, $l \in (0, l_{max})$. Then, denoting with $\Delta^l\theta(u)$ the apsidal angle of solution $u_l(t)$, for every $l \in (0, l_{max})$*

$$\frac{\pi}{2} \leq \Delta^l\theta(u) < \pi \tag{2.22}$$

Moreover

$$\lim_{l\to 0} \Delta^l\theta = \frac{\pi}{2}$$
$$\lim_{l\to l_{max}} \Delta^l\theta = \frac{\pi}{\sqrt{2}}$$

Proof

We follow the same argumentation as in the proof of theorem (7).

In the case of logarithmic potential the force acting on the particle at distance r from the center of attraction is $\varphi(r) = \frac{1}{r}$, thus the function $w_0(z)$, in the following denoted by $w(z)$, is given by $w(z) = 2\log(z)$. From formula (2.13) and (2.14), it follows that the apsidal angle of the solution $u(t)$ is

$$\Delta^l\theta(u) = \int_a^b \frac{1}{\sqrt{z-a}}\frac{1}{\sqrt{b-z}}\frac{1}{\sqrt{1+\varepsilon(z)}}$$

where

$$\varepsilon(z) = \frac{(b+a)}{(b-z)}\left[\frac{\log(z)-\log(a)}{(z-a)}\frac{(b-a)}{\log(b)-\log(a)} - 1\right]$$

and, as in the α-homogeneous case, a and b are the square of the inverse of the pericentre and apocentre of the solution $u(t)$.

Since $f(x) = \log(x)$ is a concave function it turns out that the quantity $\varepsilon(z)$ is positive for every $z \in [a, b]$, thus

$$\Delta^l\theta(u) = \int_a^b \frac{1}{\sqrt{z-a}}\frac{1}{\sqrt{b-z}}\frac{1}{\sqrt{1+\varepsilon(z)}} < \int_a^b \frac{1}{\sqrt{z-a}}\frac{1}{\sqrt{b-z}}dz = \pi\ .$$

In order to prove the lower bound we rewrite the integral in the form (2.7)

$$\Delta^l_\alpha \theta(u) = \int_a^b \frac{dz}{\sqrt{a^2 - z^2 + (b^2 - a^2)\frac{\log(z)-\log(a)}{\log(b)-\log(a)}}}$$

and show that $F_0(z) := \frac{\log(z)-\log(a)}{\log(b)-\log(a)}$ is less than $F_{-2}(z) = \frac{(z^2-a^2)}{(b^2-a^2)}\frac{b^2}{z^2}$ for every $z \in (a,b)$. Indeed, if it holds, the thesis follows from (2.8).

Define the function

$$f(z) = z\frac{\log(z) - \log(a)}{z - a}$$

then $f(z) > 0$ for every $z \in [a,b]$, $f(b) = b\frac{\log(b)-\log(a)}{(b-a)}$ and

$$\begin{aligned} f'(z) &= \frac{(z-a) - a(\log(z) - \log(a))}{(z-a)^2} \\ &= \frac{a}{(z-a)^2}\left(\frac{z}{a} - 1 - \log\left(\frac{z}{a}\right)\right) > 0 \end{aligned}$$

since $\log(x) < x - 1$ for every $x > 1$. Thus $f(z) \le f(b)$ and using $(z+a)b > (b+a)z$ we obtain

$$\frac{\log(z) - \log(a)}{\log(b) - \log(a)} \le \frac{(z-a)}{(b-a)}\frac{b}{z} < \frac{(z^2-a^2)}{(b^2-a^2)}\frac{b^2}{z^2}$$

that is the required inequality $F_0(z) < F_{-2}(z)$.

We now analize the asymptotic behaviour of the apsidal angle as the angular momentum tends to zero and l_{max}. To this aim we apply a variable changing in order to obtain a fixed end-points integral

$$\begin{aligned} \Delta^l\theta &= \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{1+\varepsilon(Ls+a)}}ds \\ \varepsilon(Ls+a) &= \frac{(L+2a)}{L(1-s)}\left[\frac{\log(Ls+a) - \log(a)}{s(\log(L+a) - \log(a))} - 1\right] \end{aligned} \tag{2.23}$$

We recall that L depends monotonically on l and, in particular, $\lim_{l\to 0} L = +\infty$ and $\lim_{l\to l_{max}} L = 0$. Moreover the parameter a, also depending on l, varies monotonically with respect to l and remains positive and bounded for all l. Therefore it's enough to perform the limit of $\varepsilon(Ls+a)$ for $L \to 0$ and $L \to \infty$.

Since

$$\frac{\log(Ls+a) - \log(a)}{s(\log(L+a) - \log(a))} = \frac{\log(1 + \frac{Ls}{a})}{s\log(1 + \frac{L}{a})}$$

for every $s \in (0,1)$, we obtain

$$\lim_{L\to 0} \varepsilon(Ls+a) = 1 \ .$$

On the other side

$$\lim_{L\to +\infty} \varepsilon(Ls+a) = \frac{1}{s} \ .$$

We have already shown that the function $\varepsilon(Ls+a) > 0$ for every fixed $s \in (0,1)$ and $L \in (0,\infty)$, thus for the Lebesgue's dominated convergence theorem, passing the limit under the integral sign, we obtain

$$\lim_{l\to l_{max}} \Delta^l\theta = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{2}}ds = \frac{\pi}{\sqrt{2}}$$
$$\lim_{l\to 0} \Delta^l\theta = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{1+\frac{1}{s}}}ds = \frac{\pi}{2}$$

□

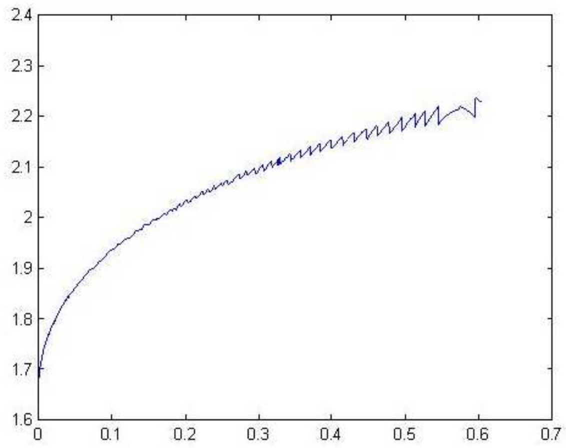


Figure 2.2: $\Delta^l\theta$ for logarithmic potential

Remark 2.2. *We note that the potential is not an homogeneous function but it is the differential equation: if $u(t)$ is a solution of system (1.1) then $v(t) = \frac{1}{\lambda}u(\lambda t)$ is a solution too. Moreover if E_u is the value of the energy of the solution $u(t)$, then $E_v = E_u - \log\lambda$. Thus without loss of generality we can take into account solutions of every energy value E.*

In the next theorem let $u(t)$ be a solution of the logarithmic potential problem with energy $E = 0$ and angular momentum l. We already know that if $\log l + \frac{1}{2} = 0$ then $u(t)$ is the circular solution, while if $\log l + \frac{1}{2} < 0$ then $u(t)$ is a bounded solution.

Theorem 9. *The apsidal angle of a solution $u(t)$ for the logarithmic potential problem of energy $E = 0$ can be written in the form*

$$\Delta^l\theta(u) = \int_0^1 \frac{z}{\sqrt{e^{2z(s-1)}s + e^{2zs}(1-s) - 1}}ds \tag{2.24}$$

where $z \in (0, +\infty)$ is a decreasing function with respect to l.

Proof
Let R_- and R_+ be the minimal and maximal values of the radial coordinate of the solution $u(t)$ and denote with $A = -\log R_-$ and $B = -\log R_+$.

Performing the variable changing $t = -\log r$ in the integral (2.21) we obtain

$$\Delta^l\theta(u) = \int_A^B \frac{-l}{\sqrt{2e^{-2t}t - l^2}}dt \tag{2.25}$$

We remind that $R_\pm$ are solutions of equation (2.20) with $E = 0$ therefore A and B are such that

$$l^2 = 2e^{-2A}A \qquad l^2 = 2e^{-2B}B$$

and

$$e^{-2(B-A)} = \frac{A}{B}$$

In order to achieve an integral with fixed end-points, we define a new variable of integration $s = \frac{t-A}{B-A}$ and apply the variable change in (2.25). Using the last relations

$$\begin{aligned}
\Delta^l\theta(u) &= \int_0^1 \frac{l(A-B)}{\sqrt{2e^{-2(B-A)s-2A}((B-A)s+A)-l^2}}ds \\
&= \int_0^1 \frac{l(A-B)}{\sqrt{2\left(\frac{A}{B}\right)^s \frac{l^2}{2A}(B-A)s + l^2\left(\frac{A}{B}\right)^s - l^2}}ds \\
&= \int_0^1 \frac{(A-B)}{\sqrt{\left(\frac{A}{B}\right)^s \frac{B}{A}s - \left(\frac{A}{B}\right)^s s + \left(\frac{A}{B}\right)^s - 1}}ds \\
&= \int_0^1 \frac{(A-B)}{\sqrt{\left(\frac{A}{B}\right)^{s-1} s - \left(\frac{A}{B}\right)^s (s-1) - 1}}ds
\end{aligned}$$

Setting $z = \frac{1}{2}\log\frac{A}{B}$ we have the formula (2.24). We note that the choice of $E = 0$ implies that, for every admissible values of angular momentum, $0 \le R_- < R_+ \le 1$. Furthermore when $l = l_{max} = e^{-\frac{1}{2}}$ the orbit is circular, $R_- = R_+ = e^{-\frac{1}{2}}$, while as l decreases, the values of R_- and R_+ moves monotonically to 0 and 1 respectively. It follows that A increases and B decreases then $z = \frac{1}{2}\log\frac{A}{B} = A - B$ increases. We conclude that z is a decreasing function with respect to l and $\lim_{l\to 0} z = +\infty$, $\lim_{l\to l_{max}} z = 0$. □

From numerical experiments, as like as in the case of α-homogeneous potential, the apsidal angle for the logarithmic potential problem seems to be monotonically increasing with respect to the angular momentum. It follows that the upperbound in (2.22) is not optimal.

2.3 The non-differentiability of the apsidal angle

We have shown that the apsidal angle is a continuous function with respect to the angular momentum l and we have found the limits when l goes to zero and l_{max}. In this section we show, in the logarithmic potential case, that the apsidal angle is not differentiable as function of angular momentum in a neighbourhood of zero.

Theorem 10. *Let $\Delta^l\theta(u)$ the apsidal angle of solution $u(t)$ with angular momentum l of system* (1.1) *in case $\alpha = 0$. Then $\Delta^l\theta(u)$ has a vertical tangent at $l = 0$.*

Proof
Consider formula (2.23) and, without loss of generality, fix $a = 1$.

$$\Delta^l\theta(u) = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{1+\varepsilon(Ls+1)}}ds$$

$$\varepsilon(Ls+1) = \frac{(L+2)}{L(1-s)}\left[\frac{\log(Ls+1)}{s(\log(L+1))} - 1\right]$$

Recall that the angular momentum l and the parameter $L = b - a$ are linked by formula

$$l^2 = \frac{2\log b - 2\log a}{b^2 - a^2} = \frac{2\log(L+1)}{(L+2)L}$$

then $L \to +\infty$ as $l \to 0$ and for small l it holds

$$l \sim \frac{\sqrt{2\log L}}{L} \tag{2.26}$$

Since for theorem 8

$$\lim_{l\to 0} \Delta^l \theta(u) = \int_0^1 \frac{1}{\sqrt{s(1-s)}} \frac{1}{\sqrt{1+\frac{1}{s}}} ds$$

the derivative of the apsidal angle with respect to l in $l = 0$ is

$$\lim_{l\to 0} \int_0^1 \frac{\frac{1}{\sqrt{s(1-s)}}\left(\frac{1}{\sqrt{1+\varepsilon(Ls+1)}} - \frac{1}{\sqrt{1+\frac{1}{s}}}\right)}{l} ds$$

Using (2.26), we rewrite the previous limit as a limit in L

$$\lim_{L\to+\infty} \int_0^1 \frac{1}{\sqrt{s(1-s)}} \frac{\left(\frac{1}{\sqrt{1+\varepsilon(Ls+1)}} - \frac{1}{\sqrt{1+\frac{1}{s}}}\right)}{\left(\sqrt{\log L}/L\right)} ds$$

In order to simplify the notation, set $\rho = \dfrac{1}{L}$ and define $f(\rho) = \dfrac{1}{\sqrt{1+\varepsilon(\frac{s}{\rho}+1)}}$. In the new variable ρ the derivative of the apsidal angle is written as

$$\lim_{\rho\to 0} \int_0^1 \frac{1}{\sqrt{s(1-s)}} \left(\frac{f(\rho)-f(0)}{\rho\sqrt{-\log\rho}}\right) ds$$

Our aim is to estimate the last limit and we will show that as ρ tends to zero the limit blows up to infinity.

The plan of the proof is the following: we first write

$$f(\rho) - f(0) = f'(\xi)\rho \qquad \xi \in (0,\rho)$$

and we show that $f'(\cdot)$ is a positive and decreasing function in a neighbourhood of the origin in order to obtain $f(\rho) - f(0) < f'(\rho)\rho$ and finally we pass to the limit.

The main part of the proof is dedicated to prove that f' is positive and decreasing at least in a right neighbourhood of zero.

In the following we use $\varepsilon(\rho)$ in place of $\varepsilon(\frac{s}{\rho}+1)$ to indicate the dependence of ε as a function of ρ. The derivative of f is given by

$$f'(\rho) = -\frac{1}{2}\frac{1}{(1+\varepsilon(\rho))^{\frac{3}{2}}}\varepsilon'(\rho)$$

Reminding the definition of $\varepsilon(\rho)$

$$\varepsilon(\rho) = \frac{(1+2\rho)}{(1-s)}\left[\frac{\log\left(\frac{s}{\rho}+1\right)}{s\log\left(\frac{1}{\rho}+1\right)} - 1\right]$$

it follows

$$
\begin{aligned}
\varepsilon'(\rho) &= \frac{2}{1-s}\left[\frac{\log\left(\frac{s}{\rho}+1\right)}{s\log\left(\frac{1}{\rho}+1\right)}-1\right] \\
&+\frac{(1+2\rho)}{(1-s)}\left[\frac{-\frac{s^2}{\rho(s+\rho)}\log(\frac{1}{\rho}+1)+\frac{s}{\rho(1+\rho)}\log(\frac{s}{\rho}+1)}{s^2\log^2(\frac{1}{\rho}+1)}\right] \\
&= \frac{2}{1+2\rho}\varepsilon(\rho)+\frac{(1+2\rho)}{(1-s)}\left[-\frac{1}{\rho(s+\rho)\log(\frac{1}{\rho}+1)}+\frac{\log(\frac{s}{\rho}+1)}{\rho s(1+\rho)\log^2(\frac{1}{\rho}+1)}\right]
\end{aligned}
\tag{2.27}
$$

Since $\varepsilon(\rho) \to \frac{1}{s}$ as $\rho \to 0$, the first term of the derivative tends to $\frac{2}{s}$, while the second term is equal to

$$
\frac{(1+2\rho)}{(1-s)}\left[\frac{s(1+\rho)(\log(\rho)-\log(1+\rho))+(\rho+s)(\log(s+\rho)-\log(\rho))}{\rho s(\rho+s)(1+\rho)\log^2(\frac{1}{\rho}+1)}\right]
$$

that, as ρ tend to zero, is asymptotic to

$$
\frac{1}{s(1-s)}\frac{\log s}{\rho\log^2(\frac{1}{\rho})} \sim \frac{\log(s)}{s(1-s)}\frac{1}{\rho\log^2(\rho)}
$$

Thus

$$
\begin{aligned}
\varepsilon'(\rho) &\sim \frac{2}{s}+\frac{\log(s)}{s(1-s)}\frac{1}{\rho\log^2(\rho)} \\
&\sim \frac{\log(s)}{s(1-s)}\frac{1}{\rho\log^2(\rho)}
\end{aligned}
$$

and substituting in the derivative of f, recalling that $s \in (0,1)$, we obtain

$$
f'(\rho) \sim -\frac{1}{2}\frac{1}{(1+\frac{1}{s})^{\frac{3}{2}}}\frac{\log(s)}{s(1-s)}\frac{1}{\rho\log^2(\rho)} \to +\infty \tag{2.28}
$$

Relation (2.28), besides showing that $f'(\rho)$ is positive in a neighbourhood of the origin, presents the asymptotic behaviour of $f'(\rho)$ as function of ρ and this estimate will be used later at the end of the proof. In order to show that $f'(\rho)$ is decreasing, let's now calculate the asymptotic behaviour of the second derivative of $f(\rho)$.

$$
f''(\rho) = \frac{1}{2}\frac{1}{(1+\varepsilon)^{\frac{3}{2}}}\left(\frac{3}{2}\frac{(\varepsilon')^2}{(1+\varepsilon)}-\varepsilon''\right)
$$

It holds

$$\begin{aligned}
\varepsilon''(\rho) = &-\frac{4}{(1+2\rho)^2}\varepsilon + \frac{2}{1+2\rho}\varepsilon' \\
&+ \frac{2}{1-s}\left(-\frac{1}{\rho(s+\rho)\log(\frac{1}{\rho}+1)} + \frac{\log(\frac{s}{\rho}+1)}{\rho s(1+\rho)\log^2(\frac{1}{\rho}+1)}\right) \\
&+ \frac{1+2\rho}{1-s}\left(\frac{(s+2\rho)\log(\frac{1}{\rho}+1) - \frac{s+\rho}{1+\rho}}{\rho^2(s+\rho)^2\log^2(\frac{1}{\rho}+1)}\right) \\
&- \frac{1+2\rho}{1-s}\left(\frac{\frac{s^2}{s+\rho}(1+\rho)\log^2(\frac{1}{\rho}+1)}{\rho^2 s^2(1+\rho)^2\log^4(\frac{1}{\rho}+1)}\right) \\
&- \frac{1+2\rho}{1-s}\left(\frac{\log(\frac{s}{\rho}+1)\left[s(1+2\rho)\log^2(\frac{1}{\rho}+1) - 2s\log(\frac{1}{\rho}+1)\right]}{\rho^2 s^2(1+\rho)^2\log^4(\frac{1}{\rho}+1)}\right)
\end{aligned}$$

For relation (2.27) the second line of the previous formula is equal to

$$2\big(\varepsilon' - \frac{2}{1+2\rho}\varepsilon\big)\frac{1}{1+2\rho}$$

then the contribution of the first and second lines is

$$\frac{4}{1-s}\left[-\frac{1}{\rho(s+\rho)\log(\frac{1}{\rho}+1)} + \frac{\log(\frac{s}{\rho}+1)}{\rho s(1+\rho)\log^2(\frac{1}{\rho}+1)}\right]$$

With straightforward calculations we obtain

$$\begin{aligned}
\varepsilon''(\rho) = &\frac{s-2\rho s+2\rho}{(1-s)\rho^2(s+\rho)^2\log(\frac{1}{\rho}+1)} \\
&- \frac{2s(1+2\rho)(1+\rho) - (s+\rho)\log(\frac{s}{\rho}+1)}{(1-s)s\rho^2(s+\rho)\log^2(\frac{1}{\rho}+1)} \\
&+ \frac{2(1+2\rho)\log(\frac{s}{\rho}+1)}{(1-s)s\rho^2(1+\rho)^2\log^3(\frac{1}{\rho}+1)}
\end{aligned}$$

and passing to the limit as $\rho \to 0$

$$\begin{aligned}
\varepsilon''(\rho) &\sim -\frac{1}{(1-s)s\rho^2\log(\rho)} - \frac{2+\log(s)-\log(\rho)}{(1-s)s\rho^2\log^2(\rho)} - \frac{2(\log(s)-\log(\rho))}{(1-s)s\rho^2\log^3(\rho)} \\
&\sim -\frac{\log(s)(2+\log(\rho))}{s(1-s)\rho^2\log^3(\rho)}
\end{aligned}$$

Thank to the asymptotic estimates of $\varepsilon'(\rho)$ and $\varepsilon''(\rho)$ we manage to achieve that $f''(\rho) < 0$. Indeed as ρ approaches zero

$$\begin{aligned}
3(\varepsilon')^2 - 2(1+\varepsilon)\varepsilon'' &\sim 3\frac{\log^2(s)}{s^2(1-s)^2\rho^2\log^4(\rho)} + 2\left(1+\frac{1}{s}\right)\frac{\log(s)(2+\log(\rho))}{s(1-s)\rho^2\log^3(\rho)} \\
&\sim 2\frac{\log(s)(1+s)}{s^2(1-s)\rho^2\log^2(\rho)}
\end{aligned}$$

and, replacing it in the formula of $f''(\rho)$, we gain $f''(\rho) \to -\infty$ as $\rho \to 0$.

This ensures that $f'(\rho)$ is decreasing in a neighbourhood of zero thus

$$f(\rho) - \frac{1}{\sqrt{1+\frac{1}{s}}} > f'(\rho)\rho$$

and again for (2.28),

$$\begin{aligned}
&\lim_{l\to 0}\int_0^1 \frac{\frac{1}{\sqrt{s(1-s)}}\left(\frac{1}{\sqrt{1+\varepsilon(Ls+1)}} - \frac{1}{\sqrt{1+\frac{1}{s}}}\right)}{l} ds \\
&> \lim_{\rho\to 0}\int_0^1 \frac{1}{\sqrt{s(1-s)}} \frac{f'(\rho)}{\sqrt{-2\log(\rho)}} ds \\
&\sim \int_0^1 \frac{1}{\sqrt{s(1-s)}}\left(-\frac{1}{2}\frac{1}{(1+\frac{1}{s})^{\frac{3}{2}}}\frac{\log(s)}{s(1-s)\rho\log^2(\rho)\sqrt{-2\log(\rho)}}\right) ds \to +\infty
\end{aligned}$$

then we can conclude that the derivative of the apsidal angle with respect to l blows up as l goes to zero

□

Chapter 3

On the regularization of the collision solutions

In the first chapter we recognized that if $\alpha > -2$ the differential equation (1.1) is singular in the origin then a solution of the α-homogeneous problem present a singularity at the collision instant. Moreover we have pointed out that the collisions are the only singularities that a solution of the central problem can present; this means that all the non collision solutions are globally defined and only the collision solutions exist for a bounded interval of time. A classical problem is to try to extend a solution of a differential equation beyond the singularities that, in the contest of the central force problem, consists in finding a method to define the trajectory beyond the collision. Usually one refers to these methods as a Regularization theory: many regularization theories could be defined in order to solve a problem and a theory may differs from another one on the different technique developed to define the extension, for instance analytical, topological or geometrical. Moreover two regularization theories may differ also for the different definition of regularization and for the different property that the extended solution has to preserve passing through the singularity.

For what the regularization of collision solutions of the α-homogeneous problem is concerned, many regularization theories have been presented. In 1907 Sundman, studying the Newtonian three body problem, attained an extension of the double collision solution using analytic continuation methods: introducing a new variable he succeeded to write the solution as a convergent power series in the new variable up to the collision instant. A detailed description of the Sundman's regularization can be found in the textbook of Siegel and Moser [22] and it is called Branch Regularization. Following the Sundman's technique the extended flow satisfies a property of smoothness with respect to time but, on the other hand, continuity with respect to the initial data in the phase space is not considered.

A different still related approach in order to define a regularization is the one followed by Levi-Civita [17]: he focused his attention on the singularities of the differential equation rather than on the singularities of the solution. As shown in the first chapter, by means of suitable transformations of space and time, the singular equation is turned into an equation without singularities and the extension is a smooth function of initial data. An extension of the Levi-Civita transformation to higher dimension is proposed in the works of Moser [20] and Kustaanheimo and Stiefel [16].

It holds that extended orbits obtained follow the Sundman's regularization and the Levi-Civita's regularization coincide. In [19] it is shown that this coincidence is a peculiar property of the gravitational interaction; indeed McGehee performed a generalization of the Sundman's and Levi-Civita's regularization to systems with degree of homogeneity $\alpha > 0$ and showed that if α belongs to a certain dense subset of rationals the collision solution can be extended by Sundman's technique but only for a few of these values of α, precisely

$$\alpha = 2\left(1 - \frac{1}{n}\right) \qquad n \in \mathbb{N}, n > 1 \tag{3.1}$$

the Levi-Civita regularization is applicable. This means that if one imposes the extension has to satisfy a better property of smoothness, for instance a continuity with respect initial data, one expects that the problem is solvable for a less number of cases.

In this chapter we take into account a different concept of regularization described in [2] due to Bellettini, Fusco and Gronchi and denoted with regularization via smoothing the potential. This method consists in perturbing the differential equation by a small parameter in order to remove the singularity in the dynamical system then in discussing how the smoothing of potential coupled with the perturbation of initial conditions lead to define a global solution of the α-homogeneous problem. In [2] the authors considered problem with degree of homogeneity $\alpha > 0$ and they found that, if the continuity with respect the initial data is ignored, the problem is regularizable via smoothing the potential for every $\alpha > 0$, $\alpha \neq 2$. On the other hand, if one forces the extension to be continuous with respect initial data, the set of values of degree α for whose the methods works is α in the form (3.1) and $\alpha > 2$.

Our idea is to use the smoothing of potential technique in order to study the regularization of the collision solutions for logarithmic potentials. Furthermore we investigate the existence of global solutions of system (1.1) for a large class of potential function $V(x)$ and we discuss whether it is possible to extend those solutions that are not global defined.

3.1 The regularization via smoothing of the potential

In this section we briefly describe the regularization method via smoothing the potential introduced in [2]. The two-body problem we are taking into account consists in finding a solution $u \in \mathbb{C}^2([0,T);\mathbb{R}^2)$ to the system

$$\begin{cases} \ddot{u} = \nabla V(|u|) \\ (u(0), \dot{u}(0)) = (q_0, p_0) \quad \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \end{cases} \tag{3.2}$$

where the potential function $V(x)$ is singular in the origin.

Agreement with definition (1.1) and (1.2) we say that a solution $u(t)$ has a collision at time t if $u(t) = 0$ and we define a singularity of the solution $u(t)$ a time $t^* > 0$ beyond which the solution can not be extended.

In the case that $V(x)$ is a potential α-homogeneous, the collisions are the only singularities that the system can develop, thus all the non collision solutions are globally defined.

We are interested on discussing whether a collision solution could be extended beyond the singularity: more precisely we analyze in which case and in which sense the method of regularization via smoothing potential allows to define a global solution of system (3.2) for initial data leading to collision.

For every singular potential $V(x)$ we define the associated smoothed potential

$$V_\varepsilon(x) = V(x + \varepsilon) \qquad \varepsilon > 0$$

and the regularized problem obtained from (3.2) replacing the potential $V(x)$ with $V_\varepsilon(x)$

$$\begin{cases} \ddot{u} = \nabla V_\varepsilon(|u|) \\ (u(0), \dot{u}(0)) = (q_0, p_0) \quad \in \mathbb{R}^2 \times \mathbb{R}^2 \end{cases} \tag{3.3}$$

Since the differential equation is no more singular in the origin, unlike (3.2), the last system admits a global solution in $\mathbb{C}^\infty((-\infty,+\infty);\mathbb{R}^2)$ for every choice of the initial data (q_0,p_0).

Let $S(V)$ be the set of initial conditions (u_0,p_0) leading to collision for the unperturbed system, that is $S(V)$ is the set of initial data of those solutions that are not global defined and for which we want to define an extension. Given $\bar{\nu}\in S(V)$, the regularization technique via smoothing the potential consists in the analysis of the solutions of the regularized system for small ε and for initial data ν nearby $\bar{\nu}$. More precisely, one investigates the existence of the asymptotic limit as $(\varepsilon,\nu)\to(0,\bar{\nu})$ of solution of (3.3) with initial data ν, the continuity of the limit trajectory with respect to initial data and its relationship with the original solution of the singular system with initial data $\bar{\nu}$. Denoting with $u(\nu)$ and $u(\varepsilon,\nu)$ the solution with initial data ν of system (3.2) and (3.3) respectively, the definition of regularization considered in [2] is the following.

Definition 3.1. *Let $V(x)$ be a singular potential and $\bar{\nu}\in S(V)$. We say that the problem (3.2) is weakly regularizable via smoothing of the potential if there exist two sequences $(\varepsilon_k)_k$, $(\nu_k)_k$ tending to 0 and $\bar{\nu}$ respectively, such that the flow*

$$\tilde{u}(\nu)=\begin{cases} u(\nu) & \nu\notin S(V)\\ \lim_{k\to\infty}u(\varepsilon_k,\nu_k) & \nu\in S(V)\end{cases}$$

is continuous with respect to ν.

In addition we say that

Definition 3.2. *The problem (3.2) is strongly regularizable via smoothing of the potential if there exists*

$$\lim_{(\varepsilon,\nu)\to(0,\bar{\nu})}u(\varepsilon,\nu)=u_0$$

and the flow

$$\tilde{u}(\nu)=\begin{cases} u(\nu) & \nu\notin S(V)\\ u_0 & \nu\in S(V)\end{cases}$$

is continuous with respect to ν.

In [2] the authors show that in the case of homogeneous potential of degree α, $V(x)=\frac{1}{|x|^\alpha}$, $0<\alpha<2$, the two body problem is weakly regularizable via smoothing of the potential if and only if α is in the form (3.1). On the other hand they showed that the problem is never strongly regularizable via smoothing of the potential. We will apply this technique in the study of the collision solutions for the two body problem with a logarithmic potential and we will show that, not only for the logarithmic potential, but for a large class $\mathcal{V}^*$ of potential, the problem is regularizable according to definition 3.2.

3.2 The potential functions sets $\mathcal{V}$ and $\mathcal{V}^*$

Definition 3.3 (The function set $\mathcal{V}$)**.** *We define $\mathcal{V}$ the set of functions $V(x)\in\mathbb{C}^\infty(\mathbb{R}^+,\mathbb{R})$ with the properties:*

i. $\lim_{x\to 0^+}V(x)=+\infty$

there exists $S>0$ such that for every $x\in(0,S)$

ii. $V'(x)<0$, $V''(x)>0$

iii. *the function* $\dfrac{V'(x)}{V''(x)}$ *is decreasing with respect to x*

and

iv. $\dfrac{d}{dx}\dfrac{V'(x)}{V''(x)}(0) < -\dfrac{1}{3}$

The property **iii** is equivalent to

$$V''' < 0 \quad \text{and} \quad (\mathrm{V}'')^2 - \mathrm{V}'\mathrm{V}''' < 0 \quad \text{in } (0, \mathrm{S})$$

while the properties **iii,iv** assure that there exist a $T > 0$ and $\eta > 0$ such that

$$\frac{V'(x)}{V''(x)} \leq -\frac{x}{3-\eta} \quad \text{in} \quad (0, \mathrm{T}) \tag{3.4}$$

Let

$$\bar{R} := \min\{T, S\}. \tag{3.5}$$

In order to better understand the meaning of the previous definition, let's now discuss some examples.

- Logarithmic potential: $V(x) = -\log(x)$.

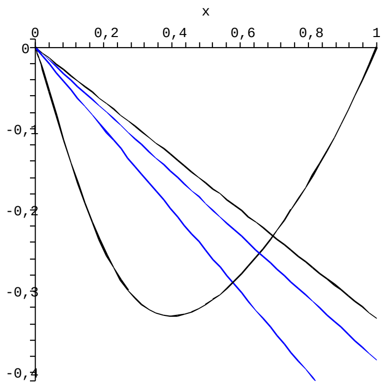
  In this case $V'(x) = -\frac{1}{x}$, $V''(x) = \frac{1}{x^2}$ thus $\frac{V'(x)}{V''(x)} = -x$ and condition **iv.** is satisfied since the derivative is equal to -1. It follows that the logarithmic potential belongs to $\mathcal{V}$ and $\bar{R} = +\infty$. The same results holds for every potential on the form $V(x) = -\log(x^\gamma)$ with $\gamma > 0$.

- α-homogeneous potential: $V(x) = x^{-\alpha}$, $\alpha > 0$

  Condition **i-iii.** are clearly satisfied for every values of $\alpha > 0$. Moreover since $V'(x) = -\frac{\alpha}{x^{\alpha+1}}$, $V''(x) = \frac{\alpha(\alpha+1)}{x^{\alpha+2}}$ one has $\frac{V'(x)}{V''(x)} = -\frac{x}{\alpha+1}$. Thus the class $\mathcal{V}$ includes the α-homogeneous potential with $0 < \alpha < 2$. As well as the previous case, $\bar{R} = \infty$.

- Weaker singular potential: $V(x) \sim \log(-\log(x))$ as $x \to 0$.

  For instance we can consider the function $V(x) = \log(-\log(x) + x)$. The limit in condition **i.** is satisfied and one can easily check that **ii.** holds for every $x \in (0, 1)$.



Moreover

$$\frac{V'(x)}{V''(x)} = \frac{(1-x)\,x\,(-\ln(x) + x)}{\ln(x) - 3\,x + 1 + x^2}$$

is decreasing in a right neighbourhood of the origin and the limit of $\dfrac{d}{dx}\left(\dfrac{V'(x)}{V''(x)}\right)$ for x leading to zero is equal to -1. Furthermore for any choice of $\eta < 2$ it happens that there exists $T < 1$, depending on η such that the function $\frac{V'(x)}{V''(x)}$ is below the line $-\frac{x}{3-\eta}$ for $x \in (0, T)$, as shown in the figure to the side. In this case the value of $\bar{R}$ is finite and less that 1.

Lemma 3.1. *Let $V(x) \in \mathcal{V}$. Then*

$$\lim_{x\to 0} V' = -\infty, \qquad \lim_{x\to 0} V'' = +\infty$$

and

$$\lim_{x\to 0} \frac{V'}{V''} = 0, \qquad \lim_{x\to 0} \frac{V}{V'} = 0$$

Proof
Since $\lim_{x\to 0^+} V(x) = +\infty$, it follows that $\liminf_{x\to 0^+} V'(x) = -\infty$ but, since the second derivative $V''(x)$ is positive definite in a neighbourhood of the origin, the function $V'(x)$ is increasing nearby $x = 0$; this implies $\lim_{x\to 0} V' = -\infty$. Similarly, repeating the same argument for $V''(x)$ in place of $V'(x)$, the sign of $V'''(x)$ implies that $\lim_{x\to 0} V'' = +\infty$.

In order to prove the remaining statement of the theorem, we note that the fraction $\frac{V'(x)}{V''(x)}$ is a negative and continuous function in $(0, S)$. Moreover, for condition **iii.** we infer that $\frac{V'(x)}{V''(x)}$ admits a finite limit l as $x \to 0^+$. Suppose by contradiction that $l \neq 0$ and set $L = \frac{1}{l}$. Then

$$L = \lim_{x\to 0^+} \frac{V''(x)}{V'(x)} = \lim_{x\to 0^+} \frac{d}{ds} \log(V'(s))\Big|_{s=x}$$

and for small x and y, $x < y$, from the Lagrange formula, one has

$$\begin{aligned} \log(V'(y)) - \log(V'(x)) &= L(y-x) + o(y-x) \\ \log\left(\frac{V'(y)}{V'(x)}\right) &= L(y-x) + o(y-x) \end{aligned}$$

Fixing y and taking the limit for $x \to 0$ we obtain that the member on the left side goes to infinity while the right one is bounded. This is a contradiction then $l = 0$. The limit of $\frac{V}{V'}$ comes from the previous one applying the formula of De L'Hôpital . □

Definition 3.4. *Denote with $\mathcal{V}^*$ the set of functions $V(x) \in \mathcal{V}$ with the further property*

v. $\lim_{\lambda\to 0} \dfrac{V(\lambda x)}{V(\lambda)} = 1$ *for every $x \geq 1$ uniformly in every compact $K = [1, M]$.*

Since the condition **v.** will be crucial for the regularization to succeed we want to discuss a little its meaning. First we observe that the function $V(x) = -\log(x)$ satisfies the above condition, indeed

$$\lim_{\lambda\to 0} \frac{\log(\lambda x)}{\log(\lambda)} = \lim_{\lambda\to 0} \left(1 + \frac{\log(x)}{\log(\lambda)}\right) = 1$$

uniformly in x on every compact K. The same result is obtained if we consider potential functions that are asymptotic to some power of the logarithmic or to $\log(-\log(x))$. On the other hand the α-homogeneous potentials don't satisfy the condition since

$$\lim_{\lambda\to 0} \frac{(\lambda x)^{-\alpha}}{\lambda^{-\alpha}} = x^{-\alpha}$$

From these examples one can be lead to define the condition **v.** as a logarithmic-type condition, that is it is satisfied only by those function whose singularity is very weak, weaker than the homogeneous one.

An other theorem we will need is the following

Proposition 3.1. *Let $V(x) \in \mathcal{V}^*$, then for every $\rho > 1$*

$$\lim_{(\delta,\varepsilon)\to(0,0)} \frac{V(\rho\delta + \varepsilon)}{V(\delta + \varepsilon)} = 1 \qquad \delta, \varepsilon > 0$$

Proof
We rewrite the above limit in the form

$$\lim_{(\delta,\varepsilon)\to(0,0)} \frac{V\left((\delta+\varepsilon)\frac{\rho\delta+\varepsilon}{\delta+\varepsilon}\right)}{V(\delta+\varepsilon)}.$$

Since $\rho > 1$, for every choice of positive values of ε and δ, it holds

$$1 \leq \frac{\rho\delta + \varepsilon}{\delta + \varepsilon} \leq \rho$$

then, from definition (3.4) of $\mathcal{V}^*$, replacing λ and M with $\delta + \varepsilon$ and ρ respectively, we infer the statement of the theorem. $\square$

3.3 A theorem of boundness

Theorem 11. *Let $V(x) \in \mathcal{V}$ and $\bar{R}$ as in (3.5). Then for every $\bar{r} < \bar{R}$ there exists an $\bar{\varepsilon}$ such that $\forall \varepsilon < \bar{\varepsilon}$ and for every $0 < y < \bar{r}$ it holds*

$$F_{(y,\bar{r})}(\varepsilon, x) := \frac{(\bar{r} + x)}{(\frac{x}{y} - 1)} \left[\left(\frac{x}{y}\right)^2 \left(\frac{V(x+\varepsilon) - V(\bar{r}+\varepsilon)}{V(y+\varepsilon) - V(\bar{r}+\varepsilon)}\right) \frac{(\bar{r}^2 - y^2)}{(\bar{r}^2 - x^2)} - 1 \right] \geq K\bar{r}$$

for every $y \leq x \leq \bar{r}$ and $K > 0$.

The proof of theorem 11 is split in two parts: we first show that there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon < \bar{\varepsilon}$ the function $F_{(y,\bar{r})}(\varepsilon, x) > F_{(y,\bar{r})}(0, x)$ for every $0 < y \leq x \leq \bar{r} < S$, then we show that $F_{(y,\bar{r})}(0, x) > K\bar{r}$ for every $0 < y \leq x \leq \bar{r} < \bar{R}$

Lemma 3.2. *Let $V(x)$ be a function satisfating the properties **i.-iii.**. Then for every choice of $0 < y < \bar{r} < S$ there exists $\bar{\varepsilon} > 0$ such that $\forall \varepsilon < \bar{\varepsilon}$ and $\forall x \in (y, \bar{r})$ it holds*

$$Q_{(y,\bar{r})}(\varepsilon, x) := \frac{V(x+\varepsilon) - V(\bar{r}+\varepsilon)}{V(y+\varepsilon) - V(\bar{r}+\varepsilon)} \geq \frac{V(x) - V(\bar{r})}{V(y) - V(\bar{r})} \tag{3.6}$$

Proof
Passing to the inverses, the relation (3.6) is equivalent to

$$\frac{V(y+\varepsilon) - V(\bar{r}+\varepsilon)}{V(x+\varepsilon) - V(\bar{r}+\varepsilon)} \leq \frac{V(y) - V(\bar{r})}{V(x) - V(\bar{r})}$$

and to

$$\frac{V(y+\varepsilon) - V(x+\varepsilon)}{V(x+\varepsilon) - V(\bar{r}+\varepsilon)} \leq \frac{V(y) - V(x)}{V(x) - V(\bar{r})}$$

For every fixed y, $\bar{r}$, with $y < \bar{r} < S$, and $x \in [y, \bar{r}]$, we define

$$g(\varepsilon) := \frac{V(y+\varepsilon) - V(x+\varepsilon)}{V(x+\varepsilon) - V(\bar{r}+\varepsilon)}.$$

Since $g(0) = \frac{V(y)-V(x)}{V(x)-V(\bar{r})}$, the thesis follows if we prove that there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon < \bar{\varepsilon}$, the inequality $g(\varepsilon) < g(0)$ holds. To this purpose it's enough to show that

$$\frac{dg}{d\varepsilon}(0) < 0 \quad \forall\ 0 < y < x < \bar{r} < S\ . \tag{3.7}$$

$$\frac{dg}{d\varepsilon}(0) = \frac{\Big(V'(y) - V'(x)\Big)\Big(V(x) - V(\bar{r})\Big) - \Big(V(y) - V(x)\Big)\Big(V'(x) - V'(\bar{r})\Big)}{\Big(V(x) - V(\bar{r})\Big)^2}$$

We define

$$G_{(y,\bar{r})}(x) := -V'(y)\Big(V(x) - V(\bar{r})\Big) - V(y)\Big(V'(\bar{r}) - V'(x)\Big) - V'(x)V'(\bar{r}) + V'(\bar{r})V(x)$$

The function $G_{(y,\bar r)}(x)$ is exactly the opposite of the numerator of the previous fraction, then relation (3.7) is equivalent to $G_{(y,\bar r)}(x) > 0$ for all $0 < y < x < \bar r < S$.

We observe that

$$G_{(y,\bar r)}(y) = G_{(y,\bar r)}(\bar r) = 0$$

and

$$\begin{aligned}\frac{dG_{(y,\bar r)}}{dx}(x) &= -V'(y)V'(x) - V''(x)V(\bar r) + V''(x)V(y) + V'(\bar r)V'(x)\\ &= V'(x)\Big(V'(\bar r) - V'(y)\Big) + V''(x)\Big(V(y) - V(\bar r)\Big)\end{aligned}$$

We want to study the sign of this derivative. For Rolle's theorem there exists at least one point $\bar x \in (y, \bar r)$ such that $\frac{dG_{(y,\bar r)}}{dx}(\bar x) = 0$ and, since $V'(\bar r) - V'(y) > 0$ because $V''(x) > 0$ for every $x < S$, the previous relation implies

$$\frac{V'(\bar x)}{V''(\bar x)} + \frac{V(y) - V(\bar r)}{V'(\bar r) - V'(y)} = 0 \tag{3.8}$$

We note that the sign the derivative of $G_{(y,\bar r)}$ is the same of the relation

$$P(x) = \frac{V'(x)}{V''(x)} + \frac{V(y) - V(\bar r)}{V'(\bar r) - V'(y)}$$

that depends on x only in the first fraction. For condition **iii.** in definition (3.3) the function $P(x)$ is decreasing then, from (3.8), we infer

$$\begin{aligned}\frac{dG_{(y,\bar r)}}{dx}(x) &> 0 \qquad \text{for} \quad x \in (y, \bar x)\\ \frac{dG_{(y,\bar r)}}{dx}(x) &< 0 \qquad \text{for} \quad x \in (\bar x, \bar r)\end{aligned}$$

Thus we conclude that for every fixed $0 < y < \bar r < S$ and for every $x \in (y, \bar r)$ it holds $G_{(y,\bar r)}(x) > 0$ that implies relation (3.7).

□

Proof of theorem 11.
We fix $0 < y < \bar r < \bar R$. For lemma 3.2 there exists $\bar\varepsilon > 0$ such that for every $0 \le \varepsilon < \bar\varepsilon$ and for every $x \in (y, \bar r)$ we have $F_{(\bar r,y)}(\varepsilon, x) \ge F_{(\bar r,y)}(0, x)$, then it's enough to show that there exists a constant K such that

$$F_{(\bar r,y)}(x) := \frac{(\bar r + x)}{(\frac{x}{y} - 1)}\left[\left(\frac{x}{y}\right)^2\left(\frac{V(x) - V(\bar r)}{V(y) - V(\bar r)}\right)\frac{(\bar r^2 - y^2)}{(\bar r^2 - x^2)} - 1\right] \ge K\bar r$$

Suppose for a moment that exists $\beta > 0$ such that

$$\frac{V(x) - V(\bar r)}{V(y) - V(\bar r)} \ge \left(\frac{\bar r^{2-\beta} - x^{2-\beta}}{\bar r^{2-\beta} - y^{2-\beta}}\right)\frac{y^{2-\beta}}{x^{2-\beta}} \tag{3.9}$$

For every fixed $0 < y < x < \bar r$ the function

$$f(\gamma) := \left(\frac{\bar r^\gamma - x^\gamma}{\bar r^\gamma - y^\gamma}\right)\frac{y^\gamma}{x^\gamma} = \frac{\left(\frac{\bar r}{x}\right)^\gamma - 1}{\left(\frac{\bar r}{y}\right)^\gamma - 1}$$

is decreasing in γ, for every real $\gamma \neq 0$ (see formula (2.19) and the further calculation), then

$$f(2 - \beta) > f(2) + C \qquad C = C(y, x, \bar r, \beta) > 0.$$

It follows
$$\frac{V(x)-V(\bar r)}{V(y)-V(\bar r)} \geq \left(\frac{\bar r^{2-\beta}-x^{2-\beta}}{\bar r^{2-\beta}-y^{2-\beta}}\right)\frac{y^{2-\beta}}{x^{2-\beta}} > \frac{(\bar r^2-x^2)}{(\bar r^2-y^2)}\frac{y^2}{x^2}+C$$

thus

$$\begin{aligned} F_{(\bar r,y)}(x) &\geq \frac{(\bar r+x)}{(\frac{x}{y}-1)}\left[\left(\frac{x}{y}\right)^2\left(\frac{(\bar r^2-x^2)}{(\bar r^2-y^2)}\frac{y^2}{x^2}+C\right)\frac{(\bar r^2-y^2)}{(\bar r^2-x^2)}-1\right] \\ &\geq y\frac{(\bar r+x)}{(x-y)}\left[\frac{x^2}{y^2}\frac{(\bar r^2-y^2)}{(\bar r^2-x^2)}C\right] \\ &\geq K\bar r \end{aligned}$$

for a suitable choice of $K>0$.

Therefore, in order to obtain the proof, it's enough to show that there exists a positive β such that relation (3.9) holds for every $x \in (y,\bar r)$. This is equivalent to

$$\frac{(V(x)-V(\bar r))}{(\bar r^{2-\beta}-x^{2-\beta})}x^{2-\beta} \geq \frac{(V(y)-V(\bar r))}{(\bar r^{2-\beta}-y^{2-\beta})}y^{2-\beta} \quad \forall x \in (y,\bar r) \tag{3.10}$$

Since for $x=y$ the previous relation is true, if we calculate the derivative on the left side member with respect the variable x and we show that such derivative is positive, the relation (3.10) holds for every $x \in (y,\bar r)$. Upon derivating, we obtain

$$\begin{aligned} &\frac{d}{dx}\left(\frac{(V(x)-V(\bar r))}{(\bar r^{2-\beta}-x^{2-\beta})}x^{2-\beta}\right) = \\ &\frac{x^{2-\beta}(\bar r^{2-\beta}-x^{2-\beta})V'(x)+(2-\beta)\bar r^{2-\beta}x^{1-\beta}(V(x)-V(\bar r))}{(\bar r^{2-\beta}-x^{2-\beta})^2} \end{aligned} \tag{3.11}$$

Let $x^{1-\beta}N(x)$ be the numerator in (3.11); one has $N(\bar r)=0$ and

$$\begin{aligned} \frac{dN}{dx}(x) &= \left(\bar r^{2-\beta}-x^{2-\beta}(3-\beta)\right)V'(x)+x(\bar r^{2-\beta}-x^{2-\beta})V''(x) \\ &= (\bar r^{2-\beta}-x^{2-\beta})\Big((3-\beta)V'(x)+xV''(x)\Big) \end{aligned}$$

For (3.4), for every choice of $0<\beta<\eta$ it follows $\frac{dN}{dx}(x) \leq 0$. Then the derivative in (3.11) is positive and relation (3.10) is true for every $x \in (y,\bar r)$. □

Corollary 3.1. *Let $V(x) \in \mathcal{V}$ with the stronger assumption*

$$\frac{V'(x)}{V''(x)} \leq -\frac{x}{2} \quad x \in (0,T) \tag{3.12}$$

Then theorem (11) *holds for $K=1$.*

Proof

The assumption (3.12) is equivalent to impose $\dfrac{d}{dx}\dfrac{V'(x)}{V''(x)}(0) < -\dfrac{1}{2}$ and with the same calculation done in the proof of the last theorem it holds

$$\frac{V(x)-V(\bar r)}{V(y)-V(\bar r)} \geq \left(\frac{\bar r-x}{\bar r-y}\right)\frac{y}{x}$$

Thus

$$\begin{aligned}
F_{(\bar{r},y)}(x) &\geq \frac{(\bar{r}+x)}{(\frac{x}{y}-1)}\left[\left(\frac{x}{y}\right)^2 \frac{(\bar{r}-x)}{(\bar{r}-y)}\frac{y}{x}\frac{(\bar{r}^2-y^2)}{(\bar{r}^2-x^2)}-1\right] \\
&\geq y\frac{(\bar{r}+x)}{(x-y)}\left[\frac{x}{y}\frac{(\bar{r}+y)}{(\bar{r}+x)}-1\right] \\
&\geq \bar{r}\ .
\end{aligned}$$

□

3.4 Setting of the problem

We first analyse the dynamical system (3.2) in the setting of the theory of Hamiltonian systems. For any potential functions $V(x)$ we recall the definition of energy E and angular momentum l of solutions of the system (3.2)

$$E = \frac{1}{2}|\dot{u}|^2 - V(|u|)$$
$$l = u \wedge \dot{u}$$

Since in the equation of motion the forcing term $\nabla V(|u|)$ is radially symmetric, it turns out that both E and l are preserved along the solutions of the system. Moreover, since the motion is planar, as well as we done in the first chapter, in the following l is used to denote the modulus of the angular momentum. We also recall that, passing to polar coordinates (r, θ), the quantities E and l are expressed in the form

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{l^2}{r^2} - V(r) \tag{3.13}$$
$$l = r^2\dot{\theta}$$

We define

$$f(r) := 2r^2(E + V(r)) \tag{3.14}$$

then the relation (3.13) is equivalent to

$$l^2 = f(r) - (r\dot{r})^2\ .$$

The last formula shows that a solution of system (3.2) of energy E and angular momentum l exists only for that values of the radial coordinate $r \geq 0$ such that $f(r) \geq l^2$. We extend the definition of minimal and maximal values of the radial coordinate given in definition 1.3 to system with a general potential. For any choice of $V(x)$, at least of class $\mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$, and for fixed values of E and l, we denote with R_+, if it exists, the minimum positive value of r such that $f(r) = l^2$ and $f'(r) < 0$, and with R_-, if it exists, the minimum positive value of $r < R_+$ such that $f(r) = l^2$ and $f'(r) > 0$. Otherwise we set $R_+ = +\infty$ and $R_- = 0$ if they are not defined by the above definition. The values of R_- and R_+ depend strictly on the potential $V(x)$ and on the values of l and E. We illustrate in figure 3.1 the different behaviour of the function $f(r)$ corresponding to the choice of different potentials.

In the first case the potential is $V(x) = \frac{1}{x^3}$ and a negative value of E is chosen. We note that $f(r)$ tends to $+\infty$ as r tend to zero thus, for any choice of l, the minimal value of radial coordinate is zero. This means that all the trajectories with negative energy lead to collision.

In the other case we drawn the function $f(r)$ in case of Kepler and logarithmic potential. The considerations about the values of R_+ and R_- are the same done in section §1.1 then we don't discuss anymore. We only point out that, unlike the first case, $f(r)$ tends to zero as r goes to zero, therefore the collision solution occurs only in case of zero angular momentum. In order to remark this difference, we give the following definition.

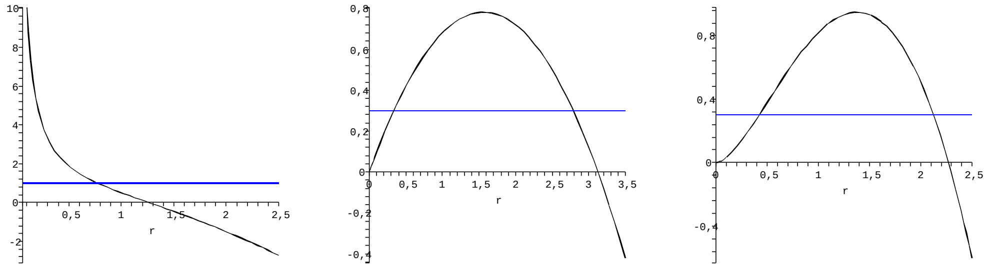


Figure 3.1:

Definition 3.5. *We say that a potential function $V(x)$, singular in the origin, is of* ***weak type*** *if*

$$\lim_{x \to 0^+} x^2 V(x) = 0$$

Otherwise we say that $V(x)$ is a ***strong type*** *potential.*

It comes out that the potential $V(x) = x^{-3}$ is of strong type, while the Kepler and logarithmic potentials are of weak type. The definition 3.5 is motivated by the fact that if $V(x)$ is of strong type the condition to have the angular momentum equal zero is not necessary for a solution $u(t)$ of system (3.2) to end into collision. Otherwise, as it will be shown in the following theorem, if the potential is of weak type, a collision solution occurs if and only if the value of the angular momentum is zero.

Theorem 12. *If $V(x)$ is a potential of weak type, a solution $u(t)$ of the dynamical system $\ddot{u} = \nabla V(|u|)$ ends into a collision if and only if the angular momentum is zero.*

Proof
Denote with E and l the energy and the angular momentum of the solution $u(t)$ and let $f(r)$ be the function defined in (3.14), then a solution exists only for the values of radial coordinate r that satisfy

$$l^2 \le f(r)$$

Suppose $l = 0$: since $V(x)$ tends to infinity as x goes to zero, for every value of E there exists a neighbourhood of the origin where $E + V(r) > 0$ then the solution presents a collision.

Conversely, since $V(x)$ is a weak type potential, it follows that $f(r) \to 0$ as $x \to 0^+$ thus for every value of $l \neq 0$ there exist a neighbourhood of the origin where $l^2 > f(r)$. This means that the collision can not be attained on $u(t)$. □

A similar classification of the set of singular potentials can be found in [11] in which a potential $V(x)$ is said to satisfy a strong force condition at a point x_0 if V tends to infinity as x tends to x_0 and also there exists a function U with infinitely deep wells at x_0, such that

$$V(x) \ge |\nabla U(x)|^2$$

in a neighbourhood of x_0. This classification arises from the study of the dynamical system $\ddot{u} = \nabla V(u)$ by a variational approach and it is shown that the action integral satisfies the Palais and Smale condition provided V satisfies the strong force condition. Without going deeper into details, we want only to notice that, among the homogeneous potentials, the set of potentials with the property to be of strong type and the set of potentials satisfying the strong force condition coincide.

Theorem 13. *Every $V(x) \in \mathcal{V}$ is a weak type potential.*

Proof
For relation (3.4) there exists $\eta > 0$ such that

$$\frac{V''}{-V'} \leq \frac{3-\eta}{x}$$

for every x in a neighbourhood of the origin.

The last relation is equivalent to $-\frac{d}{dx}\log(-V') \leq \frac{3-\eta}{x}$, then, by integration it follows an asymptotic estimate for the derivative of $V(x)$

$$-\int_x^1 \frac{d}{dt}\log(-V')dt \leq \int_x^1 \frac{3-\eta}{t}dt$$
$$\log(-V'(x)) \leq \log(x^{-(3-\eta)}) + C$$
$$-V'(x) \leq \frac{C}{x^{3-\eta}}$$

Again by integration we have

$$V(x) \leq \frac{C}{x^{2-\eta}} + C_1 \tag{3.15}$$

and we conclude

$$\lim_{x\to 0} x^2 V(x) = 0 \ .$$

□

Our intent is to discuss for which potentials $V(x)$ in the class $\mathcal{V}$ and in which sense the singular solution could be extended beyond the singularity following the technique of the smoothing of the potential.

The last two theorems show that if the potential function $V(x) \in \mathcal{V}$, a solution of the system (3.2) has a collision and, as a consequence, it is not globally defined if and only if the angular momentum is zero. Therefore the set $S(V)$ of initial conditions $\bar{\nu} = (\bar{q}_0, \bar{p}_0)$ that make the solutions to be singular consists in those $(\bar{q}_0, \bar{p}_0)$ satisfying $l = |\bar{q}_0 \wedge \bar{p}_0| = 0$.

The regularized system we are taking into account is the same as (3.3)

$$\begin{cases} \ddot{u} = \nabla V_\varepsilon(|u|) \\ (u(0), \dot{u}(0)) = (q_0, p_0) \qquad \in \mathbb{R}^2 \times \mathbb{R}^2 \end{cases}$$

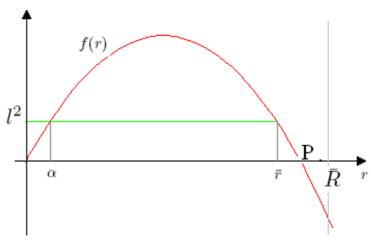
where the initial data $\nu = (q_0, p_0)$ depends on the potential $V(x)$ and on the value of energy E of the collision solution as follows.

Let $V(x) \in \mathcal{V}$ and E fixed. Define $P > 0$ the first positive solution of equation $f(r) = 0$, $P := +\infty$ if such value does not exists, and denote with

$$R := \min(\bar{R}, P) \ . \tag{3.16}$$

where $\bar{R}$ is defined in (3.5).

Case 1 $R = P$.



In this case the solutions of the unperturbed system are bounded in a ball centered in the origin of radius $P < \bar{R}$ and P represents the maximal value of the radial coordinate of the collision solution. Thus, without loss of generality, we can set the initial condition $\bar{\nu}$ leading to collision as

$$\bar{\nu} = (\bar{q}_0, 0), \qquad |\bar{q}_0| = P \tag{3.17}$$

Therefore, for small enough value of l we define the initial condition $\nu = (q_0, p_0)$ of the regularized system as

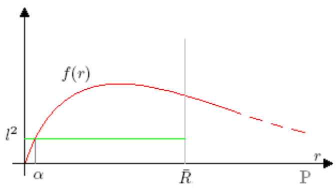
$$\begin{array}{ll} q_0 \parallel \bar{q}_0 & |q_0| = \bar{r} \\ p_0 \cdot q_0 = 0 & |p_0|^2 = \frac{l^2}{\bar{r}^2} \end{array} \tag{3.18}$$

where $\bar{r}$ depends on ε and on l and it's defined as the largest solution of the energy equation $E = \frac{1}{2}p_0^2 - V_\varepsilon(r)$. From this definition it follows that the solution of system (3.3) with initial condition ν admits angular momentum equal to l, energy E and the values of R_- and R_+ are respectively α and $\bar{r}$, where α is the solution different of $\bar{r}$ of the equation

$$\frac{1}{2}\frac{l^2}{r^2} - V_\varepsilon(r) = \frac{1}{2}\frac{l^2}{\bar{r}^2} - V_\varepsilon(\bar{r})$$

Case2 Otherwise, if $R = \bar{R}$ we chose as initial condition leading to collision the couple

$$\bar{\nu} = (\bar{q}_0, \bar{p}_0), \qquad |\bar{q}_0| = \bar{R}, \qquad |\bar{p}_0|^2 = 2(E + V(\bar{R})) \tag{3.19}$$



where the initial velocity $\bar{p}_0$ is directed toward the center of attraction.
For small enough l let the initial condition $\nu = (q_0, p_0)$ be defined as

$$\begin{array}{ll} q_0 = \bar{q}_0, & p_0 = (v_1, v_2) \\ v_1^2 = 2E - \frac{l^2}{\bar{R}^2} + 2V_\varepsilon(\bar{R}) & v_2^2 = \frac{l^2}{\bar{R}^2} \end{array} \tag{3.20}$$

where v_1, v_2 are the radial and the orthogonal part of the velocity. The choice of initial condition is done in such a way that the corresponding solution has energy E, angular momentum l and the value α of the minimal radial coordinate is the solution, less then $\bar{R}$, of equation

$$\frac{1}{2}\frac{l^2}{r^2} - V_\varepsilon(r) = E$$

Moreover $R_+ > \bar{R}$ or infinity and $R_- \to 0$ as $l \to 0$.

In any case, for every potential function $V(x) \in \mathcal{V}$, given a collision solution $\bar{u}(t)$ of system (3.2) starting from $\bar{\nu} = (\bar{q}_0, \bar{p}_0)$ in the form (3.17) or (3.19), the initial conditions $\nu = (q_0, p_0)$ defined in (3.18) and (3.20) depend only on the choice of the smoothing parameter ε and on the value of the angular momentum l we impose to the solution of the regularized system, indeed the value of the energy E is fixed and it is equal to the energy of the collision solution we are going to extend. From the definition it follows

$$\lim_{(\varepsilon,l)\to(0,0)} \nu = \bar{\nu} \tag{3.21}$$

Denoting with $u_{\varepsilon,\nu}(t)$ the solution of the regularized system (3.3) leading from the initial data ν, we investigate the existence of the limit of $u_{\varepsilon,\nu}(t)$ as $(\varepsilon, \nu) \to (0, \bar{\nu})$ and its relation with $\bar{u}(t)$. In other words, since (3.21) holds, we study the convergence of the solutions of the regularized system as the angular momentum l tends to zero together with the vanishing of the perturbation ε. Thus it could be convenient use the notation $(\varepsilon, l) \to (0, 0)$ in place of $(\varepsilon, \nu) \to (0, \bar{\nu})$.

Given a potential function $V(x) \in \mathcal{V}$ and a collision solution $\bar{u}(t)$ of system (3.2) with energy E and initial condition $\bar{\nu}$, denote with (T_-, T_+) the maximal interval of existence of the solution.

Definition 3.6. *Let $V(x) \in \mathcal{V}$. We say that the two-body problem with potential $V(x)$ is regularizable if for every collision solution $\bar{u}$ there exists*

$$\lim_{(\varepsilon,l)\to(0,0)} u_{\varepsilon,l}(t) = u_0(t) \qquad \varepsilon > 0$$

and

$$u_0(t) = \bar{u}(t) \qquad t \in (T_-, T_+)$$

The theorem we will prove is the following:

Theorem 14 (Main Theorem)**.** *For every $V(x) \in \mathcal{V}^*$ the two-body problem with potential $V(x)$ is regularizable according to definition 3.6.*

Before starting the proof of the theorem we recall the definition of the apsidal angle of a solution $u_{\varepsilon,l}(t)$ of system (3.3) with angular momentum l. As usual, denoting with R_- and R_+ the values of pericentre and apocentre of the orbit $u_{\varepsilon,l}(t)$, the apsidal angle is given by

$$\Theta_{\varepsilon,l}(u) = \int_{R_-}^{R_+} \frac{l}{r^2 \dot{r}} dr \tag{3.22}$$

It's convenient to rewrite the formula of the apsidal angle for the solution $u_{\varepsilon,l}(t)$ according to the value of R in (3.16):

if $R = P$, **case 1**, all the regular orbits $u_{\varepsilon,l}$ are bounded in the ball of radius $\bar{R}$ and, reminding the definitions of $\bar{r}$ and α, the value of apsidal angle is

$$\Theta_{\varepsilon,l}(u) = \int_{\alpha}^{\bar{r}} \frac{l}{r^2 \dot{r}} dr$$

otherwise, in **case 2**, for l and ε small enough the orbit $u_{\varepsilon,l}$ is not bounded in a ball of radius $\bar{R}$ and it could be also unbounded. Thus the apocentre R_+, possibly infinity, is larger that $\bar{R}$ and we write the apsidal angle of the solution $u_{\varepsilon,l}(t)$ as

$$\Theta_{\varepsilon,l}(u) = \int_{\alpha}^{\bar{R}} \frac{l}{r^2 \dot{r}} dr + \int_{\bar{R}}^{R^+} \frac{l}{r^2 \dot{r}} dr$$

where α is the pericentre.

3.5 Proof of Main Theorem and property of the extended flow

This section is devoted to the proof of the main theorem and to the study of the regularity of the extended flow with respect initial data.

The plan of the proof of theorem 14 is the following. In theorem 15 we gain the existence of the limit of the apsidal angle $\Theta_{\varepsilon,l}(u)$ as $(\varepsilon, l) \to (0, 0)$: to this aim we first prove in lemma 3.3 the $\mathbb{L}^1$ boundness of the integrand in (3.22) then we apply the dominated convergence theorem and pass to the limit under the integral sign. The result we obtain suggest to define the extension $u_0(t)$ of the collision solution $\bar{u}(t)$ beyond the singularity as a transmission solution. We show, in theorem 16, that this definition of extension is optimal in the sense that the sequence of paths $u_{\varepsilon,l}(t)$, solutions of the regularized problem, converges for $(\varepsilon, l) \to (0, 0)$ to $u_0(t)$ for every t.

Moreover, we analize the properties of the Poincaré section of the extended flow in the phase space: in theorems 17 we state the continuity of the Poincaré section whit respect the initial data while in theorem 18 we prove that the Poincaré section is not differentiable with respect initial data in the neighbourhood of initial data leading to collision.

Theorem 15. *Let $V(x) \in \mathcal{V}^*$ and $\bar{u}(t)$ any collision solution of the two body-problem with initial condition $\bar{\nu} = (\bar{q}_0, \bar{p}_0)$. Then there exists*

$$\lim_{(\varepsilon,l)\to(0,0)} \Theta_{\varepsilon,l}(u)$$

and such limit is $\frac{\pi}{2}$.

Proof
Let $V(x) \in \mathcal{V}$ and denote with E the energy of the collision solution $\bar{u}(t)$. According with the previous setting, for every sufficiently small $\varepsilon > 0$ and l, let $u_{\varepsilon,l}(t)$ be the solution of the regularized system (3.3) with initial data ν and $\Theta_{\varepsilon,l}(u)$ the corresponding apsidal angle. Reminding the definition of $\bar{R}$ and R_+, we define

$$\beta = \min\{\bar{R}, R_+\}$$

and

$$\Delta_{\varepsilon,l}(u) = \int_{R_-}^{\beta} \frac{l}{r^2 \dot{r}} dr$$

It follows that if the orbit $u_{\varepsilon,l}(t)$ is bounded in a ball of radius $\bar{R}$, **case 1**, then $\Theta_{\varepsilon,l}(u) = \Delta_{\varepsilon,l}(u)$, otherwise $\Theta_{\varepsilon,l}(u) = \Delta_{\varepsilon,l}(u) + \int_{\bar{R}}^{R^+} \frac{l}{r^2 \dot{r}} dr$. The introduction of β and $\Delta_{\varepsilon,l}(u)$ is motivated by the fact that the main contribution to the value of the apsidal angle is given by those values of the radial coordinate near the pericentre. More precisely we will prove that the remaining term $\int_{\bar{R}}^{R^+} \frac{l}{r^2 \dot{r}} dr$ tends to zero uniformly as $(\varepsilon, l) \to (0,0)$, thus in both the cases, the convergence of $\Delta_{\varepsilon,l}(u)$ implies the convergence of $\Theta_{\varepsilon,l}(u)$.

In order to deal with the convergence of $\Delta_{\varepsilon,l}(u)$ we first rewrite the integrand in a different way. From the conservation of energy

$$\dot{r}^2 = 2(E + V_\varepsilon(r)) - \frac{l^2}{r^2}$$

thus, replacing the radial velocity and extracting the roots R_- and β from the denominator, we infer

$$\begin{aligned}
\Delta_{\varepsilon,l}(u) &= \int_{R_-}^{\beta} \frac{1}{r\sqrt{\frac{2r^2}{l^2}(E + V_\varepsilon(r)) - 1}} dr \\
&= \int_{R_-}^{\beta} \frac{1}{r\sqrt{(r - R_-)(\beta - r)}} \sqrt{\frac{(r - R_-)(\beta - r)}{\frac{2r^2}{l^2}(E + V_\varepsilon(r)) - 1}}
\end{aligned}$$

In **case 1**, the parameter β is equal to R_+, the maximal value of radial coordinate of $u_{\varepsilon,l}$, thus replacing $E = \frac{1}{2}\frac{l^2}{R_+^2} - V_\varepsilon(R_+)$ and performing the variable changing $\rho = \frac{r}{R_-}$ we obtain

$$\begin{aligned}
\Delta_{\varepsilon,l}(u) &= \int_1^{\frac{R_+}{R_-}} \frac{1}{\rho\sqrt{(R_+ - \rho R_-)(\rho - 1)}} \sqrt{\frac{(R_+ - \rho R_-)(\rho - 1)}{\left(\frac{\rho^2 R_-^2}{R_+^2} - 1\right) + \frac{2R_-^2\rho^2}{l^2}\Big(V_\varepsilon(\rho R_-) - V_\varepsilon(R_+)\Big)}} d\rho \\
&= \int_1^{\frac{R_+}{R_-}} \frac{1}{\rho\sqrt{(R_+ - \rho R_-)(\rho - 1)}} \sqrt{K_1} d\rho \qquad (3.23)
\end{aligned}$$

In **case 2**, β is equal to $\bar{R}$ but, since $\bar{R}$ is not an apsid of the orbit $u_{\varepsilon,l}(t)$, the best we can do is to express $E = \frac{1}{2}|\vec{v}|^2 - V_\varepsilon(\bar{R})$, where the vector $\vec{v}$ is the initial velocity (3.20). Anyway, replacing the formula of the energy and performing the same change of variables as before, we can write

$$\begin{aligned}
\Delta_{\varepsilon,l}(u) &= \int_1^{\frac{\bar{R}}{R_-}} \frac{1}{\rho\sqrt{(\bar{R} - \rho R_-)(\rho - 1)}} \sqrt{\frac{(\bar{R} - \rho R_-)(\rho - 1)}{\frac{2R_-^2\rho^2}{l^2}(\frac{1}{2}|\vec{v}|^2 - V_\varepsilon(\bar{R}) + V_\varepsilon(\rho R_-)) - 1}} d\rho \\
&= \int_1^{\frac{\bar{R}}{R_-}} \frac{1}{\rho\sqrt{(\bar{R} - \rho R_-)(\rho - 1)}} \sqrt{K_2} d\rho \qquad (3.24)
\end{aligned}$$

We proceed with the proof of the theorem as follows: first we exhibit, for every $V(x) \in \mathcal{V}$, an uniform bound for the functions K_1 and K_2, provided ε small enough is taken, then we apply the Lebesgue's theorem in order to get the limit of $\Delta_{\varepsilon,l}(u)$ as $(\varepsilon, l) \to (0,0)$ and we'll prove that such limit exists if $V(x) \in \mathcal{V}^*$.

Lemma 3.3. *Let $V(x) \in \mathcal{V}$, then for ε small enough the functions K_1 and K_2 are bounded by a constant in their domain.*

Proof
We start with the analysis of the function K_1: since the quantities R_- and R_+ are extremal points for the radial component of the motion $r(t)$, they are solutions of the energy equation $E = \frac{1}{2}\frac{l^2}{r^2} - V_\varepsilon(r)$. Subtracting the one from the other, we obtain the relation

$$\frac{2R_-^2}{l^2} = \frac{1}{(V_\varepsilon(R_-) - V_\varepsilon(R_+))}\frac{R_+^2 - R_-^2}{R_+^2}$$

After replacing in the formula of K_1 the term $\frac{2R_-^2}{l^2}$ with the previous one, it holds

$$\begin{aligned}
K_1 &= \frac{(R_+ - \rho R_-)(\rho - 1)}{\left(\frac{\rho^2 R_-^2}{R_+^2} - 1\right) + \rho^2 \frac{\left(V_\varepsilon(\rho R_-) - V_\varepsilon(R_+)\right)}{\left(V_\varepsilon(R_-) - V_\varepsilon(R_+)\right)} \frac{(R_+^2 - R_-^2)}{R_+^2}} \\
&= \frac{(\rho - 1)(R_+ - \rho R_-)}{\left(\frac{R_+^2 - R_-^2\rho^2}{R_+^2}\right)\left(\rho^2 \frac{\left(V_\varepsilon(\rho R_-) - V_\varepsilon(R_+)\right)}{\left(V_\varepsilon(R_-) - V_\varepsilon(R_+)\right)} \frac{(R_+^2 - R_-^2)}{(R_+^2 - R_-^2\rho^2)} - 1\right)} \\
&= \frac{R_+^2}{\left(\frac{R_+ + \rho R_-}{\rho - 1}\right)\left(\rho^2 \frac{\left(V_\varepsilon(\rho R_-) - V_\varepsilon(R_+)\right)}{\left(V_\varepsilon(R_-) - V_\varepsilon(R_+)\right)} \frac{(R_+^2 - R_-^2)}{(R_+^2 - R_-^2\rho^2)} - 1\right)}
\end{aligned}$$

On the other hand, for what that concerns K_2, from definition (3.20), one has $|\vec{v}|^2 = \dfrac{l^2}{\bar{R}^2} + v_1^2$, therefore

$$\begin{aligned}
K_2 &= \frac{(\bar{R} - \rho R_-)(\rho - 1)}{\frac{2R_-^2\rho^2}{l^2}\left(\frac{1}{2}\frac{l^2}{\bar{R}^2} + \frac{1}{2}v_1^2 + V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R})\right) - 1} \\
&= \frac{(\bar{R} - \rho R_-)(\rho - 1)}{\frac{R_-^2\rho^2}{l^2}v_1^2 + \left(\frac{R_-^2\rho^2}{\bar{R}^2} - 1\right) + \frac{2R_-^2\rho^2}{l^2}\left(V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R})\right)}
\end{aligned}$$

Subtracting the energy formula $E = \frac{1}{2}|\dot{u}_{\varepsilon,l}|^2 - V_\varepsilon(u_{\varepsilon,l})$ evaluated in R_- from the same evaluated in $\bar{R}$ we infer

$$\frac{1}{2}\frac{l^2}{R_-^2} - V_\varepsilon(R_-) = \frac{1}{2}\frac{l^2}{\bar{R}^2} + \frac{1}{2}v_1^2 - V_\varepsilon(\bar{R})$$

It descends the relation

$$\frac{2R_-^2}{l^2} = \frac{1}{(V_\varepsilon(R_-) - V_\varepsilon(\bar{R}) + \frac{1}{2}v_1^2)}\frac{\bar{R}^2 - R_-^2}{\bar{R}^2}$$

that, replaced into K_2, implies

$$K_2 = \frac{(\bar{R} - \rho R_-)(\rho - 1)}{\rho^2\left(\frac{\bar{R}^2 - R_-^2}{\bar{R}^2}\right)\frac{\frac{1}{2}v_1^2}{V_\varepsilon(R_-) - V_\varepsilon(\bar{R}) + \frac{1}{2}v_1^2} + \left(\frac{R_-^2\rho^2}{\bar{R}^2} - 1\right) + \rho^2\left(\frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R})}{V_\varepsilon(R_-) - V_\varepsilon(\bar{R}) + \frac{1}{2}v_1^2}\right)\frac{\bar{R}^2 - R_-^2}{\bar{R}^2}}$$

$$= \frac{(\bar{R} - \rho R_-)(\rho - 1)}{\left(\frac{R_-^2\rho^2}{\bar{R}^2} - 1\right) + \rho^2\left(\frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R}) + \frac{v_1^2}{2}}{V_\varepsilon(R_-) - V_\varepsilon(\bar{R}) + \frac{v_1^2}{2}}\right)\frac{\bar{R}^2 - R_-^2}{\bar{R}^2}}$$

In order to get rid of the dependence on v_1, we note that for every $0 < a < b$, the function $f(z) = \frac{a+z}{b+z}$ is increasing for positive z. The condition **ii.** in definition (3.3) implies that, for every small enough ε, the function $V_\varepsilon(x)$ is decreasing with respect to x for every $x < \bar{R}$. This yields the relations $V_\varepsilon(\bar{R}) < V_\varepsilon(\rho R_-) < V_\varepsilon(R_-)$, thus replacing $a = V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R})$, $b = V_\varepsilon(R_-) - V_\varepsilon(\bar{R})$ and $z = \frac{v_1^2}{2}$ it follows

$$\frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R}) + \frac{v_1^2}{2}}{V_\varepsilon(R_-) - V_\varepsilon(\bar{R}) + \frac{v_1^2}{2}} > \frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R})}{V_\varepsilon(R_-) - V_\varepsilon(\bar{R})} \qquad \forall \rho \in \left(1, \frac{\bar{R}}{R_-}\right),\ \forall v_1^2$$

In this way we obtain a bound for K_2

$$\begin{aligned} K_2 &< \frac{(\bar{R} - \rho R_-)(\rho - 1)}{\left(\frac{R_-^2\rho^2}{\bar{R}^2} - 1\right) + \rho^2 \left(\frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R})}{V_\varepsilon(R_-) - V_\varepsilon(\bar{R})}\right) \frac{\bar{R}^2 - R_-^2}{\bar{R}^2}} \\ &= \frac{\bar{R}^2}{\frac{(\bar{R}+\rho R_-)}{\rho - 1}\left(\rho^2 \left(\frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\bar{R})}{V_\varepsilon(R_-) - V_\varepsilon(\bar{R})}\right) \frac{\bar{R}^2 - R_-^2}{\bar{R}^2 - R_-^2\rho^2} - 1\right)} \end{aligned}$$

Recalling the definition $V_\varepsilon(\cdot) = V(\cdot + \varepsilon)$ and by means of the substitutions $x = \rho R_-$, $y = R_-$ we have

$$K_1 = \frac{R_+^2}{\frac{(R_+ + x)}{(\frac{x}{y} - 1)}\left(\left(\frac{x}{y}\right)^2 \left(\frac{V(x+\varepsilon) - V(R_+ + \varepsilon)}{V(y+\varepsilon) - V(R_+ + \varepsilon)}\right) \frac{R_+^2 - y^2}{R_+^2 - x^2} - 1\right)} \qquad x \in (y, R_+)$$

and

$$K_2 < \frac{\bar{R}^2}{\frac{(\bar{R} + x)}{\frac{x}{y} - 1}\left(\left(\frac{x}{y}\right)^2 \left(\frac{V(x+\varepsilon) - V(\bar{R} + \varepsilon)}{V(y+\varepsilon) - V(\bar{R} + \varepsilon)}\right) \frac{\bar{R}^2 - y^2}{\bar{R}^2 - x^2} - 1\right)} \qquad x \in (y, \bar{R})$$

Finally, applying theorem (11), we prove an uniform bound for $K1$ and K_2 provided ε is small enough

$$K_1 < CR_+ \quad \forall \rho \in \left(1, \frac{R_+}{R_-}\right), \quad K_2 < C\bar{R} \quad \forall \rho \in \left(1, \frac{\bar{R}}{R_-}\right)$$

□

Reminding the definition of $\beta = \min\{\bar{R}, R_+\}$ and denoting with K either K_1 or K_2, without longer distinguish **case 1,2**. we can write

$$K \leq \frac{\beta^2}{\frac{(\beta + x)}{\frac{x}{y} - 1}\left(\left(\frac{x}{y}\right)^2 \left(\frac{V(x+\varepsilon) - V(\beta + \varepsilon)}{V(y+\varepsilon) - V(\beta + \varepsilon)}\right) \frac{\beta^2 - y^2}{\beta^2 - x^2} - 1\right)} \qquad x \in (y, \beta)$$

thus

$$K(\rho) < C\beta \quad \forall \rho \in \left(1, \frac{\beta}{R_-}\right)$$

Continue the proof of theorem 15.

The boundness of function K_1, K_2 is the first step in order to apply the dominated converge theorem and to obtain the limit of the integral $\Delta_{\varepsilon,l}(u)$. We now exhibit an $\mathbb{L}^1$ uniform bound for the integrand function, then we show in which cases the limit for $(\varepsilon, l) \to (0, 0)$ exists. First we recall the values of the integrals

$$\int_1^\xi \frac{1}{x\sqrt{(x-1)(1-\frac{x}{\xi})}} dx = \pi \quad \xi > 1 \tag{3.25}$$

and

$$\int_{\gamma}^{\delta} \frac{1}{x\sqrt{(x-1)(1-sx)}}dx = 2\arctan\left[\sqrt{\frac{x-1}{1-sx}}\right]\Bigg|_{\gamma}^{\delta} \tag{3.26}$$

for every $1 \le \gamma \le \delta \le \infty$ and $s < \frac{1}{\delta}$.

We note that the first integral implies that all the integrals (3.23) and (3.24) are bounded. On the other hand in order to apply the Lebesgue dominated convergence theorem we need an uniform bound of all the integrands, independently on the values of ε, l and $R_{\pm}$. To this aim, we write the integral $\Delta_{\varepsilon,l}(u)$ as follow

$$\Delta_{\varepsilon,l}(u) = \int_1^{\frac{\beta}{R_-}} \frac{1}{\rho\sqrt{(\beta-\rho R_-)(\rho-1)}}\sqrt{K}d\rho$$

$$= \int_1^{\sqrt{\frac{\beta}{R_-}}} \frac{1}{\rho\sqrt{(\beta-\rho R_-)(\rho-1)}}\sqrt{K}d\rho + \int_{\sqrt{\frac{\beta}{R_-}}}^{\frac{\beta}{R_-}} \frac{1}{\rho\sqrt{(\beta-\rho R_-)(\rho-1)}}\sqrt{K}d\rho$$

$$= I_1 + I_2$$

therefore

$$\lim_{(\varepsilon,l)\to(0,0)} \Delta_{\varepsilon,l}(u) = \lim_{(\varepsilon,l)\to(0,0)} I_1 + \lim_{(\varepsilon,l)\to(0,0)} I_2$$

The motivation of this further partition of the integral will be clear later. For the moment we check that the integral I_2 is infinitesimal as $(\varepsilon, l) \to (0,0)$, that is equivalent to say that the apsidal angle has no significant contribution far from the pericentre.

For the boundness of K, $K < C\beta$,

$$I_2 = \frac{1}{\sqrt{\beta}}\int_{\sqrt{\frac{\beta}{R_-}}}^{\frac{\beta}{R_-}} \frac{1}{\rho\sqrt{(1-\rho\frac{R_-}{\beta})(\rho-1)}}\sqrt{K}d\rho < C\int_{\sqrt{\frac{\beta}{R_-}}}^{\frac{\beta}{R_-}} \frac{1}{\rho\sqrt{(1-\rho\frac{R_-}{\beta})(\rho-1)}}d\rho$$

and for (3.25), (3.26) it follows

$$\begin{aligned} I_2 &< \pi - \int_1^{\sqrt{\frac{\beta}{R_-}}} \frac{1}{\rho\sqrt{(1-\rho\frac{R_-}{\beta})(\rho-1)}}d\rho = \pi - 2\arctan\left[\sqrt{\frac{\rho-1}{1-\frac{\rho R_-}{\beta}}}\right]\Bigg|_1^{\sqrt{\frac{\beta}{R_-}}} \\ &= 2\left(\frac{\pi}{2} - \arctan\sqrt{\frac{\sqrt{\beta}}{\sqrt{R_-}}}\right) \\ &= 2\arctan\sqrt{\frac{\sqrt{R_-}}{\sqrt{\beta}}} \approx 2\sqrt[4]{\frac{R_-}{\beta}} \end{aligned}$$

where, in the last passage, the relation $\arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2}$ is used. Then, passing to the limit, we remind that $R_- \to 0$ as $(\varepsilon, l) \to (0,0)$ and we infer

$$\lim_{(\varepsilon,l)\to(0,0)} I_2 = 0\ .$$

It remains to prove the existence of the limit for I_1.

For every $\varepsilon > 0$ and l let $F_{(\varepsilon,l)}(\rho) : \mathbb{R}^+ \to \mathbb{R}^+$ be the function

$$F_{(\varepsilon,l)} := \frac{1}{\rho\sqrt{(\rho-1)(\beta-\rho R_-)}}\sqrt{K}\ \chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]}$$

thus the limit of I_1 is equivalent to

$$\lim_{(\varepsilon,l)\to(0,0)} \int_1^\infty F_{(\varepsilon,l)} d\rho$$

We note that every function $F_{\varepsilon,l}$ are unbounded only for ρ approaching $\rho = 1$, since the partition we done in the integral $\Delta_{\varepsilon,l}(u)$ and the corresponding multiplication for the characteristic function $\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]}$ makes the function $F_{\varepsilon,l}$ to be regular in $\rho = \frac{\beta}{R_-}$. Moreover, for the boundness of K

$$\begin{aligned} F_{(\varepsilon,l)}(\rho) &< \frac{C}{\rho\sqrt{(\rho-1)(1-\frac{\rho R_-}{\beta})}}\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]} \\ &< \frac{C}{\rho\sqrt{(\rho-1)\Big(1-\sqrt{\frac{R_-}{\beta}}\Big)}}\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]} \\ &< \frac{C'}{\rho\sqrt{\rho-1}} \quad \text{for R}_- \text{ small enough} \end{aligned}$$

then all the functions $F_{(\varepsilon,l)}$ are dominated by a function $\tilde{F} \in \mathbb{L}^1([1,\infty])$. We apply the Lebesgue theorem computing the pointwise limit of $F_{(\varepsilon,l)}$ as $(\varepsilon, l) \to (0,0)$.

Recalling the definition of K_1, in **case 1** we obtain

$$\lim_{(l,\varepsilon)\to(0,0)} \frac{1}{\rho\sqrt{(\rho-1)(\beta-\rho R_-)}}\sqrt{K_1}\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]} =$$

$$\lim_{(l,\varepsilon)\to(0,0)} \frac{1}{\rho\sqrt{(\rho+1)(\rho-1)}}\sqrt{\frac{\beta^2}{(\beta^2-R_-^2\rho^2)\left(\rho^2\frac{(V_\varepsilon(\rho R_-)-V_\varepsilon(\beta))}{(V_\varepsilon(R_-)-V_\varepsilon(\beta))}\frac{(\beta^2-R_-^2)}{(\beta^2-R_-^2\rho^2)}-1\right)}}\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]}$$

For every $V(x) \in \mathcal{V}^*$, since R_- goes to zero as l goes to zero, from theorem (3.1) we obtain

$$\lim_{(l,\varepsilon)\to(0,0)} F_{(\varepsilon,l)} = \frac{1}{\rho\sqrt{\rho^2-1}}$$

uniformly in ε and l. Thus for the Lebesgue theorem

$$\lim_{(l,\varepsilon)\to(0,0)} \int_1^\infty F_{(\varepsilon,l)} d\rho = \frac{\pi}{2}.$$

With same calculations we get the proof also in **case 2**, since as l goes to zero, v_1^2 disappears.

□

The value of the apsidal angle $\Theta_{\varepsilon,l}$ is sufficient to determine the behaviour of the solution $u_{\varepsilon,l}(t)$: in fact, expressing the motion $u_{\varepsilon,l}(t) = (r_{\varepsilon,l}(t), \theta_{\varepsilon,l}(t))$ in polar coordinates, we have that the function $r_{\varepsilon,l}(t)$ oscillates periodically between the maximal and the minimal value, while the corresponding increment of the angle $\theta_{\varepsilon,l}(t)$ between two consecutive maxima of $r_{\varepsilon,l}(t)$ is given by $2\Theta_{\varepsilon,l}$.

Since for $(\varepsilon, l) \to (0,0)$ the apsidal angle $\Theta_{\varepsilon,l}$ tends to $\frac{\pi}{2}$, the pointwise limit of the sequence of trajectories $u_{\varepsilon,l}$ is a straight line trajectory that crosses the origin.

This suggest to extend a collision solution $\bar{u}(t)$ beyond the singularity replacing symmetrically the solution itself forward the collision point in the same direction.

Definition 3.7. *Let $\bar{u}(t)$, $t \in [0, T_0)$, be a collision path, T_0 the collision instant. Define the **transmission solution** $u_0(t)$, $t \in [0, 2T_0]$ as*

$$\begin{cases} u_0(t) = \bar{u}(t) & t \in [0, T_0] \\ u_0(T_0+\tau) = -u_0(T_0-\tau) & \tau \in [0, T_0] \end{cases}$$

For the moment the path $u_0(t)$ is just a definition of an extension of the collision solution $\bar{u}(t)$ beyond the singular time T_0. In order to complete the proof of theorem 14 we have to show that, for every $t \in [0, 2T_0]$, the sequence $\{u_{\varepsilon,l}(t)\}$ converges to $u_0(t)$ as $(\varepsilon, l) \to (0,0)$.

To this aim we define $\mathcal{F} = \{y = (x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2\}$ as the phase space of planar motion and we consider the initial value problems defined on $\mathcal{F}$ equivalent to systems (3.2) and (3.3)

$$P(0) = \begin{cases} y' = f(y) \\ x(0) = x_0 \in (\mathrm{R}^2 \backslash 0) \times \mathbb{R}^2 \\ \dot{x}(0) = \dot{x}_0 \in \mathrm{R}^2 \end{cases}, \quad P(\varepsilon) = \begin{cases} y' = f_\varepsilon(y) \\ x(0) = x_0 \in (\mathrm{R}^2) \times \mathbb{R}^2 \\ \dot{x}(0) = \dot{x}_0 \in \mathrm{R}^2 \end{cases}$$

where $f(x, \dot{x}) = (\dot{x}, \nabla V(|x|))$ and $f_\varepsilon(x, \dot{x}) = (\dot{x}, \nabla V_\varepsilon(|x|))$.

Obviously, the flow generated by the solutions of system $P(0)$ is not complete since a solution $y(t) = (x(t), \dot{x}(t))$ has a singularity in T_0 whenever $|x(T_0)| = 0$. We extend the singular path $y(t)$ beyond the singular time T_0 according with the above definition of transmission solution defining $y(t) = (x(t), \dot{x}(t))$ for $t > T_0$ as

$$x(T_0 + \tau) = -x(T_0 - \tau)$$
$$\dot{x}(T_0 + \tau) = \dot{x}(T_0 - \tau)$$

In this way we can define the flow $\Phi(y_0, t) : \mathcal{F} \times \mathbb{R}^+ \to \mathcal{F}$ as the solution of the system $P(0)$ at time t with initial data y_0 also for those initial data $\bar{y}$ leading to collision except at the collision time T_0, where the velocity component $\dot{x}(T_0)$ is undefined.

Moreover we denote with $\Phi_\varepsilon(y_0, t) : \mathcal{F} \times \mathbb{R}^+ \to \mathcal{F}$ the flow associated to the perturbed system $P(\varepsilon)$. We will study the convergence property of the flow $\Phi_\varepsilon(y_0, t)$ as $\varepsilon \to 0$ and y_0 approaches a collision initial data $\bar{y}$. Unlike what we showed for the solutions $u_{\varepsilon,l}$ of the system (3.3), now we can not expect that the regular flow Φ_ε converges to a limit flow as $\varepsilon \to 0$ for every initial data $y_0 \in \mathcal{F}$ and for every t, because the flow Φ_ε is defined on the phase space and, even if the sequence of the configurations $x_\varepsilon(t)$ would converge for every t and for every initial data, the limit can not be achieved by the sequence $\dot{x}_\varepsilon(T_0)$, if a colliding initial data is considered.

Therefore, for fixed T, we define the Poincaré map $\Phi_\varepsilon(y_0, T)$ as the solution at time T of the system $P(\varepsilon)$ with initial value y_0. Suppose $\bar{y}$ be a initial data leading to collision for the problem $P(0)$ and T_0 be the collision instant, the purpose of the next section is to show that, for every $T \neq T_0$, the map $\Phi_\varepsilon(y_0, T)$ converges to $\Phi(\bar{y}, T)$ as $(\varepsilon, y_0) \to (0, \bar{y})$. We note that the continuity of the Poincaré map implies the proof of theorem 14.

3.5.1 The continuity of the Poincarè map

In order to underline the dependence with respect to initial data y_0 and the parameter ε rather than the time T, we denote the Poincarè map with $\Phi_T(y_0, \varepsilon)$ in place of $\Phi_\varepsilon(y_0, T)$. We want to study the continuity of the map $\Phi_T(y_0, \varepsilon)$ to $\Phi_T(\bar{y})$ when ε goes to zero and $\bar{y}$ is a initial data leading to collision. Our aim is to prove that

$$\lim_{\substack{y_0 \to \bar{y} \\ \varepsilon \to 0}} \Phi_T(y_0, \varepsilon) = \Phi_T(\bar{y})$$

for every positive $T \neq T_0$.

If $T < T_0$ no problem arises since we are considering the flow $\Phi(\bar{y}, t)$ before the singular time, thus the above limit comes from the classical theorem of continuity with respect initial data of ordinary differential equations.

On the other hand, for $T > T_0$ the continuity of the Poincaré map is stated in the following theorem.

Theorem 16. *Let $V(x) \in \mathcal{V}^*$ and suppose $\bar{y} = (\bar{x}, \dot{\bar{x}})$ be an initial condition leading to collision for the system $P(0)$ at time T_0. Then*

$$\lim_{\substack{y_0 \to \bar{y} \\ \varepsilon \to 0}} \Phi_T(y_0, \varepsilon) = \Phi_T(\bar{y})$$

for every $T > T_0$ such that $\dot{x}(T) \neq 0$.

In the proof of the theorem we will need the following classical lemma

Lemma 3.4. *Let $H(r, p)$ and $H_0(p)$ be real continuous functions with respect to a set of parameters p, and suppose H be strictly increasing as function of r. Then for every T the function $r = r(T, p)$, implicit solution of equation*

$$T = H_0(p) + H\Big(r(T, p), p\Big)$$

is continuous as function of p.

Proof of theorem (16)
As usual we set (r, θ) the polar coordinates of the plane then a point $y = (x, \dot{x})$ in the phase space is replaced by

$$x = (r \cos\theta, r \sin\theta)$$
$$\dot{x} = (\dot{r} \cos\theta - r\dot{\theta} \sin\theta, \dot{r} \sin\theta + r\dot{\theta} \cos\theta)$$

From the definition of $\Phi_T(y_0)$ and $\Phi_T(y_0, \varepsilon)$ remain well defined the functions $r_T(y_0), \theta_T(y_0)$, $\dot{r}_T(y_0), \dot{\theta}_T(y_0)$ and $r_T(y_0, \varepsilon), \theta_T(y_0, \varepsilon)$, $\dot{r}_T(y_0, \varepsilon), \dot{\theta}_T(y_0, \varepsilon)$ denoting, respectively, the values of the radial and angular coordinate and their velocity at time $T \neq T_0$ of a solution of system $P(0)$ and $P(\varepsilon)$ with initial data y_0. The continuity of $\Phi_T(y_0, \varepsilon)$ is equivalent to the continuity of each of the previous functions.

Recall the definition of the energy and angular momentum integral

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{l^2}{r^2} - V(r)$$
$$l^2 = r^2\dot{\theta}$$

Let $\bar{y} = (\bar{r}, \bar{\theta}, \dot{\bar{r}}, \dot{\bar{\theta}})$ be an initial data in the phase space leading to collision with nonzero initial velocity, $\dot{\bar{r}} < 0$, and denote with $\bar{E} = \frac{1}{2}\dot{\bar{r}}^2 - V(\bar{r})$ the energy of the collision solution.

A point $y \in \mathcal{F}$, $y = (r, \theta, \dot{r}, \dot{\theta})$, tends to $\bar{y}$ if it holds

$$|r - \bar{r}| \to 0 \ , \qquad |\theta - \bar{\theta}| \to 0 \mod 2\pi$$
$$(E, l) \to (\bar{E}, 0)$$

We start showing the continuity of the function $r_T(y_0, \varepsilon)$ as $y_0 \to \bar{y}$ and $\varepsilon \to 0$. To this aim we note that the value of $r_T(y_0, \varepsilon)$ is governed by the differential equation $\dot{r} = \sqrt{2(E + V_\varepsilon(r)) - \frac{l^2}{r^2}}$, thus $r_T(y_0, \varepsilon)$ is a function of the initial position r_0, the couple E, l and the parameter ε. Define $\mathcal{T}_0(E, l, r_0, \varepsilon)$ as the time necessary to the solution $r(y_0, t)$ to reach the minimal value $r_- = r_-(E, l)$, then

$$\begin{aligned} T &= \int_{r_-}^{r_0} \frac{1}{\sqrt{2(E + V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}} d\rho + \int_{r_-}^{r_T(y_0,\varepsilon)} \frac{1}{\sqrt{2(E + V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}} d\rho \\ &= \mathcal{T}_0(E, l, r_0, \varepsilon) + \mathcal{T}(r_T(y_0, \varepsilon), E, l) \end{aligned}$$

<u>Claim</u> *The function $\mathcal{T}_0$ and $\mathcal{T}$ are continuous respect to r_0, E, l, ε.*

Suppose for the moment that the claim is true, since the function $\mathcal{T}$ is strictly increasing with respect to $r_T(y_0, \varepsilon)$, for lemma (3.4), the function $r_T(y_0, \varepsilon)$ is continuous with respect to the set of parameter E, l, r_0, ε. Therefore

$$\lim_{\substack{y_0 \to \bar{y} \\ \varepsilon \to 0}} r_T(y_0, \varepsilon) = r_T(\bar{y})$$

The continuity of $\theta_T(y_0, \varepsilon)$ is equivalent to the continuity of $\Delta\theta_T(y_0, \varepsilon) := \theta_T(y_0, \varepsilon) - \theta_0$. For the definition of transmission solution $\Delta\theta_T(\bar{y}) = \pi$, while, in theorem (15), we showed that the apsidal angle associated to a solution of the perturbed system tends to $\frac{\pi}{2}$ whenever the initial data tends to a colliding one and the parameter ε tends to zero. Thus we gain

$$\lim_{\substack{y_0 \to \bar{y} \\ \varepsilon \to 0}} \theta_T(y_0, \varepsilon) = \theta_T(\bar{y})$$

The continuity of $\dot{r}_T(y_0, \varepsilon)$ and $\dot{\theta}_T(y_0, \varepsilon)$ follows immediately by the continuity of $r_T(y_0, \varepsilon)$ from the relations

$$\dot{r}_T(y_0, \varepsilon) = 2\sqrt{E + V_\varepsilon(r_T(y_0, \varepsilon)) - \frac{1}{2}\frac{l^2}{r_T(y_0, \varepsilon)^2}}$$

$$\dot{\theta}_T(y_0, \varepsilon) = \frac{l}{r_T(y_0, \varepsilon)^2}$$

□

Proof of the claim

Denoting with p and $\bar{p}$ the sets of parameter $p = (E, l, r_0, \varepsilon)$ and $\bar{p} = (\bar{E}, 0, \bar{r}, 0)$, we want to show that

$$\lim_{p \to \bar{p}} \mathcal{T}_0(p) = \mathcal{T}_0(\bar{p})$$

We want to apply the dominated convergence theorem and pass the limit under the integral sign in

$$\lim_{p \to \bar{p}} \int_{r_-}^{r_0} \frac{1}{\sqrt{2(E + V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}} d\rho$$

To this aim we have to exhibit an $\mathbb{L}^1$ bound for the integrand function. We observe that the integrand become unbounded only for ρ approaching to r_- since we have chosen the initial value $\dot{r} < 0$ and, as a consequence, we can suppose $\dot{r}_0 < 0$.

Let $\beta \in (r_-, r_0)$ and write

$$\int_{r_-}^{r_0} \frac{1}{\sqrt{2(E + V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}} d\rho = \int_{r_-}^{\beta} \frac{1}{\sqrt{2(E + V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}} d\rho + \int_{\beta}^{\tilde{r}} \frac{1}{\sqrt{2(E + V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}} d\rho$$
$$= I + II$$

Since for every $\rho \in [\beta, r_0]$ the integrand is bounded, it follows

$$II \leq C_1(r_0 - \beta)$$

In order to show the $\mathbb{L}^1$ boundness of the first integrand, using the definition of the energy integral, we rewrite I in the form

$$I = \int_{r_-}^{\beta} \frac{1}{\sqrt{\dot{r}_0^2 + \frac{l^2}{r_0^2} - 2V_\varepsilon(r_0) + 2V_\varepsilon(\rho) - \frac{l^2}{\rho^2}}} d\rho =$$
$$= \int_{r_-}^{\beta} \frac{1}{\sqrt{\dot{r}_0^2 + l^2\left(\frac{r^2 - r_0^2}{r^2 r_0^2}\right) + 2\big(V_\varepsilon(\rho) - V_\varepsilon(r_0)\big)}} d\rho$$

Again from the definition of energy, evaluated in r_0 and in the pericentre r_-,

$$E = \frac{1}{2}\dot{r}_0^2 + \frac{1}{2}\frac{l^2}{r_0^2} - V_\varepsilon(r_0) = \frac{1}{2}\frac{l^2}{r_-^2} - V_\varepsilon(r_-)$$

it follows

$$l^2 = \left(\frac{r_-^2 - r_0^2}{r_0^2 r_-^2}\right)\left(2\big(V(r_0) - V(r_-)\big) - \dot{r}_0^2\right)$$

and, replacing the last relation in the integrand, we obtain

$$\begin{aligned} I &= \int_{r_-}^{\beta} \frac{1}{\sqrt{\dot{r}_0^2\left[1 - \frac{r_-^2(r_0^2-\rho^2)}{\rho^2(r_0^2-r_-^2)}\right] + \frac{r_-^2}{\rho^2}\frac{(r_0^2-\rho^2)}{(r_0^2-r_-^2)}2\big(V_\varepsilon(r_0) - V_\varepsilon(r_-)\big) + 2\big(V_\varepsilon(\rho) - V_\varepsilon(r_0)\big)}} d\rho \\ &\leq \int_{r_-}^{\beta} \frac{1}{\sqrt{2\big(V_\varepsilon(\rho) - V_\varepsilon(r_0)\big)}} \frac{1}{\sqrt{1 - \frac{r_-^2}{\rho^2}\frac{(r_0^2-\rho^2)}{(r_0^2-r_-^2)}\frac{V_\varepsilon(r_-)-V_\varepsilon(r_0)}{V_\varepsilon(\rho)-V_\varepsilon(r_0)}}} d\rho \end{aligned}$$

For theorem 11 there exists positive constant K such that, for every ε small enough,

$$1 - \frac{r_-^2}{\rho^2}\frac{(r_0^2 - \rho^2)}{(r_0^2 - r_-^2)}\frac{V_\varepsilon(r_-) - V_\varepsilon(r_0)}{V_\varepsilon(\rho) - V_\varepsilon(r_0)} > \frac{K r_0(\rho - r_-)}{r_-(r_0 + \rho) + K r_0(\rho - r_-)}$$

thus

$$I < C\int_{r_-}^{\beta} \frac{1}{\sqrt{\rho - r_-}} \frac{1}{\sqrt{2\big(V(\beta) - V(r_0)\big)}} < C_1 \int_{r_-}^{\beta} \frac{1}{\sqrt{\rho - r_-}} d\rho < \infty$$

In order to obtain an uniform bound for every choice of r_-, we perform the variable changing $z = \rho - r_-$ and conclude

$$I < C_2 \int_0^{\beta} \frac{1}{\sqrt{z}} dz < \infty$$

We can now apply the Lebesgue theorem and we obtain

$$\lim_{p \to \bar{p}} \int_{r_-}^{r_0} \frac{1}{\sqrt{2(E + V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}} d\rho = \int_0^{\bar{r}} \frac{1}{\sqrt{2(\bar{E} + V(\rho))}} d\rho$$

thus the function $\mathcal{T}_0$ and, for the same reason, the function $\mathcal{T}$ are continuous with respect the parameter p.

□

3.5.2 The continuity of the Poincarè section

Let again $\bar{y}$ be an initial condition leading to collision for the system $P(0)$, T_0 the collision instant and for a choice of $T > T_0$ we denote $y_1 = \Phi(\bar{y}, T)$ according to the definition 3.7 of extension of collision solution. Given Σ an hyperplane in the phase space passing through y_1 and transversally to the flow, we will show that for every initial data y_0 near $\bar{y}$, there exists a time $t = \tau(y_0)$ when the trajectory $\Phi(y_0, t)$ intersect the hyperplane Σ. The trace $S(y_0) = \Phi(y_0, \tau(y_0))$ drawn on Σ as the data y_0 changes is called the **Poincaré section** of the flow on Σ. In the next theorem we prove the continuity of the map $\tau(y_0)$ and the continuity of the Poincaré section in a neighbourhood of $\bar{y}$.

Theorem 17. *Let $V(x) \in \mathcal{V}^*$. Then there exists a $\delta > 0$ and a continuous function $\tau(y_0)$ defined in a δ-neighbourhood of $\bar{y}$, $N_\delta(\bar{y})$, such that $\tau(\bar{y}) = T$ and*

$$\Phi(y_0, \tau(y_0)) \in \Sigma$$

Moreover the Poincaré section is continuous in $N_\delta(\bar{y})$.

Proof
Let F a vector in the phase space such that $F \cdot f(y_1) > 0$ and consider the hyperplane

$$\Sigma = \{y : (y - y1) \cdot F = 0\}$$

It is obvious by definition of y_1 that $(\Phi(\bar{y}, T) - y1) \cdot F = 0$ and since

$$\frac{d}{dt}(\Phi(\bar{y}, t) - y1) \cdot F\Big|_{t=T} = f(y_1) \cdot F > 0$$

it holds that there exists an $\xi > 0$ such that

$$\begin{aligned}(\Phi(\bar{y}, T - \xi) - y1) \cdot F < 0\\ (\Phi(\bar{y}, T + \xi) - y1) \cdot F > 0\end{aligned}$$

For theorem (16) and for the sign permanence theorem it follows that there exists a $\delta > 0$ such for every $|\bar{y} - y_0| < \delta$

$$\begin{aligned}(\Phi(y_0, T - \xi) - y1) \cdot F < 0\\ (\Phi(y_0, T + \xi) - y1) \cdot F > 0\end{aligned}$$

For every fixed y_0, the function $(\Phi(y_0, t) - y1) \cdot F$ is increasing in t: indeed

$$\frac{d}{dt}(\Phi(y_0, t) - y1) \cdot F = f(\Phi(y_0, t)) \cdot F$$

and the continuity of the vector space $f(y)$ far from the collision points and the continuity of the orbit $\Phi(y_0, t)$ with respect to both the variables imply that $f(\Phi(y_0, t)) \cdot F > 0$. Therefore for every $y_0 \in N_\delta(\bar{y})$ there exists a time $\tau(y_0)$ such that $\Phi(y_0, \tau(y_0)) \in \Sigma$ and $\tau(\bar{y}) = T$. In order to prove the continuity of $\tau(y_0)$ we define

$$H(y_0, \tau) = (\Phi(y_0, \tau) - y_1) \cdot F$$

then $\tau(y_0)$ is the implicit solution of $H(y_0, \tau(y_0)) = 0$. Since H is continuous in y_0 and it is continuous and increasing with respect to τ, for lemma (3.4) the function $\tau(y_0)$ is continuous. Moreover, for composition of continuous functions we infer the continuity of the Poincaré section. □

In the following we want to generalise for a large class of potential $V(x)$ the result obtained in §2.3 about the non-differentiability of the apsidal angle of a solution of the logarithmic potential problem when the angular momentum goes to zero.

Theorem 18. *Let $V \in \mathcal{V}^*$ with the further assumption*

$$\lim_{\rho \to 0} \frac{1}{\sqrt{2V(\rho)}} \frac{d}{dx}\left(\frac{V(\lambda x)}{V(x)}\right)\Big|_{x=\rho} = -\infty \quad \lambda > 1 \tag{3.27}$$

then the apsidal angle $\Delta^l\theta$ is not differentiable as l goes to zero.

Proof
As in the proof of theorem 7 let $\varphi(r) = -V'(r)$ the force at distance r directed to the origin, $\psi(z) = -\frac{1}{z^2}V'\left(\frac{1}{z}\right)$ and $w(z) = 2\int \psi(z)dz = 2V\left(\frac{1}{z}\right)$. The apsidal angle is given by

$$\Delta^l\theta = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{1}{\sqrt{1+\varepsilon(Ls+a)}}ds$$

$$\varepsilon(Ls+a) = \frac{(L+2a)}{L(1-s)}\left[\frac{V(\frac{1}{Ls+a}) - V(\frac{1}{a})}{s(V(\frac{1}{L+a}) - V(\frac{1}{a}))} - 1\right]$$

when l and L are linked by

$$l^2 = \frac{w(L+a) - w(a)}{(L+2a)L} = 2\frac{V(\frac{1}{L+a}) - V(\frac{1}{a})}{(L+2a)L}$$

then $L \to \infty$ when $l \to 0$ and $l^2 \sim \frac{2V(\frac{1}{L})}{L^2}$.
As in the proof of theorem 10 fix $a = 1$ and write the incremental quotient

$$\frac{\Delta^l\theta - \Delta^0\theta}{l} = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{\left(\frac{1}{\sqrt{1+\varepsilon(Ls+1)}} - \frac{1}{\sqrt{1+\frac{1}{s}}}\right)}{\left(\frac{\sqrt{2V(\frac{1}{L})}}{L}\right)}ds$$

Set $\rho = \frac{1}{L}$ and $f(\rho) = \frac{1}{\sqrt{(1+\varepsilon(\rho))}}$ and take the limit

$$\lim_{\rho\to 0}\int_0^1 \frac{1}{\sqrt{s(1-s)}}\left(\frac{f(\rho) - f(0)}{\rho\sqrt{2V(\rho)}}\right)ds = \lim_{\rho\to 0}\int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{f'(\rho)}{\sqrt{2V(\rho)}}ds$$

We show that $\lim_{\rho\to 0}\frac{f'(\rho)}{\sqrt{2V(\rho)}} = \infty$ for a.e. $s \in (0,1)$. Indeed

$$f'(\rho) = -\frac{1}{2}\frac{1}{(1+\varepsilon(\rho))^{\frac{3}{2}}}\varepsilon'(\rho)$$

and, since

$$\varepsilon(\rho) = \frac{1+2\rho}{1-s}\left[\frac{V\left(\frac{\rho}{s+\rho}\right) - V(1)}{s\left(V\left(\frac{\rho}{1+\rho}\right) - V(1)\right)} - 1\right]$$

it follows

$$\varepsilon'(\rho) = \frac{2}{1-s}\left[\frac{V\left(\frac{\rho}{s+\rho}\right) - V(1)}{s\left(V\left(\frac{\rho}{1+\rho}\right) - V(1)\right)} - 1\right] + \frac{1+2\rho}{s(1-s)}\cdot$$
$$\left[\frac{\left(V'\left(\frac{\rho}{s+\rho}\right)\frac{s}{(s+\rho)^2}\right)\left(V\left(\frac{\rho}{1+\rho}\right) - V(1)\right) - \left(V\left(\frac{\rho}{s+\rho}\right) - V(1)\right)V'\left(\frac{\rho}{1+\rho}\right)\frac{1}{(1+\rho)^2}}{\left(V\left(\frac{\rho}{1+\rho}\right) - V(1)\right)^2}\right]$$

We observe that the first term of the derivative is equal to $\frac{2}{1+2\rho}\varepsilon(\rho)$ and tends to $\frac{2}{s}$ as ρ tends to zero, then as $\rho \to 0$

$$\begin{aligned}\varepsilon'(\rho) &\sim \frac{2}{s} + \frac{1}{s(1-s)}\left[\frac{V'\left(\frac{\rho}{s}\right)\frac{1}{s}V(\rho) - V\left(\frac{\rho}{s}\right)V'(\rho)}{V(\rho)^2}\right] \\ &\sim \frac{2}{s} + \frac{1}{s(1-s)}\frac{d}{dx}\left(\frac{V\left(\frac{x}{s}\right)}{V(x)}\right)\Bigg|_{x=\rho}\end{aligned}$$

Thus the assumption (3.27) implies that

$$\lim_{\rho\to 0}\frac{f'(\rho)}{\sqrt{2V(\rho)}} =$$

$$-\frac{1}{2}\frac{1}{(1+\frac{1}{s})^{\frac{3}{2}}}\lim_{\rho\to 0}\frac{1}{\sqrt{2V(\rho)}}\left(\frac{2}{s} + \frac{1}{s(1-s)}\frac{d}{dx}\left(\frac{V\left(\frac{x}{s}\right)}{V(x)}\right)\Bigg|_{x=\rho}\right) = +\infty$$

This means that the function $\Delta^l\theta$ is not differentiable in $l = 0$. $\square$

3.6 Variational property of the collision solutions

In this section we join a variational approach to the study of the dynamical system (3.2). We recall that the system of differential equations we are taking into account is a conservative hamiltonian system where the energy is the hamiltonian function. Till now the hamiltonian structure of the system and the existence of conserved quantities, like the energy and the angular momentum, played a crucial role in all the results we have proved. On the other hand the central problem can also be approached using its characterization as lagrangian dynamical system. In this formulation a classical solution of the dynamical system is a solution of the Euler-Lagrange equation associated to the lagrangian function

$$\mathcal{L}(u,\dot{u}) = \frac{1}{2}|\dot{u}(t)|^2 + V(|u(t)|$$

The variational approach consists in finding solutions of the motion as critical point of the action functional

$$\mathcal{A}(u) = \int_T \mathcal{L}(u,\dot{u})dt$$

The principle of least action states that if $u(t)$ is a stationary point of the action functional in a suitable open set of paths, then $u(t)$ is a solution for the dynamical system. See the Appendix for more details.

From the point of view of calculus of variation a critical point of the action may not to be a minimum, indeed also saddle or maxima are stationary points of the functional. However in the variational framework, may be relevant to know if a critical point of the action functional is or not a local minimum. Indeed the knowledge of the topology of the action functional in the neighbourhood of a critical point could be useful in order to find other solutions or in the study of the stability of the system.

In the following we want to extend some results contained in [1] and in the reference therein, about the collisionless property of minimal paths for the two body problem when a homogeneous potential and the logarithmic potential is considered. In particular we will prove that every collision path can not be a minimum for the action functional whenever a potential function $V(x) \in \mathcal{V}$ is chosen. To this aim we have first to discuss on the asymptotic estimates of the solution of the dynamical system as it approaches a collision point.

3.6.1 Asymptotic estimate

The purpose of this section is to analyse the asymptotic behaviour of a solution as it approaches a collision. We consider problems with potentials $V(x)$ belonging to $\mathcal{V}$ and, without loss of generality, we assume that the collision is achieved for $t = 0$. The function $V(x) = -\log(x)$ can be considered the prototype for potentials in $\mathcal{V}$ and the analysis of the asymptotics for the logarithmic potential can be found, with more generality, in [1]. In this paper the authors show the following estimates for the radial coordinate and its velocity as t tends to zero

$$r(t) \sim t\sqrt{-2\log(t)}$$
$$\dot{r}(t) \sim -\sqrt{-2\log(t)}$$

In order to extend these estimates to the larger class fo potential $\mathcal{V}$, we remind that, for theorem 13, every potential function $V(x) \in \mathcal{V}$ is of weak type, thus, for theorem 12, a collision solution occurs if and only if the angular momentum is zero. Therefore, from the energy integral associated to a collision solution,

$$E = \frac{1}{2}\dot{r}^2 - V(r)$$

we infer the relation

$$\lim_{t\to 0} \frac{\dot{r}(t)}{-\sqrt{2V(r(t))}} = 1$$

since the potential $V(x)$ diverges to infinity as the configuration approach the collision. From the last relation it follows the asymptotic estimate for the radial velocity

$$\dot{r} \sim -\sqrt{2V(r)} \tag{3.28}$$

and by integration

$$\int_0^{r(t)} \frac{1}{\sqrt{2V(\rho)}} d\rho = \int_0^t d\tau = t$$

In general an explicit estimate of the behaviour of the radial coordinate and its velocity with respect to time t can not be achieved. However, denoting with $H(r)$ the integral function

$$H(r) = \int_0^r \frac{1}{\sqrt{2V(\rho)}} d\rho$$

we have, at least in a neighbourhood of the collision, the implicit formula

$$r(t) = H^{-1}(t) \ .$$

since $H(r)$ is positive and increasing as function of r small enough.

3.6.2 The non-minimality of collision solutions

The aim of the following theorems is to give a characterization of the extended collision solutions of system (3.2) from a variational point of view.

Since a classical collision solution $u(t)$ is a path joining the initial position P to the origin in finite time T_0, we regard the transmission solution $u_0(t)$, extension of $u(t)$, as a path joining two different point P, Q, different from the origin, in finite time T.

Thus, denoting with $\mathcal{A}_T$ the action functional

$$\mathcal{A}_T(u) = \int_0^T \left(\frac{1}{2}|\dot{u}|^2 + V(|u|)\right) dt$$

defined on the set of paths

$$\Lambda = \{u(t), \dot{u} \in \mathbb{L}^2, u(0) = P, u(T) = Q\}$$

we ask ourselves if the collision path $u_0(t)$ is, or not, a minimal path for $\mathcal{A}_T$.

Before proving the theorem which states the collisionless property for the minimal paths of the action, we show that any collision solution $u(t)$ of the dynamical system belongs to $H^1(T, \mathbb{R}^2)$ and the action $\mathcal{A}(u)$ is finite, provided a potential $V(x) \in \mathcal{V}$ is considered.

Theorem 19. *Let $V(x) \in \mathcal{V}$, $u(t)$ a collision solution of system* (3.2)*. Then $u \in H^1((t_1, t_2), \mathbb{R}^2)$ and $\mathcal{A}(u) < \infty$, where $[t_1, t_2)$ in the interval time of existence of solution $u(t)$ and t_2 denotes the collision instant.*

Proof
By definition the action functional is sum of two positive parts and the boundness of the action implies $u \in H^1((t_1, t_2), \mathbb{R}^2)$.

Since a collision solution for the two body problem with potential $V(x) \in \mathcal{V}$ is a radial solution, we have

$$\mathcal{A}(u) = \int_{t_1}^{t_2} \left(\frac{1}{2}\dot{r}^2(t) + V(r(t)) \right) dt$$

The integrand function diverges to infinity as t approaches the collision time t_2, since both the velocity and the potential blow up when a collision occurs. Thanks to the energy integral, we write

$$\begin{aligned} \mathcal{A}(u) &= \int_{t_1}^{t_2} \Big(E + 2V(r(t)) \Big) dt \\ &\leq C \int_{t_1}^{t_2} V(r(t))\, dt \end{aligned}$$

After a change of coordinates we write the action in the form

$$\mathcal{A}(u) \leq C \int_0^R \frac{V(r)}{-\dot{r}} dr$$

then, thanks to the estimate (3.28) and (3.15), we conclude

$$\mathcal{A}(u) \leq C \int_0^R \sqrt{V(r)}\, dr \leq \int_0^R \frac{1}{r^{1-\frac{\eta}{2}}} dr < \infty$$

□

Theorem 20. *For every $V(x) \in \mathcal{V}^*$, let $u_0 : [-T, T] \to \mathbb{R}^2$ be the extension of a collision solution of system* (3.2) *according with the definition 3.7 of transmission solution. Then u_0 is not a minimal path for the action functional $\mathcal{A}_\Lambda$, where*

$$\Lambda = \{u(t) : \dot{u} \in \mathbb{L}^2([-T, T]), u(-T) = u_0(-T), u(T) = u_0(T)\}$$

denotes the set of paths joining the end points of u_0.

Proof
Let $u(t)$ be a collision solution for the system (3.2), defined on the time interval $[-T, 0)$ where $t = 0$ denotes the collision instant. We extend the collision path $u(t)$ beyond the singularity according to definition 3.7 of transmission solution; the extended path, denoted with $u_0(t)$, is defined for $t \in [-T, T]$ and belongs to the same line for all the time. We note that, being a classical solution out of the collision time, the path u_0 is a critical point for the action with respect to variation

with compact support in $[-T, 0)$ and in $(0, T]$. In order to prove that the minima for the action $\mathcal{A}$ among all the paths in Λ are collisionless, we perform a variation on the trajectory u_0 that removes the collision and makes the action decrease.

Let $u_1(t) = u_0(t) + v^\delta(t)$ where $v^\delta(t)$ is the standard variation

$$v^\delta(t) = \begin{cases} \delta & |t| < T_1 \\ \frac{(T-t)}{(T_1-T)}\delta & T_1 < |t| < T \end{cases}$$

directed orthogonal to $u_0(t)$.

Let us compute the difference $\Delta\mathcal{A} = \mathcal{A}(u_0) - \mathcal{A}(u_1)$ and show that for every δ sufficiently small $\Delta\mathcal{A}$ is positive: this means that $u_0(t)$ is not a local minimum of the action functional.

$$\begin{aligned} \Delta\mathcal{A} = \int_{-T}^{T} \Big(\mathcal{L}(u_0) - \mathcal{L}(u_1)\Big) dt &= \int_{-T}^{T} \left(\frac{1}{2}|\dot{u}_0|^2 + V(|u_0|)\right) - \left(\frac{1}{2}|\dot{u}_1|^2 + V(|u_1|)\right) dt \\ &= \int_{-T}^{T} \frac{1}{2}\left(|\dot{u}_0|^2 - |\dot{u}_1|^2\right) dt + \int_{-T}^{T} V(|u_0|) - V(|u_1|) dt \\ &= \Delta\mathcal{K} + \Delta V \end{aligned}$$

We study separately the kinetic and the potential contribute.

Since the variation v^δ is directed orthogonally to u_0, we gain

$$|\dot{u}_1(t)|^2 = \begin{cases} |\dot{u}_0(t)|^2 & |t| \le T_1 \\ |\dot{u}_0(t)|^2 + \frac{\delta^2}{(T-T_1)^2} & T_1 < |t| \le T \end{cases}$$

then

$$\Delta\mathcal{K} = -2\int_{T_1}^{T} \frac{1}{2}\frac{\delta^2}{(T-T_1)^2} dt = -\frac{\delta^2}{(T-T_1)}$$

One should expect that the variation $\Delta\mathcal{K}$ is negative: indeed the path $u_1(t)$ is given by $u_0(t)$ adding a motion with nonzero velocity; on the other hand, for every t, the configuration $u_1(t)$ is further than $u_0(t)$ from the origin then the potential contribute ΔV will be positive. We show that, for every δ small enough, the contribution of the potential part is bigger than the penalising contribution, due to the kinetic part.

$$\begin{aligned} \Delta V = 2\int_0^T V(|u_0|) - V(|u_1|)\, dt &> 2\int_0^{T_1} V(|u_0|) - V(|u_1|)\, dt \\ &\ge \int_0^{T_1} V(|u_0|) - V\big(\sqrt{u_0^2 + \delta^2}\big)\, dt \end{aligned}$$

For every t fixed

$$V(|u_0|) - V\big(\sqrt{u_0^2+\delta^2}\big) = -\int_0^1 \frac{d}{d\xi} V\left(\sqrt{u_0^2 + \xi\delta^2}\right) d\xi$$

Hence

$$\Delta V \ge 2\int_0^{T_1}\int_0^1 -\frac{V'\big(\sqrt{u_0^2+\xi\delta^2}\big)}{2\sqrt{u_0^2+\xi\delta^2}}\delta^2\, d\xi\, dt$$

$$\ge \delta^2 \int_0^{T_1}\int_0^1 -\frac{V'\big(\sqrt{u_0^2+\xi\delta^2}\big)}{\sqrt{u_0^2+\xi\delta^2}}\, d\xi\, dt = \delta^2\int_0^{T_1} f_\delta(t) dt = \delta^2 R_\delta$$

For δ sufficiently small the functions $f_\delta(t)$ are positive and

$$\int_0^{T_1} \lim_{\delta\to 0} f_\delta(t)\, dt = \int_0^{T_1} \frac{-V'(|u_0(t)|)}{|u_0(t)|}\, dt = +\infty \tag{3.29}$$

The last relation follows from the fact that the collision solution u_0 is a radial solution, thus the radial differential equation $\ddot{r}(t) = -V'(r(t))$ is satisfied. Indeed

$$\int_0^{T_1} \frac{-V'(|u_0(t)|)}{|u_0(t)|}\, dt > C \int_0^{T_1} -V'(|u_0(t)|)\, dt = C \lim_{t\to 0} \dot{r}(t) = +\infty$$

By relation (3.29) and Fatou's Lemma we conclude that

$$\lim_{\delta\to 0} \int_0^{T_1} \int_0^1 -\frac{V'\big(\sqrt{u_0^2+\xi\delta^2}\big)}{\sqrt{u_0^2+\xi\delta^2}}\, d\xi\, dt = +\infty$$

hence

$$\Delta\mathcal{A} = \Delta\mathcal{K} + \Delta\mathcal{V} = \delta^2\big(-\frac{1}{T-T_1} + R_\delta\big) > 0$$

for every δ small enough. □

3.7 Conclusions about the regularization of collision solutions

At the end of this chapter we recollect all the results we have obtained and we try to give a complete picture of the problem of the extension of the flow for the two body problem.

We concentrate our analysis in the case that the potential function $V(x)$ belongs to a class of potential $\mathcal{V}$ whose elements have the main property of being singular in the origin, with a singularity weaker than the inverse square, i.e. $\lim_{|x|\to 0} x^2 V(|x|) = 0$. We remark that, at least among the α-homogeneous functions, the potentials of strong type, according to definition 3.5, satisfy the strong force condition introduced by Gordon([11],[10]).

The collisions play a fundamental role in the phase portrait of the flow associated to the two body problem as they are the responsible of the singularities of the flow. Thus a theory of regularization of the flow consists in defining an extension of the collision solutions in such a way that the extended flow satisfies some properties of smoothness.

The results achieved in the previous chapter about the asymptotic behaviour of the apsidal angle of a solution for the logarithmic potential problem, suggest to apply the technique of regularization via smoothing of the potential in order to extend the flow for logarithmic type potentials. Indeed we proved that the extended flow obtained defining the solution beyond the singularity as a transmission solution, attains some good properties of smoothness provided the potential functions satisfies a certain logarithmic type condition. In particular we showed that for every potential $V(x) \in \mathcal{V}^*$ it is possible to extend the flow beyond the singularity in such a way that the extended flow not only is continuous with respect initial data in the space of configurations, but also the continuity of the Poincaré map and continuity of the Poincaré section in the phase space are attained. Since the same regularization of collision solutions can not be applied for system with α-homogeneous potential, it's reasonable to think that the weakness of the singularity of the potential make the system to be regularized with the method of smoothing of the potential. Moreover we proved that for some potentials in the class $\mathcal{V}^*$, included the logarithmic potential, the Poincaré map is not differentiable with respect initial data at the collision solution: this means that for all these potentials does not exists a change of coordinates, in the spirit of Levi-Civita, that transforms the singular differential system into a regular one. Thus the possibility to define a transformation in space and time that removes the singularity from the equation of motion is not connected to the local behaviour of the potential at the singularity, but to an algebraic condition on the power of homogeneity of the potential.

However, for what that concerns potentials with a logarithmic type singularity, the definition of transmission solution and the good topological properties held by the extended flow, may induce us to think that, in some way, the particle, after falling into the collision, emerges in the same direction without notice the presence of the singularity in the equation of motion. More precisely one could expect that the contribution of the singularity of the potential is negligible from a variational point of view and, as a consequence, the collision solutions are minimal paths for the action functional. We proved that this expectation could not be confirmed, indeed for any potential function in $\mathcal{V}^*$, the minimal paths for the action are free of collisions. This result joins similar one about the non minimality of the collision solution obtained in case of logarithmic potential and α-homogeneous potential, $0 < \alpha < 2$, see [1], and for strong force type potentials, see [10]. On the other hand, if the gravitational potential is considered, the collision solution could be a local minimizer for the action [11]; I think that this peculiar phenomena is connected to the fact that the differential equation describing the Kepler motion is regularizable by Levi-Civita transformation.

Part II

Variational methods for the N-centre problem

Chapter 4

The classical approach to the two-center problem

The two center problem, one of the most famous integrable problem in classical mechanics, describes the motion of a particle in the field of two fixed newtonian centers of attraction. First investigated by Euler in 1760, this problem may be viewed as a limiting case of a restricted three-body problem, in which a particle of negligible mass moves in the gravitational field of two massive bodies. As Euler showed, this system is completely integrable in elliptic coordinates and explicit solutions are given in Jacobi's work "Vorlesungen über Dynamik".

Besides the importance of this problem in celestial mechanic, the system plays a important role in atomic physics model of a diatomic molecule in which the two center are atomic nuclei and the test particle is an electron. Starting from Pauli's dissertation, the system became a model for one-electron molecules such as H_2^+ (symmetric case) or HHe^{2+} (asymmetric case), see [25] for more references.

In this chapter we will study the planar two center problem in the symmetric case: first we show the integrability structure of the system and all the possible types of motion and the bifurcation diagram are described. Then we gain a variational approach and we prove the existence of a special periodic solution of eight shape.

4.1 The integrability of the system

Given a planar fixed cartesian coordinate system, we assume two fixed centers to be located at positions $C_{1,2} = (\pm c, 0)$ on the x-axis. The equation of motion of a test particle restricted on a plane, moving in the mutual keplerian attraction of the two centres is

$$-\ddot{u} = \mu_1 \frac{u - C_1}{|u - C_1|^3} + \mu_2 \frac{u - C_2}{|u - C_2|^3} \qquad \mu_{1,2} > 0 \tag{4.1}$$

The constants $\mu_{1,2}$ indicate the strengths of the attraction of the fixed points, for instance in the gravitational case $\mu_i = GM_i$, where G is the gravitational constant and M_1, M_2 are the masses of the two fixed bodies. The case $\mu_1 = \mu_2$ is said symmetric case, asymmetric case otherwise.

The hamiltonian associated to the system (4.1) is

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \mu_1 \frac{1}{\sqrt{(x-c)^2 + y^2}} - \mu_2 \frac{1}{\sqrt{(x+c)^2 + y^2}} = T + U$$

and denotes the energy of the moving particle. Now we perform a changing of coordinates in order to obtain a second conserved quantity and to establish the integrability of the system. (see [27])

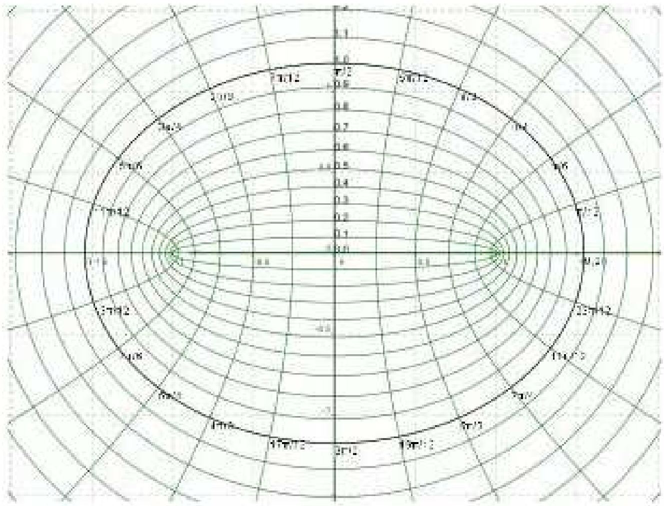


Figure 4.1:

Since any ellipse or hyperbola with the two centres as foci is a possible orbit when one of the centres acts alone, by Bonnet's theorem the same orbits are admissible solutions when both the centres of forces are acting. This suggest to replace the cartesian coordinates (x, y) by elliptic coordinates (ξ, η) defined by

$$x = c \cosh \xi \cos \eta \quad y = c \sinh \xi \sin \eta$$

Since

$$\frac{x^2}{c^2 \cosh^2 \xi} + \frac{y^2}{c^2 \sinh^2 \xi} = 1$$

and

$$\frac{x^2}{c^2 \cos^2 \eta} - \frac{y^2}{c^2 \sin^2 \eta} = 1$$

the coordinate lines $\xi = constant$ and $\eta = constant$ represent, respectively, ellipses and hyperbola whose foci are the centres of attraction.

The potential energy U , expressed in terms of ξ, η , becomes

$$U = -\mu_1 \frac{1}{c(\cosh \xi - \cos \eta)} - \mu_2 \frac{1}{c(\cosh \xi + \cos \eta)}$$

and the kinetic energy T is given by

$$T = \frac{1}{2}(\cosh^2 \xi - \cos^2 \eta)(\dot{\xi}^2 + \dot{\eta}^2)$$

Denoting with $\mathcal{L} = T - U$ the lagrangian function, the Euler-Lagrange equations associated to the motion of the coordinate ξ are given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}}\right) = \frac{\partial \mathcal{L}}{\partial \xi}$$

$$c^2 \frac{d}{dt}\left((\cosh^2 \xi - \cos^2 \eta)\dot{\xi}\right) = c^2 \cosh \xi \sinh \xi (\dot{\xi}^2 + \dot{\eta}^2) - \frac{\partial U}{\partial \xi}$$

We multiply both sides for $2(\cosh^2\xi - \cos^2\eta)\dot{\xi}$ then

$$\begin{aligned}
&c^2\frac{d}{dt}\Big((\cosh^2\xi-\cos^2\eta)^2\dot{\xi}^2\Big)\\
&= -2(\cosh^2\xi-\cos^2\eta)\frac{\partial U}{\partial\xi}\dot{\xi}^2 + 2\dot{\xi}\cosh\xi\sinh\xi 2(H-U)\\
&= -2(\cosh^2\xi-\cos^2\eta)\frac{\partial U}{\partial\xi}\dot{\xi}^2 + 2\dot{\xi}(H-U)\frac{\partial}{\partial\xi}(\cosh^2\xi-\cos^2\eta)\\
&= 2\dot{\xi}^2\frac{\partial}{\partial\xi}\big((\cosh^2\xi-\cos^2\eta)(H-U)\big)\\
&= 2\dot{\xi}^2\frac{\partial}{\partial\xi}\Big((\cosh^2\xi-\cos^2\eta)H+\frac{\mu_1}{c}(\cosh\xi+\cos\eta)+\frac{\mu_2}{c}(\cosh\xi-\cos\eta)\Big)\\
&= 2\frac{d}{dt}\left(H\cosh^2\xi+\frac{\mu_1+\mu_2}{c}\cosh\xi\right)
\end{aligned}$$

Integrating we obtain

$$\frac{c^2}{2}(\cosh^2\xi-\cos^2\eta)^2\dot{\xi}^2 = H\cosh^2\xi\frac{\mu_1+\mu_2}{c}\cosh\xi - G \tag{4.2}$$

where G is a constant of integration.

In the coordinates (ξ,η) the energy H satisfies the relation

$$\begin{gathered}
H(\cosh^2\xi-\cos^2\eta)=\\
\frac{c^2}{2}(\cosh^2\xi-\cos^2\eta)^2(\dot{\xi}^2+\dot{\eta}^2)-\frac{\mu_1}{c}(\cosh\xi+\cos\eta)-\frac{\mu_2}{c}(\cosh\xi-\cos\eta)
\end{gathered}$$

thus, subtracting formula (4.2) from the previous one, we obtain

$$\frac{c^2}{2}(\cosh^2\xi-\cos^2\eta)^2\dot{\eta}^2 = -H\cos^2\eta+\frac{\mu_1-\mu_2}{c}\cos\eta+G \tag{4.3}$$

From relations (4.2),(4.3) we have

$$\frac{(d\xi)^2}{H\cosh^2\xi\frac{\mu_1+\mu_2}{c}\cosh\xi-G} = \frac{(d\eta)^2}{-H\cos^2\eta+\frac{\mu_1-\mu_2}{c}\cos\eta+G}$$

then, introducing an auxiliary variable s, it holds

$$\begin{aligned}
s &= \int\left(H\cosh^2\xi\frac{\mu_1+\mu_2}{c}\cosh\xi-G\right)^{-\frac{1}{2}}d\xi\\
s &= \int\left(-H\cos^2\eta\frac{\mu_1-\mu_2}{c}\cos\eta+G\right)^{-\frac{1}{2}}d\eta
\end{aligned}$$

These are elliptic integrals and, formally, one can express the motion of the coordinates ξ and η as function of the parameter s,

$$\xi=\chi(s),\quad \eta=\Psi(s)$$

which determine the orbit of the particle.

4.2 Bifurcation diagram for symmetric case

In the following we set $\mu_1 = \mu_2 = \frac{1}{2}$ and $c = 1$.

In elliptic coordinates the equations of motion are given by

$$(\cosh^2 \xi - \cos^2 \eta)^2 \dot{\xi}^2 = 2(H \cosh \xi + \cosh \xi - G)$$
$$(\cosh^2 \xi - \cos^2 \eta)^2 \dot{\eta}^2 = 2(-H \cos^2 \eta + G)$$

We observe that the velocities $\dot{\xi}$, $\dot{\eta}$, are singular on the collision configurations $\xi = \eta = 0$. We introduce an appropriate scaling of time,

$$dt = (\cosh^2 \xi - \cos^2 \eta) ds$$

in such a way that the previous relations become

$$\begin{aligned} \xi'^2 &= 2(H \cosh^2 \xi + \cosh \xi - G) \\ \eta'^2 &= 2(-H \cos^2 \eta + G) \end{aligned} \tag{4.4}$$

With the substitutions

$$u = \cosh \xi \ , \qquad v = \cos \eta$$

we obtain the equations

$$\frac{u'^2}{(u^2 - 1)} = 2(Hu^2 + u - G)$$
$$\frac{v'^2}{(1 - v^2)} = 2(G - Hv^2)$$

that, in non-collision case, can be written in the form

$$u'^2 = 2(u^2 - 1)(Hu^2 + u - G) = 2(u^2 - 1)P(u) \tag{4.5}$$
$$v'^2 = 2(1 - v^2)(G - Hv^2) = 2(1 - v^2)Q(v) \tag{4.6}$$

Classically allowed motions require the squared velocities to be positive. We discuss this occurrence in terms of the zeros of the polynomials $P(u), Q(v)$. We concentrate our discussion only for negative energy orbits, $H < 0$: we will see that this choice implies the boundness of the orbits. We now describe all the possible bounded motions in term of the constant H, G and we emphasise the connections between orbital topology and sectors of the (G, H)-plane. For a detailed description of the bifurcation diagram for the asymmetric and three dimensional case see [25], [26].

A necessary condition for the existence of an orbit is that the polinomials $P(u)$ is positive for some $u \geq 1$ and $Q(v)$ is positive for some $v \in [-1, 1]$ and the motion is bounded in the (u, v) rectangular where the polinomials are positive. We first study the equation of motion relative to the coordinate v (4.6): if $G \leq 0$ the polinomial $Q(v)$ has real roots $v_{1,2} = \pm\sqrt{\frac{G}{H}}$, while $Q(v) > 0$ for every v in case $G > 0$. The lines $\mathcal{L}_v^1 = \{H = G\}$ and $\mathcal{L}_v^2 = \{G = 0\}$ divide the (G, H) halfplane $H < 0$ in three sectors, see fig (4.2):

in sector I , $\sqrt{\frac{G}{H}} > 1$ then $Q(v) < 0$ for every $v \in [-1, 1]$ hence no motion is possible. On line $\mathcal{L}_v^1$ it holds $H = G$ then only the collision motions $v = 1$, $v = -1$ are admitted;

in region II the roots $v_{1,2}$ move in the interval $[-1, 1]$ and oscillatory motions in the v-range $[v_1, 1]$ and $[-1, -v_1]$ become possible: these correspond to motions that swing around one of the two centres. This situation persists as long as $G \neq 0$: on line $\mathcal{L}_v^2$ it holds $v_1 = v_2 = 0$ hence the axial solution rises.

Finally, in the third region III, $Q(v)$ is positive for all v and motion exists for every physical value $v \in [-1, 1]$; in this situation the particle swing around the two centres.

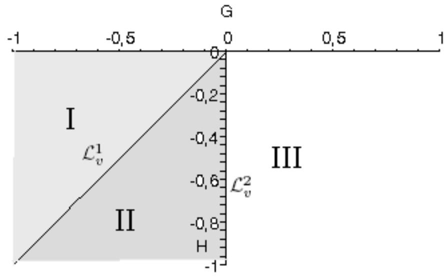


Figure 4.2:

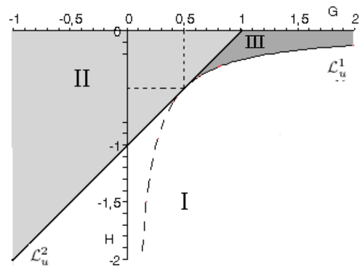


Figure 4.3:

We now study the equation of motion for the coordinate u: the motion exists in correspondence of that values of coordinate $u \geq 1$ such that $P(u) \geq 0$.

The restriction we done on the sign of the constant H, $H < 0$, implies that the function $P(u)$ is a concave parabola then all the possible motions are bounded. The topology of the orbits depends on how many solutions $u > 1$ the equation $P(u) = 0$ has; more precisely, if the discriminant of the polynomial $P(u)$ is negative, i.e. $1 + 4HG < 0$, the previous equation has no real solutions then $P(u) < 0$ for all u and no motion exists.

Otherwise, suppose $H \geq -\frac{1}{4G}$, then $P(u) = 0$ has two real solutions, say for instance u_1, u_2, with $u_1 < u_2$. If $u_1 < 1$ and $u_2 < 1$ then $P(u) < 0$ for all $u > 1$ hence again no motion exists, while if $u_1 < 1$ and $u_2 > 1$ a motion in the u-range $[1, u_2]$ is possible and, finally, if both the solutions u_1, u_2 are bigger than one , the motion exists in the strip $[u_1, u_2]$. Let us now derive algebraic conditions on H and G for each of the cases listed below. The solutions of $P(u) = 0$ are

$$u_1 = \frac{-1 + \sqrt{1 + 4HG}}{2H} \qquad u_2 \frac{-1 - \sqrt{1 + 4HG}}{2H}$$

The conditions on H and G that ensure $u_2 > 1$ are

$$\begin{cases} 1 + 4HG \geq 0 \\ H < -\dfrac{1}{2} \\ H > G - 1 \end{cases} \cup \begin{cases} 1 + 4HG \geq 0 \\ H > -\dfrac{1}{2} \end{cases}$$

while both the solutions $u_1, u_2 > 1$ if

$$\begin{cases} 1 + 4HG \geq 0 \\ H > -\dfrac{1}{2} \\ H < G - 1 \end{cases}$$

The hyperbola $\mathcal{L}_u^1 = \{H = -\frac{1}{4G}\}$ and the line $\mathcal{L}_u^2 = \{h = G - 1\}$ are boundary in the (G, H) half-plane $H < 0$ of three regions shown in figure (4.3): in region I no motion is possible, in region II the motion exists in the u-range $[1, u_2]$ with u_2 leading to infinity as H tends to zero. On the other hand on the separating line $\mathcal{L}_u^2$ we have two different behaviour: if $G < \frac{1}{2}$, that is $\mathcal{L}_u^2$ separates region I and II, only the collision motion is possible ($u_2 = 1$),while $u_2 = \frac{G}{1-G}$ if $\frac{1}{2} < G < 1$. In region III the solutions u_1 and u_2 are both bigger then 1, then an oscillatory motion in the u interval $[u_1, u_2]$ is possible.

To get the complete bifurcation diagram of the bounded motions, we superimpose the critical line of v- and u- motion : this leads to figure (4.4). The admissible region of the (G, H) plane that correspond real motions is partioned in three sectors by critical lines. For each of this sectors the orbits $(x(s), y(s))$ have a different topology, see figure (4.5):

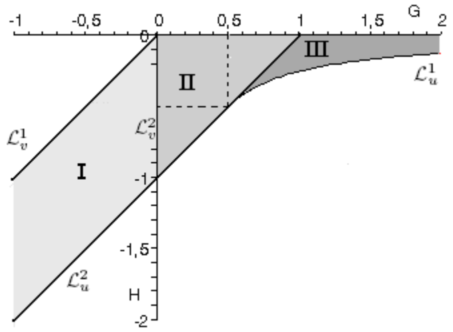


Figure 4.4:

I *Satellite orbits.* These orbits wind around one of the centres and occupy two disconnected regions between the ellipse $\xi = \xi_2$ and two branches of the hyperbola $\eta = \eta_1$ where $\cosh \xi_2 = u_2$ and $\cos \eta_1 = v_1$

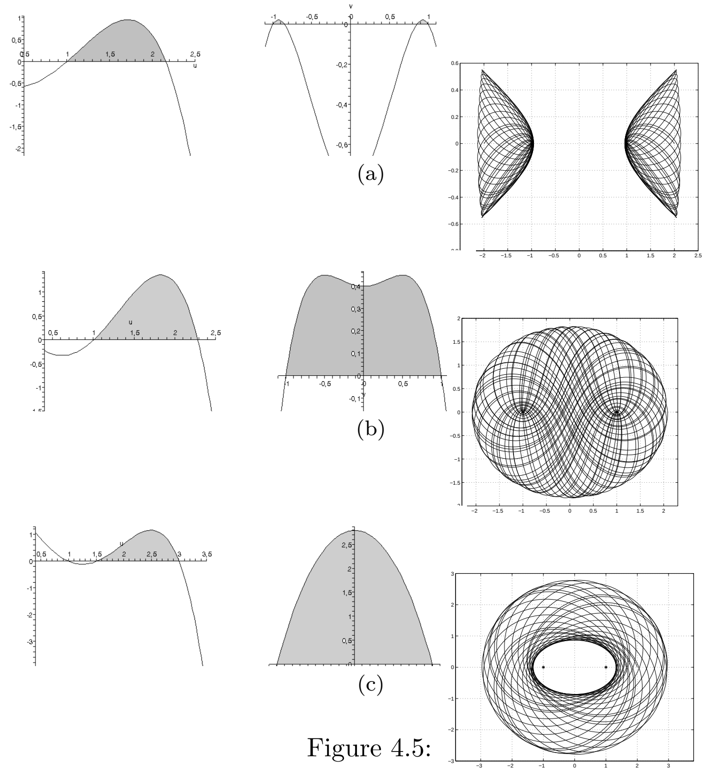
II *Lemniscate orbits.* These orbits encircle both the centres and are bounded in the region within the ellipse $\xi = \xi_2$. These orbits intersect the line joining the centres and except the periodic orbits, they are quasi-periodic and fill the entire region inside the boundary ellipse.

III *Planetary orbits.* In this case the particle rings around the two centres and the orbits are enclosed by two ellipses, $\xi = \xi_1$ and $\xi = \xi_2$ where, as usual, $\cosh \xi_i = u_i$. Beside the periodic one, all the orbits are quasi-periodic and explore the entire region.

On critical lines the orbits change topology and could present collision. In particular on line $\mathcal{L}_v^1$ the constants are equal, $H = G$ then $v \equiv \pm 1$ and $u \in [1, u_2]$. It corresponds to collision orbits on the x-axis with $x \geq 1$ or $x \leq -1$. On line $\mathcal{L}_u^2$ the constants satisfy $H = G - 1$ then if $G < \frac{1}{2}$ it holds $u \equiv 1$. The orbits are pendular libration along the line joining the centres on the x-axis and they are of two types: one with v running between limits $v_{max} = \pm 1$ $(G > 0)$ and a second with

v running from $|v_{min}| < 1$ to $v_{max} = \pm 1$ ($G < 0$). An orbit in either of these classes present a collision with one of the centres. On the boundary line $\mathcal{L}_u^1$ the roots of the polinomial $P(u)$ are equal and bigger then 1. The motion is periodic and lies on an ellipse $\xi = const$.

Passing through the separation lines $\mathcal{L}_v^2$ and $\mathcal{L}_u^2$ with $\frac{1}{2} < G < 1$ the topology of region P of the plane when the orbits live changes. In sector I P is composed by two disconnected portions, while in sector II P becomes simply connected. Crossing the line $\mathcal{L}_u^2$ and entering in sector III, the region P remains connected but not simply connected.



(a)

(b)

(c)

Figure 4.5:

4.3 Periodic solutions

In elliptic coordinates (ξ, η) the system of motion for the symmetric two-center problem is given by equations (4.4). Then every bounded solution is the superposition of an hanarmonic oscillator in the coordinate ξ and a harmonic oscillator for the coordinate η.

The periods of the ξ and η motions are

$$T_\xi = \oint \left(2(H\cosh^2\xi + \cosh\xi - G\right)^{-\frac{1}{2}} d\xi$$

$$T_\eta = \oint \left(2(G - H\cos\eta\right)^{-\frac{1}{2}} d\eta$$

We define the winding number α of the trajectory as

$$\alpha = \frac{T_\eta}{T_\xi}$$

When α is a rational number m/n a periodic orbit occurs, otherwise quasi-periodic.

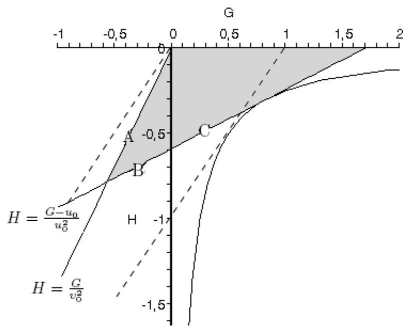


Figure 4.6:

A first class of periodic orbits is given for $\alpha = 0, \infty$ that is $\xi = const$ and $\eta = const$. For every $\xi_0 > 0$ and $\eta_0 \in (0, 2\pi)$ there exist a periodic solution on the coordinate lines $\xi = \xi_0$ and $\eta = \eta_0$: we show the values of the parameters H, G that correspond to that solutions.

As before let $u_0 = \cosh \xi_0$ and $v_0 = \cos \eta_0$, then the bifurcation diagram for the motion constrained to pass by the point (u_0, v_0) is depicted in figure (4.6). The segment $H = \frac{G}{v_0^2}$ and $H = \frac{G - u_0}{u_0^2}$ sign the edge of the admissible region. If the constant point (G, H) belongs to the boundary of that region then a periodic orbit passing by (u_0, v_0) occurs; in particular on the segment A the periodic orbits consists in an arc of hyperbola, while on the segment B the orbit draws an arc of ellipse and on the segment C a closed ellipse. In the last case we say that the orbits is of class $\alpha = 1 : 0$. The stability analysis of these periodic orbits is given in [25].

In figure 4.7 we present some examples of periodic orbits related to different winding numbers: in particular the eight-shape periodic orbit corresponds to $\alpha = 1 : 2$ and for every integer m, n there exists periodic orbits of winding number $\alpha = m : n$. In [9, 8] a systematic classification of the periodic orbits is given and the convergent properties are studied.

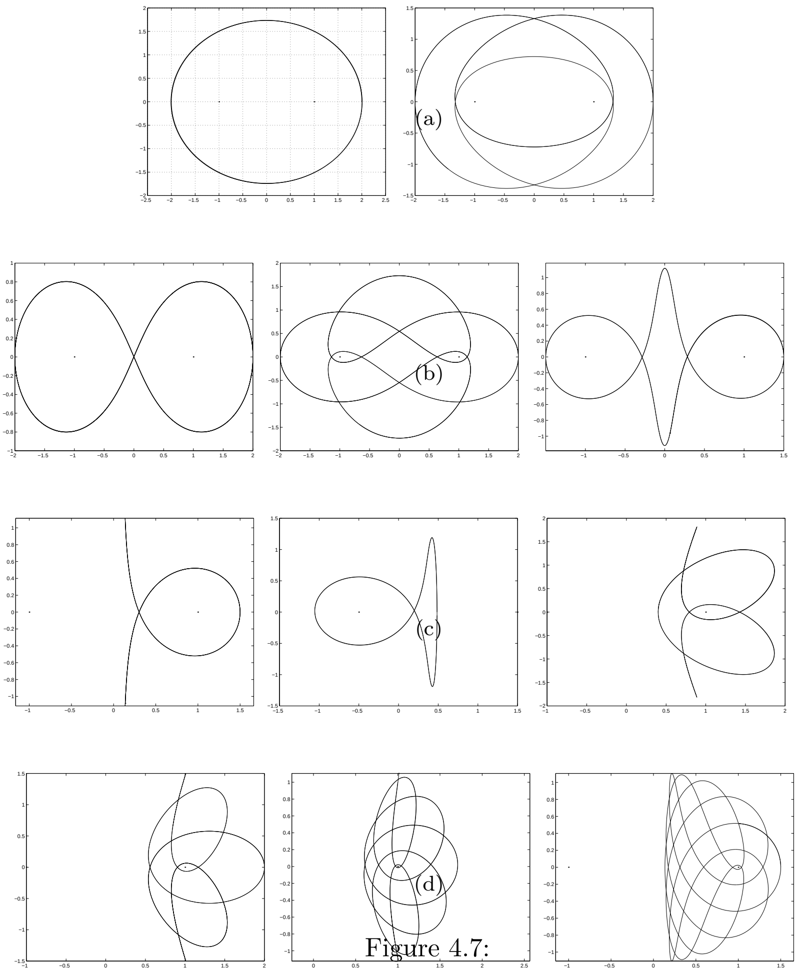


Figure 4.7:

Chapter 5

A variational approach to the study of the planar N-centre problem

Classically, the N-centre problem consists in the study of the dynamics of a test particle moving under the gravitational influence of N fixed masses.

Although in the literature both the planar and spatial and both attractive and repelling cases are considered, we focus our analysis only in the planar case, when the test particle is forced to move in the euclidean plane, and in purely attractive case, that is all the centres are attractive and, when it is not specified, we suppose the force of attraction to be gravitational.

The N-centre problem is considered to be a very interesting problem for many reasons: first of all, the N-centre problem is strictly connected to the $N+1$-body problem, indeed it can be considered as a limit case of the $N+1$-body problem when one of the body, the satellite, moves faster than the other N bodies, called primaries, see for instance [14, 15]. Moreover the N-centre problem is also related to models in molecular physics, where the centres play the role of massive components in atoms or the gravitational force is replaced by the coulombic interaction, [25, 9, 8, 24].

Except the 1-center problem and the 2-center problem, the N-centre problem is not completely integrable; this was proved for the planar case by Bolotin [4], and in the spatial case by Bolotin, Negrini [6, 5] and Knauf, Klein [15, 14]. In these last works the authors consider scattering orbits in the force field of N attracting coulombic singularities and they show also an unexpected feature of the N-central problem: the existence of a set of bounded orbits at arbitrarily high energies.

In the study of the N-centre problem, as well as the N-body problem, the collisions with the centres form a major technical obstacle in the understanding of the dynamics since they are the source of the non-completeness of the flow. In the following we will handle this problem from a variational point of view: we will use the standard tolls of calculus of variation together with some ideas, as the Levi-Civita transformation and certain topological constraints, and we will prove the existence of non collision solutions for the N-centre problem for particular displacements of the centres in the plane. Our approach will be quite different from the one followed in the above mentioned works: we look for solutions of the problem by means of a minimizing procedure among certain sets of paths with fixed period rather than fixed energy and our solutions are expected to be of low energy; moreover the collinear configuration of centres, the one avoided in [14, 15], is the situation we are most interested.

Before studying the motion of a particle attracted by more than one centres, we want to deepen a little more the one-centre problem: the next results can be viewed as an extension, or a fulfilment, of the results obtained in the first part of this thesis about the collisionless property of the minimal path for the action functional associated to an homogeneous potential problem.

5.1 An homogeneous perturbation of the 1-centre problem

Consider the differential equation

$$-\ddot{x} = \frac{x}{|x|^{2+\alpha}} + \frac{x}{|x|^{2+\gamma}} \tag{5.1}$$

where $\alpha \geq 1$, $\gamma < \alpha$; this system can be considered as a singular perturbation of the α-homogeneous potential problem considered before. We want to deal about the variational property of collision paths and we will show that the presence of the perturbation, even if singular, does not change the results obtained for the unperturbed system.

To this aim, denoting with P_1, P_2 two different points, different from the origin, and with θ the internal angle $\widehat{P_1OP_2} = \theta > 0$, $\theta \leq \pi$, we define $\hat{\Lambda}^\theta_{i,f}$ as the set of all non-collision paths joining P_1 to P_2 such that the total variation of the angle is $2\pi - \theta$. Moreover we use $\Lambda^\theta_{i,f}$ to indicate the union of $\hat{\Lambda}^\theta_{i,f}$ with the collision path joining P_1 to P_2.

The equation (5.1) is the Euler-Lagrange equation associated to the action functional

$$\mathcal{A}(q) = \int_0^T L(q(t), \dot{q}(t))dt$$

where $t \mapsto q(t)$ is a path in the plane and $L(q, \dot{q})$ is the Lagrangian function defined as

$$L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 + \frac{1}{\alpha}\frac{1}{|q|^\alpha} + \frac{1}{\gamma}\frac{1}{|q|^\gamma} = \frac{K}{2} + U$$

The functional

$$\mathcal{A}^\theta_{i,f}(q) : \Lambda^\theta_{i,f} \to \mathbb{R}^+ \cup \{+\infty\}, \quad \mathcal{A}^\theta_{i,f}(q) = \int_0^T \left(\frac{K}{2} + U\right) dt$$

is the restriction of the lagrangian action to $\Lambda^\theta_{i,f}$.

Theorem 21. *If $\alpha \geq 1$ the functional $\mathcal{A}^\theta_{i,f}(q)$ has at least a minimizer and all the minimizers are free of collisions.*

Proof
The existence of a minimizer follows from the coercivity and the lower semicontinuity of the action functional (see for instance [11]).

For every $\varepsilon > 0$ we define

$$\Lambda^\theta_{i,f}(\varepsilon) = \{q \in \Lambda^\theta_{i,f} \ : \ |q(t)| \geq \varepsilon\}$$

$$c(\varepsilon) = \min_{q \in \Lambda^\theta_{i,f}(\varepsilon)} \mathcal{A}^\theta_{i,f}(q)$$

and

$$\mathcal{M}(\varepsilon) = \left\{q \in \Lambda^\theta_{i,f}(\varepsilon) \ : \mathcal{A}^\theta_{i,f}(q) = c(\varepsilon), \ \min|q(t)| = \varepsilon\right\}$$

The minimizers of the action $\mathcal{A}^\theta_{i,f}$ are free of collisions if and only if the set $\mathcal{M}(\varepsilon)$ is empty for ε small enough. For the sake of absurdity we suppose there exists a sequence $\{\varepsilon_n\}_{n=1}^\infty$ converging to zero and $q_n \in \Lambda^\theta_{i,f}$ such that $\min|q_n(t)| = \varepsilon_n$ and $\mathcal{A}^\theta_{i,f}(q_n) = c(\varepsilon_n)$.

For every $n \in \mathbb{N}^+$ the path $q_n(t)$ has the following properties:

i. $q_n(t) \in \mathcal{C}^1([0, T_n])$, $q_n(0) = P_1$, $q_n(T_n) = P_2$ and there exists $t_n^- \leq t_n^+$ such that

$$\begin{aligned} |q_n(t)| = \varepsilon_n \qquad & if\ t \in [t_n^-, t_n^+] \\ |q_n(t)| > \varepsilon_n \qquad & if\ t \in [0, T_n] \setminus [t_n^-, t_n^+] \end{aligned}$$

ii. $q_n(t)\Big|_{[0,t_n^-]}$ and $q_n(t)\Big|_{[t_n^+,T_n]}$ are $\mathcal{C}^2$ functions and are solutions of the equation (5.1).

iii. the energy H_n and the angular momentum L_n of $q_n(t) = \rho_n(t)e^{i\phi_n(t)}$

$$H_n = \frac{1}{2}\left(\dot{\rho}_n^2 + \rho^2\dot{\phi}^2\right) - \frac{1}{\alpha}\frac{1}{\rho^\alpha} - \frac{1}{\gamma}\frac{1}{\rho^\gamma}$$
$$L_n = \rho^2\dot{\phi}$$

are constant for every $t \in [0, T_n]$ and $\{H_n\}_{n=1}^\infty$ is uniformly bounded.

The first property is a consequence of a result of [18] which concerns extremals of lagrangians with convex obstacle, while the other points follow from the extremality of q_n with respect to time reparametrization and with respect to compact supported variations. We refer to [23] for a detailed proof. Moreover it's not difficult to show that

Lemma 5.1. *If $u(t)$ is a solution of equation* (5.1) *then for every $\varepsilon > 0$ the function $u_\varepsilon(t) = \frac{1}{\varepsilon}u(\varepsilon^{\frac{2+\alpha}{2}}t)$ solves the equation*

$$-\ddot{u} = \frac{u}{|u|^{2+\alpha}} + \varepsilon^{\alpha-\gamma}\frac{u}{|u|^{2+\gamma}} \tag{5.2}$$

For every n we decompose the path $q_n(t)$ as follows:

- $q_{1,n}(t) = q_n(t)\Big|_{[0,t_n^-]}$ the path from P_1 till the point when q_n reaches the constraint B_{ε_n}
- $q_{2,n}(t) = q_n(t)\Big|_{[t_n^-,t_n^+]}$ the path on the constraint
- $q_{3,n}(t) = q_n(t)\Big|_{[t_n^+,T_n]}$ the path from the constraint to P_2.

and, at the same time, we write the total variation of the angle

$$2\pi - \theta = \alpha_{1,n} + \alpha_{2,n} + \alpha_{3,n} \tag{5.3}$$

where $\alpha_{i,n}$ is tha angle covered by $q_{i,n}$.

We now consider the path $q_{1,n}(t)$ and, without loosing of generality, we apply a time translation such that $|q_{1,n}(0)| = \varepsilon_n$, $q_{1,n}(T_n) = P_1$.

For every n let $u_n(t) = r_n(t)e^{i\psi_n(t)}$ be defined by

$$u_n(t) = \frac{1}{\varepsilon_n}q_{1,n}\left(\varepsilon_n^{\frac{2+\alpha}{2}}t\right) \qquad t \in \left[0, T_n\varepsilon_n^{-\frac{2+\alpha}{2}}\right]$$
$$r_n(t) = \frac{1}{\varepsilon_n}\rho_{1,n}\left(\varepsilon_n^{\frac{2+\alpha}{2}}t\right)\ , \quad \psi_n(t) = \phi_{1,n}\left(\varepsilon_n^{\frac{2+\alpha}{2}}t\right)$$

thus $u_n(0) = 1$ and, for the previous lemma, u_n is a solution of equation (5.2).

The energy integral h_n and the angular momentum l_n of the solution $u_n(t)$ are given, respectively, by

$$\begin{aligned} h_n &= \frac{1}{2}|\dot{u}_n(t)|^2 - \frac{1}{\alpha}\frac{1}{|u_n(t)|^\alpha} - \frac{\varepsilon^{\alpha-\gamma}}{\gamma}\frac{1}{|u_n(t)|^\gamma} \\ &= \frac{1}{2}\left|\frac{1}{\varepsilon_n}\dot{q}_n(\varepsilon_n^{\frac{2+\alpha}{2}}t)\varepsilon_n^{\frac{2+\alpha}{2}}\right|^2 - \frac{\varepsilon_n^\alpha}{\alpha}\frac{1}{\left|q_n(\varepsilon_n^{\frac{2+\alpha}{2}}t)\right|^\alpha} - \frac{\varepsilon_n^\alpha}{\gamma}\frac{1}{\left|q_n(\varepsilon_n^{\frac{2+\alpha}{2}}t)\right|^\gamma} \\ &= \varepsilon_n^\alpha H_n \end{aligned}$$

and

$$\begin{aligned} l_n = r_n^2 \dot{\psi}_n &= \frac{1}{\varepsilon_n^2} \rho_n(\varepsilon_n^{\frac{2+\alpha}{2}} t)^2 \dot{\phi}(\varepsilon_n^{\frac{2+\alpha}{2}} t) \varepsilon_n^{\frac{2+\alpha}{2}} \\ &= \varepsilon_n^{\frac{\alpha-2}{2}} L_n \end{aligned}$$

Since $h_n \to 0$ as $n \to \infty$, the sequence of paths $u_n(t)$ tends to the parabolic solution. Moreover, since $q_n(t)$ is tangent to the constraint B_{ε_n}, it follows $\dot{\rho}_n(0) = 0$ and, from the energy integral, we infer

$$H_n = \frac{1}{2} \frac{L_n^2}{\varepsilon_n^2} - \frac{1}{\alpha} \frac{1}{\varepsilon_n^\alpha} - \frac{1}{\gamma} \frac{1}{\varepsilon_n^\gamma}$$

This implies that

$$l_n^2 = L_n^2 \varepsilon_n^{\alpha-2} = 2\varepsilon_n^\alpha H_n + \frac{2}{\alpha} + \frac{2}{\gamma} \varepsilon_n^{\alpha-\gamma} \to \frac{2}{\alpha}$$

as $n \to \infty$.

The sequence $\{u_n\}_{n=1}^\infty$ uniformly converges on every closed interval of $\mathbb{R}^+$ to the path $\bar{u} : \mathbb{R}^+ \to \mathbb{R}^2$ with the property of being of class $\mathcal{C}^2$ on $[0, +\infty]$ and a parabolic solution of the perturbed kepler problem (5.2) with energy $\bar{h} = 0$ and angular momentum $\bar{l} = \sqrt{\frac{2}{\alpha}}$.

Let (r, ψ) be the polar coordinates of the plane motion and express the path $\bar{x}$ in this reference, $\bar{x} = re^{i\psi}$. Since the angular momentum $\bar{l}$ is a positive conserved quantity, the function $s \mapsto \psi(s)$ is strictly increasing. Denoting with $\psi_0 = \psi(0)$, $\psi(+\infty) = \lim_{s\to\infty} \psi(s)$ and with $\Delta\psi = \psi(+\infty) - \psi_0$, from the energy integral it follows

$$\begin{aligned} \Delta\psi &= \lim_{n\to+\infty} \int_1^{R^+} \frac{l_n}{r^2 \sqrt{2\left(H_n + \frac{1}{\alpha}\frac{1}{r^\alpha} + \frac{\varepsilon_n^{\alpha-\gamma}}{\gamma}\frac{1}{r^\gamma}\right) - \frac{l_n^2}{r^2}}} dr \\ &= \int_1^{+\infty} \frac{\sqrt{\frac{2}{\alpha}}}{r^2 \sqrt{\frac{2}{\alpha}\frac{1}{r^\alpha} - \frac{2}{\alpha}\frac{1}{r^2}}} dr = \frac{\pi}{2-\alpha} \end{aligned}$$

The sequence ψ_n uniformly converges to the function ψ on every closed interval in $\mathbb{R}^+$. If $[0, A]$ is such a closed interval, for n sufficiently large, we have

$$\psi_n(A) - \psi_n(0) \le \psi_n\left(\varepsilon_n^{-\frac{2+\alpha}{2}} T_n\right) = \alpha_{1,n}$$

Repeating all the reasoning for the path $q_{3,n}$ we find a function $\bar{\psi}$ with the same property of ψ and sequence $\bar{\psi}_n$ uniformly converging to $\bar{\psi}$ on every closed interval of $\mathbb{R}^+$ such that, for every interval $[0, B]$

$$\bar{\psi}_n(B) - \bar{\psi}_n(0) \le \alpha_{3,n}$$

Adding the terms $\alpha_{1,n}$ and $\alpha_{3,n}$, from (5.3) it results

$$(\psi_n(A) - \psi_n(0)) + (\bar{\psi}_n(B) - \bar{\psi}_n(0)) \le \alpha_{1,n} + \alpha_{3,n} \le 2\pi - \theta$$

Taking the limit for $n \to +\infty$

$$(\psi(A) - \psi(0)) + (\bar{\psi}(B) - \bar{\psi}(0)) \le 2\pi - \theta$$

since A, B are arbitrarily chosen, as $A, B \to +\infty$

$$\Delta\psi + \Delta\bar{\psi} = \frac{2\pi}{2-\alpha} \le 2\pi - \theta$$

This gives a contradiction and the theorem is proved. □

5.2 On the existence of 8-shaped orbits

The aim of this section is to develop a theory and some useful tools in order to deal with the N-centre problem. On the other hand, if one look at what is written few lines below, he may recognise that again a perturbation of the one-centre problem is considered. The reason on why we persist in studying the 1-centre problem consists on the fact that the global behaviour of a solution of the N-centre problem strictly depends on what happens when such solution passes near one of the centres. Thus we want to deepen the analysis of the variational properties of the dynamical system in the neighbourhood of one of the center when the contribution to the force field due to the presence of the others centres is regard to be a smooth perturbation.

Keeping in mind the 2-centres problem, as a model of all the systems in which the centers are placed in symmetric configurations with respect an axis, we assume the perturbation to be a symmetric function with respect the axis itself. More precisely we are interested on the study of the differential system

$$\ddot{u} = -\frac{u}{|u|^3} + f(u) \tag{5.4}$$

where the function $f(x, y) = (f_1, f_2)$ is a smooth function and satisfies $f(x, -y) = (f_1, -f_2)(x, y)$. The main difference between the last perturbed 1-centre problem and the one considered in the previous chapter is that, in this case, the perturbation is not rotationally invariant thus, as a consequence, the angular momentum is not more conserved on the trajectory of the system. It follows that the same technique, namely the blow-up technique, is not more applicable in order to avoid the collision on minimal paths.

However we will prove that the axial symmetry of the perturbation is sufficient to establish a precise characterization of the collision orbits of the system (5.4).

To this aim we start writing the action functional associated to the system (5.4): denoting with $F \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$ the potential associted to the perturbation f, $f = \nabla F$, we define the action functional $\mathcal{A}$ as

$$\mathcal{A}(u) = \int_0^T \left\{ \frac{1}{2}|\dot{u}|^2 + \frac{1}{|u|} - F(u) \right\} dt$$

We denote the x-axis with π and with π_+, π_- the two open components of π disconnected by the origin. For fixed T we consider the following sets of paths

$$\Gamma_{P,Q} = \left\{ u(t),\ \dot{u} \in L^2([0,T]) :\ u(0) = P, u(T) = Q \right\}$$

where $P \in \pi_+$ and $Q \in \pi_-$ are two fixed points. We may allow one of the points, say Q, to vary on the closure of its connected component; consequently we consider

$$\Gamma_P = \left\{ u(t),\ \dot{u} \in L^2([0,T]) :\ u(0) = P \in \pi_+, u(T) \in \overline{\pi_-} \right\}$$

Finally we may allow both the end-points to vary in the closure of their components, hence we consider the set of paths

$$\Gamma = \left\{ u(t),\ \dot{u} \in L^2([0,T]) :\ u(0) \in \overline{\pi_+}, u(T) \in \overline{\pi_-} \right\}$$

We use the notation $\Gamma_\star$ in order to indicate each one of the previuos sets and let $\mathcal{A}_{\Gamma_\star}$ be the restriction of $\mathcal{A}$ on $\Gamma_\star$.

Moreover, for every $\varepsilon > 0$ we denote with $\Gamma_\star^\varepsilon \subset \Gamma_\star$ as the set of paths $u(t)$ joining the points P and Q remaining always at least at distance ε from the singularity:

$$\Gamma_\star^\varepsilon = \left\{ u \in \Gamma_\star,\ \min_{t \in [0,T]} |u(t)| \geq \varepsilon \right\}$$

We term $u_\varepsilon(t)$ any (local) minimizer of $\mathcal{A}_{\Gamma_\star}$ on $\Gamma_\star^\varepsilon$ and

$$c(\varepsilon) = \min_{u \in \Gamma_*^\varepsilon} \mathcal{A}(u)$$

Such minimizer exists because every set $\Gamma_\star^\varepsilon$ is weakly closed in H^1 and every functional $\mathcal{A}_{\Gamma_\star}$ is weakly lower semicontinuous and coercive (see [7], [23]).

For each $\varepsilon > 0$ two alternatives may occur. The path $u_\varepsilon(t)$ does not intersect the constraint $\mathcal{B}_0(\varepsilon)$ for all time, in this case $u_\varepsilon(t)$ is a local minimizer of the action in the interior of $\Gamma_\star$.

Otherwise, there exists a nonempty set of instants $\{t_i\} \in [0,T], i = 0,1,...$ such that

$$\begin{aligned} |u_\varepsilon(t)| > \varepsilon \quad t \in \Delta T_\varepsilon \\ |u_\varepsilon(t)| = \varepsilon \quad t \notin \Delta T_\varepsilon \end{aligned}$$

where

$$\Delta T_\varepsilon = \bigcup_j (t_{2j}, t_{2j+1}) \tag{5.5}$$

A priori the set $\{t_i\}$ could have an infinite number of elements and then the interval ΔT_ε could be an infinite union of open intervals. In the following lemma we prove that this alternative is not possible and some other properties of $u_\varepsilon(t)$ are given.

Lemma 5.2. *For every $\varepsilon > 0$ the path $u_\varepsilon(t)$ has the following properties:*

i) $u_\varepsilon(t)$ is a $\mathcal{C}^1([0,T])$ path

ii) $u_\varepsilon(t)\big|_{\Delta T_\varepsilon}$ is a $\mathcal{C}^2$ path and it's solution of equation (5.4).

iii) the energy $h_\varepsilon = \dfrac{1}{2}|\dot u_\varepsilon|^2 - \dfrac{1}{|u_\varepsilon|} + F(u_\varepsilon)$ is constant for $t \in [0,T]$ and the set $\{h_\varepsilon, \varepsilon < \bar\varepsilon\}$ is uniformly bounded.

iv) the set of instants $\{t_i\}$ is finite and $(\Delta T_\varepsilon)^c = \bigcup_{j=0...N} [t_{2j+1}, t_{2j+2}]$

v) for ε small enough, for any $j \in \{0,1,..,N\}$ it holds that for any couple of instants $t_{1,2} \in [t_{2j+1}, t_{2j+2}]$, $t_1 \neq t_2$, then $u_n(t_1) \neq u_n(t_2)$

vi) $\dot u_\varepsilon(t_j)$ are tangent to the boundary of the constraint $\mathcal{B}_0(\varepsilon)$ and $\frac{1}{2}|\dot u_\varepsilon(t_j)|^2\varepsilon \to 1$ as $\varepsilon \to 0$

Proof

i) It's a consequence of a result of [18] which concerns extremals of Lagrangian with convex obstacle. In this case the obstacle is the set

$$\{u \in \mathbb{R}^2, |u| \leq \varepsilon\}$$

ii) It follows from the extremality of u_ε with respect to variations whose support is contained in ΔT_ε.

iii) Let $V(u) = -\dfrac{1}{|u|} + F(u)$ then $h_\varepsilon = \dfrac{1}{2}|\dot u_\varepsilon|^2 + V(u_\varepsilon)$ and

$$\mathcal{A}_{\Gamma_T}(u_\varepsilon) = \int_0^T \left(\frac{1}{2}|\dot u_\varepsilon|^2 - V(u_\varepsilon)\right) dt$$

Let τ be a reparametrization of the interval $[0,T]$ defined by the equation

$$t = \tau - \gamma\delta(\tau) \tag{5.6}$$

where $\delta \in \mathcal{C}_c^\infty(0,T)$ and γ is any constant small enough such that $\tau - \gamma\delta(\tau)$ is monotonically increasing.

Denote $\tau = \varphi(t)$ the inverse of the previous relation and $u_\varepsilon^\delta(t) = u_\varepsilon(\varphi(t))$.

$$\begin{aligned}\mathcal{A}_{\Gamma_T}(u_\varepsilon^\delta) &= \int_0^T \Big(\frac{1}{2}|\dot{u}_\varepsilon^\delta(t)|^2 - V(u_\varepsilon^\delta(t))\Big)dt \\ &= \int_0^T \Big(\frac{1}{2}\Big|\frac{d}{dt}(u_\varepsilon(\varphi(t)))\Big|^2 - V\big(u_\varepsilon(\varphi(t))\big)\Big)dt\end{aligned}$$

We now perform the variable changing defined in (5.6) and obtain

$$\begin{aligned}\mathcal{A}_{\Gamma_T}(u_\varepsilon^\delta) &= \int_0^T \Big[\frac{1}{2}\Big|\frac{d}{d\tau}u_\varepsilon(\tau)\Big|^2\big(1-\gamma\delta'(\tau)\big)^{-2} - V(u_\varepsilon(\tau))\Big]\big(1-\gamma\delta'(\tau)\big)d\tau \\ &= \int_0^T \Big(\frac{1}{2}\Big|\frac{d}{d\tau}u_\varepsilon(\tau)\Big|^2\big(1-\gamma\delta'(\tau)\big)^{-1} - V\big(u_\varepsilon(\tau)\big)\big(1-\gamma\delta'(\tau)\big)\Big)d\tau\end{aligned}$$

Since $\big(1-\gamma\delta'(\tau)\big)^{-1} = 1+\gamma\delta'(\tau)+o(\gamma)$ as γ tends to zero, for γ small enough we can write

$$\begin{aligned}\mathcal{A}_{\Gamma_T}(u_\varepsilon^\delta) &= \int_0^T \Big(\frac{1}{2}\Big|\frac{d}{d\tau}u_\varepsilon(\tau)\Big|^2\big(1+\gamma\delta'(\tau)+o(\gamma)\big) - V\big(u_\varepsilon(\tau)\big)\big(1-\gamma\delta'(\tau)\big)\Big)d\tau \\ &= \int_0^T \Big(\frac{1}{2}\Big|\frac{d}{d\tau}u_\varepsilon(\tau)\Big|^2(1+\gamma\delta'(\tau)) - V\big(u_\varepsilon(\tau)\big)\big(1-\gamma\delta'(\tau)\big)\Big)\,d\tau + o(\gamma)\end{aligned}$$

For the minimality of u_ε

$$\begin{aligned}0 &\le \mathcal{A}_{\Gamma_T}(u_\varepsilon^\delta) - \mathcal{A}_{\Gamma_T}(u_\varepsilon) \\ &= \int_0^T \Big(\frac{1}{2}\Big|\frac{d}{d\tau}u_\varepsilon(\tau)\Big|^2 + V\big(u_\varepsilon(\tau)\big)\Big)\gamma\delta'(\tau)\,d\tau + o(\gamma)\end{aligned}$$

It follows that for every choice of $\delta(\tau) \in \mathcal{C}_c^\infty[0,T]$

$$\int_0^T \left(\frac{1}{2}|\dot{u}_\varepsilon(\tau)|^2 + V(u_\varepsilon(\tau))\right)\delta'(\tau)d\tau \ge 0$$

Since the previous relation holds for function $\delta(\tau)$ and $-\delta(\tau)$ we conclude that

$$\int_0^T \left(\frac{1}{2}|\dot{u}_\varepsilon(\tau)|^2 + V(u_\varepsilon(\tau))\right)\delta'(\tau)d\tau = 0$$

Integrating by parts, recording that the functions $\delta(\tau)$ are of compact support, we obtain

$$\int_0^T \frac{d}{d\tau}\left(\frac{1}{2}|\dot{u}_\varepsilon(\tau)|^2 + V(u_\varepsilon(\tau))\right)\delta(\tau)d\tau = 0$$

and, from the arbitrarity of $\delta(\tau)$, it follows

$$\frac{d}{d\tau}\left(\frac{1}{2}|\dot{u}_\varepsilon(\tau)|^2 + V(u_\varepsilon(\tau))\right) = \frac{d}{d\tau}h_\varepsilon = 0$$

Moreover

$$|h_\varepsilon| \le \frac{1}{2}|\dot{u}_\varepsilon|^2 + \frac{1}{|u_\varepsilon|} + |F(u_\varepsilon)|$$

thus

$$\begin{aligned}\left|\int_0^T h_\varepsilon dt\right| &\le \int_0^T |h_\varepsilon|dt \le \int_0^T \left(\frac{1}{2}|\dot{u}_\varepsilon|^2 + \frac{1}{|u_\varepsilon|} + |F(u_\varepsilon)|\right)dt \\ &\le \mathcal{A}(u_\varepsilon) + 2\int_0^T |F(u_\varepsilon)|dt \le \mathcal{A}(u_\varepsilon) + C\end{aligned}$$

and, since $h_\varepsilon(t)$ is constant, by integration

$$|h_\varepsilon| \leq \frac{\mathcal{A}(u_\varepsilon)+C}{T} \leq \frac{\mathcal{A}(u_{\bar\varepsilon})+C}{T}$$

This shows that the set $\{h_\varepsilon, \varepsilon < \bar\varepsilon\}$ is uniformly bounded.

iv) In order to prove that the instants t_i are of finite number we show that the measure of every interval $\Delta T_i = (t_{i+1} - t_i)$ where $|u_\varepsilon(t)| > \varepsilon$ is positive and bounded from below.
We term $I = \frac{1}{2}|u_\varepsilon|^2$. Since $u_\varepsilon(t)\big|_{\Delta T_i}$ is a classical solution of equation (5.4), deriving $I(t)$ twice we obtain the Lagrange-Lacobi identity:

$$\begin{aligned}\frac{d^2}{dt^2}I &= |\dot u_\varepsilon|^2 + u_\varepsilon \cdot \ddot u_\varepsilon \\ &= |\dot u_\varepsilon|^2 - \frac{1}{|u_\varepsilon|} + f(u_\varepsilon)\cdot u_\varepsilon \\ &= \frac{1}{|u_\varepsilon|} + 2h_\varepsilon + f(u_\varepsilon)\cdot u_\varepsilon - 2F(u_\varepsilon)\end{aligned}$$

Since h_ε is bounded and the term $f(u_\varepsilon)\cdot u_\varepsilon - 2F(u_\varepsilon) = o\left(\frac{1}{|u_\varepsilon|}\right)$ as $|u_\varepsilon| \to 0$ follows that there exists $R > 0$ and two positive constant C_1, C_2 such that if $|u_\varepsilon| \leq R$

$$\frac{d^2}{dt^2}I > 0$$
$$\frac{C_1}{\sqrt{I}} < \frac{d^2}{dt^2}I < \frac{C_2}{\sqrt{I}}$$

The first of the previous relations implies the existence of a time $t_i^* \in \Delta T_i$ where $|u(t_i^*)| = R$, while from the second one it follows

$$\frac{d}{dt}\underbrace{\left(\frac{1}{2}\dot I^2 - C_2\sqrt{I}\right)}_{W(t)} < 0 \quad t \in (t_i, t_i^*)$$

Since $\dot I(t_i) = 0$ and $I(t_i) = \dfrac{1}{2}\varepsilon^2$, we have $W(t_i) = -\dfrac{C_2}{\sqrt{2}}\varepsilon$ and for $t \in [t_i, t_i^*]$

$$W(t) = \frac{1}{2}\dot I^2 - C_2\sqrt{I} < -\frac{C_2}{\sqrt{2}}\varepsilon$$

Thus

$$\begin{aligned}\frac{1}{2}\dot I^2 &< \frac{C_2}{\sqrt{2}}(\sqrt{2}\sqrt{I} - \varepsilon) \\ &< \frac{C_2}{\sqrt{2}}(R - \varepsilon)\end{aligned}$$

and

$$\frac{1}{2}|R^2 - \varepsilon^2| = |I(t_i^*) - I(t_i)| \leq \int_{t_i}^{t_i^*} |\dot I(t)|dt \leq C\sqrt{R-\varepsilon}(t_i^* - t_i)$$

We can conclude that

$$|\Delta T_i| > (t_i^* - t_i) \geq C\frac{(R^2 - \varepsilon^2)}{\sqrt{R-\varepsilon}} > \bar C$$

Thus for every ε the number of instants $\{t_i\}$ is finite and ΔT_ε is the union of a finite number of open intervals, hence its complementary is the union of a finite number of closed intervals.
Precisely $(\Delta T_\varepsilon)^c = \bigcup\limits_{j=0\ldots N} [t_{2j+1}, t_{2j+2}]$ where N depends on ε.

v) For every $t \in [t_{2j+1}, t_{2j+2}]$ it holds $|u_\varepsilon(t)| = \varepsilon$ then the boundness of h_ε and the boundness of $F(u_\varepsilon)$ imply that, for ε small enough, $|\dot{u}_\varepsilon|^2 \sim \frac{1}{\varepsilon} > 0$. This, together with the fact that the velocity $\dot{u}_\varepsilon$ is tangent to the boundary of $\mathcal{B}_0(\varepsilon)$ and it is a continuous function, point i), implies that an inversion of motion could not occur on the constraint.

vi) The vector $\dot{u}_\varepsilon(t_1)$ is tangent to the constraint for i).

From the conservation of energy for every ε we have

$$h\varepsilon = \frac{1}{2}|\dot{u}_\varepsilon(t_1)|^2 - \frac{1}{\varepsilon} + F(u_\varepsilon(t_1))$$

Passing to the limit for $\varepsilon \to 0$, the boundness of F together with the boundness of $\{h\varepsilon\}$ implies

$$\frac{1}{2}|\dot{u}_\varepsilon(t_1)|^2\varepsilon \to 1$$

□

We now investigate the limit property of u_ε as ε tends to zero.
Let $\{\varepsilon_n\}_{n=1}^\infty$ a sequence such that $\lim_{n\to 0} \varepsilon_n = 0$ and denote $u_n = u_{\varepsilon_n}$.
We first show that there exists a uniform converging subsequence of $\{u_n\}_{n=1}^\infty$.

Proposition 5.1. *From every sequence $\{\varepsilon_n\}_{n=1}^\infty$, with $\lim_{n\to\infty} \varepsilon_n = 0$ it is possible to extract a subsequence, still denoted by $\{\varepsilon_n\}_{n=1}^\infty$, such that there exists*

$$u_0 = \lim_{n\to\infty} u_n$$

Proof
Since the functional $\mathcal{A}$ is coercive the sequence $\{u_n\}_{n=1}^\infty$ is bounded in H^1 and equicontinuous then, by Ascoli-Arzelá theorem, it possesses an uniformly converging subsequence, still denoted by $\{u_n\}_{n=1}^\infty$. Let

$$u_0 = \lim_{n\to\infty} u_n$$

□

Moreover Γ_* is weakly closed in H^1 then the functional $\mathcal{A}$ admits a minimizer in Γ_*. Let u_{min} such minimizer and denote

$$c(0) = \min_{u\in\Gamma_*} \mathcal{A}(u) = \mathcal{A}(u_{min})$$

Some properties of u_0 are given in the following lemma.

Lemma 5.3. *The path u_0 has the following properties*

i) the quantity $h_0 = \frac{1}{2}|\dot{u}_o|^2 - \frac{1}{|u_0|} + F(u_0)$ is conserved on the path u_0 and

$$h_0 = \lim_{n\to\infty} h_{\varepsilon_n}$$

ii) $u_0(t)\big|_{[0,T]\setminus\{t_k^0\}}$ is a $\mathcal{C}^2$ path and solve the equation (5.4)

iii) $|u_0(t)| = 0$ at most in a finite number of instants $\{t_k^0\}$

iv) it holds

$$\mathcal{A}(u_0) = c(0)$$

Proof
If the functional $\mathcal{A}$ attains the minimal value $c(0)$ on collisionless paths then there exists $\bar{n}$ such that the sequence $\{u_n\}_{n=1}^\infty$ is constant for $n \geq \bar{n}$ and $u_0 = u_{\bar{n}} = u_{min}$. In this case the proof of the statement is the same done in the previous lemma.

Otherwise suppose that u_{min} is a collision path and, consequently, u_0 is a collision path.

i) The boundness of the action $\mathcal{A}(u_0)$ implies that the path $u_0(t)$ has collisions at most on a subset of $[0,T]$ of zero measure and since u_0 is a continuous function, being the uniform limit of continuous functions, the complement of this subset is given by a union of at most countable sequence of open intervals. Let (t_1,t_2) be one of such intervals, then for every closed interval $[a,b]\subset(t_1,t_2)$, there exists $\bar{\varepsilon}$ such that $|u_n|>\bar{\varepsilon}$ for every $t\in[a,b]$. It follows that

$$\frac{1}{|u_n|}-F(u_n)\to\frac{1}{|u_0|}-F(u_0)$$
$$\frac{u_n}{|u_n|^3}\to\frac{u_0}{|u_0|^3}$$

uniformly in $[a,b]$. Since u_n solve the equation (5.4) the convergence of the sequence $\{u_n\}$ is of class $\mathcal{C}^2$ and therefore $\frac{1}{2}|\dot{u}_n|^2\to\frac{1}{2}|\dot{u}_0|^2$. We conclude that $h_{\varepsilon_n}\to h_0$ and h_0 is conserved on the path u_0 for every $t\in[a,b]$.

ii) Derivating the quantity $h_0=\frac{1}{2}|\dot{u}_0|^2-\frac{1}{|u_0|}$ with respect to t in $[a,b]$, since h_0 is constant, we obtain

$$\ddot{u}_0+\frac{u_0}{|u_0|^3}-f(u_0)=0$$

iii) Since u_0 solve the equation (5.4) for $t\in[a,b]$, it is a critical point for the action $\mathcal{A}$ with respect to variations whose support is compact contained in $[a,b]$. We repeat the argument done in point iii) of lemma 5.2 in order to prove that every interval (t_i,t_{i+1}) where $|u_0(t)|>0$ has a length bounded below by a constant, hence they are in a finite number. If follows that the set of collision times is a finite number of instants $\{t_k\}$.

iv) Let

$$c(0)=\min_{u\in\Gamma_\star}\mathcal{A}(u)$$

and suppose that the value $c(0)$ is achieved by $u_{min}(t)$. We know that u_{min} has a collision at most for a finite number of instant $\{t_k\}_{k=0}^N$ and by asymptotic estimates, see for instance [1], it holds

$$\begin{cases}|u_{min}(t)|\simeq C|t-t_k|^{\frac{2}{3}}\\ |\dot{u}_{min}(t)|\simeq C|t-t_k|^{-\frac{1}{3}}\end{cases}\tag{5.7}$$

as $t\to t_k^{\pm}$.

For $\varepsilon>0$ sufficiently small, let $\gamma_k^+(\varepsilon)$ and $\gamma_k^-(\varepsilon)$ be positive solutions of

$$|u_{min}(t_k+\gamma_k^+(\varepsilon))|=\varepsilon\qquad|u_{min}(t_k-\gamma_k^-(\varepsilon))|=\varepsilon$$
$$[t_k-\gamma_k^-(\varepsilon),t_k+\gamma_k^+(\varepsilon)]\cap\left(\cup_{j=0}^N\{t_j\}\right)=t_k\quad\forall k=0...N$$

Denote $I_k(\varepsilon)$ the interval $[t_k-\gamma_k^-(\varepsilon),t_k+\gamma_k^+(\varepsilon)]$.

For every ε small enough, it holds

$$|u_{min}(t)|\geq\varepsilon\quad\forall t\in[0,T]\setminus\left(\bigcup_{k=0}^N I_k(\varepsilon)\right)\tag{5.8}$$

Moreover, for every k let α_k and β_k be such that

$$u_{min}(t_k-\gamma_k^-(\varepsilon))=\varepsilon e^{i\alpha_k}\quad u_{min}(t_k+\gamma_k^+(\varepsilon))=\varepsilon e^{i\beta_k}$$

and define the path $\bar{u}_\varepsilon(t)$ as

$$\bar{u}_\varepsilon(t)\begin{cases}u_{min}(t) & t\in[0,T]\setminus\left(\bigcup_{k=0}^N I_k(\varepsilon)\right)\\ \\ \varepsilon e^{i\phi_k^\varepsilon(t)} & t\in I_k(\varepsilon)\ ,\ k=1...N\end{cases}\tag{5.9}$$

where

$$\phi_k^\varepsilon(t) = \frac{\beta_k\big(t - t_k + \gamma_k^-(\varepsilon)\big) - \alpha_k\big(t - t_k - \gamma_k^+(\varepsilon)\big)}{\gamma_k^+(\varepsilon) + \gamma_k^-(\varepsilon)}$$

From relation (5.8) it follows

$$\min_{t\in[0,T]} |\bar{u}_\varepsilon(t)| = \varepsilon$$

then $\bar{u}_\varepsilon \in \Gamma_\star^\varepsilon$ and differs from u_{min} only in the union of I_k, while from the asymptotic estimates (5.7) we infer

$$\gamma_k^\pm(\varepsilon) \simeq \varepsilon^{\frac{3}{2}}$$

Denoting $V(u) = \frac{1}{|u|} - F(u)$, the difference $\Delta\mathcal{A} = \mathcal{A}(u_{min}) - \mathcal{A}(\bar{u}_\varepsilon)$ is expressed by

$$\Delta\mathcal{A} = \sum_{k=0}^{N} \int_{I_k(\varepsilon)} \frac{1}{2}\Big(|\dot{u}_{min}|^2 - |\dot{\bar{u}}_\varepsilon|^2\Big) + \Big(V(u_{min}) - V(\bar{u}_\varepsilon)\Big)\, dt$$

From the estimate (5.7) we have

$$\int_{I_k(\varepsilon)} |\dot{u}_{min}|^2 dt = C \int_{I_k(\varepsilon)} |t - t_k|^{-\frac{2}{3}} dt = C|I_k(\varepsilon)|^{\frac{1}{3}} = C\sqrt{\varepsilon}$$

where C indicates a generic constant. Moreover, from the relation (5.9), $\dot{\bar{u}}_\varepsilon(t) = \varepsilon \frac{\beta_k - \alpha_k}{\gamma_k^+(\varepsilon)+\gamma_k^-(\varepsilon)}$ for every $t \in I_k(\varepsilon)$, thus

$$\int_{I_k(\varepsilon)} |\dot{\bar{u}}_\varepsilon|^2 dt = C \int_{I_k(\varepsilon)} \frac{\varepsilon^2}{|\gamma_k^+(\varepsilon) + \gamma_k^-(\varepsilon)|^2} dt = C \int_{I_k(\varepsilon)} \frac{1}{\varepsilon} dt = C\frac{1}{\varepsilon}|I_k(\varepsilon)| = C\sqrt{\varepsilon}$$

On the other hand

$$\int_{I_k(\varepsilon)} V(u_{min}) - V(\bar{u}_\varepsilon) dt \sim \int_{I_k(\varepsilon)} C|t - t_k|^{-\frac{2}{3}} - \frac{1}{\varepsilon} dt \sim |I_k(\varepsilon)|^{\frac{1}{3}} + |I_k(\varepsilon)|\frac{1}{\varepsilon} \sim C\sqrt{\varepsilon}$$

We glean

$$\mathcal{A}(u_{min}) - \mathcal{A}(\bar{u}_\varepsilon) = \mathscr{O}(\sqrt{\varepsilon})$$

and, since $c(\varepsilon) \leq \mathcal{A}(u)$ for every $u \in \Gamma_*^\varepsilon$, it implies

$$\limsup_{\varepsilon\to 0^+} c(\varepsilon) \leq c(0)$$

Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence of positive real number converging to zero such that

$$c(\varepsilon_n) \to \liminf_{\varepsilon\to 0^+} c(\varepsilon)$$

and let $\{u_n\}_{n=1}^\infty$ be a sequence in $\Gamma_\star$ such that

$$\min_{t\in[0,T]} |u_n(t)| = \varepsilon_n \qquad \mathcal{A}(u_n) = c(\varepsilon_n)$$

The sequence $\{u_n\}_{n=1}^\infty$ is bounded in H^1 because the action $\mathcal{A}(u_n)$ is bounded and $\mathcal{A}$ is coercive, then we can extract a subsequence that converges weakly to $\bar{u} \in \Gamma_\star$. Since the weak convergence in H^1 implies the uniform convergence, $\bar{u}$ is a collision path and $\bar{u} \in \Gamma_\star$. From the weak lower semicontinuity of $\mathcal{A}$ we infer

$$\mathcal{A}(\bar{u}) \leq \liminf_{\varepsilon\to 0^+} c(\varepsilon)$$

then

$$c(0) \leq \mathcal{A}(\bar{u}) \leq \liminf_{\varepsilon\to 0^+} c(\varepsilon) \leq \limsup_{\varepsilon\to 0^+} c(\varepsilon) \leq c(0)$$

We conclude that $c(0) = \lim_{\varepsilon\to 0^+} c(\varepsilon)$.

□

Definition 5.1 (Levi-Civita transform)**.** *For every complex-valued, continuous function $u(t)$ we define the set of continuous functions $w(s)$ such that*

$$\begin{aligned} u(t) &= w^2(s(t)) \\ dt &= |w(s)|^2 ds \end{aligned}$$

We denote $u \mapsto \Psi(u)$ such multivalued transformation.

We will use the notation 'Levi-Civita space' or 'parameters space' in order to indicate the space where $w(s)$ lives and 'physical space' the space where $u(t)$ is defined. Moreover we will denote with $(\cdot)'$ the derivative with respect to s.

The Levi Civita transform, first proposed in [17], is a classical method used to regularize the singularity in the Kepler problem and to define the trajectories beyond the collision. Generalisations to higher dimension and to potential with different degree of homogeneity have been proposed in [16] and [19].

We now study the Levi Civita transform from the point of view of calculus of variation, in order to gather from the variational properties of a path in the parameter space some properties of the associated path in the physical space.

Lemma 5.4. *Let $u(t) : [0, T] \to \mathbb{C}$ a local minimizer path for the action functional $\mathcal{A}$. Then every $w \in \Psi(u)$ is a local minimizer of*

$$\tilde{\mathcal{A}}(w) = \int_0^S \tilde{\mathcal{L}}(w, w')ds$$

on the constraint

$$T = \int_0^S |w|^2 ds$$

where

$$\tilde{\mathcal{L}}(w, w') = (2|w'|^2 + 1 - F(w^2)|w|^2)$$

Proof

We write the action integrand $\mathcal{L}dt$ in terms of (w, s).

$$\begin{aligned} \mathcal{L}dt &= \left(\frac{1}{2}|\dot{u}|^2 + \frac{1}{|u|} - F(u)\right) dt \\ &= \left(2|w|^2|\dot{w}|^2 + \frac{1}{|w|^2} - F(w^2)\right) dt \\ &= \left(2|w|^4|\dot{w}|^2 + 1 - F(w^2)|w|^2\right) \frac{dt}{|w|^2} \\ &= (2|w'|^2 + 1 - F(w^2)|w|^2)ds = \tilde{\mathcal{L}}ds \end{aligned}$$

Then in the parameter space the action functional $\mathcal{A}(u)$ is replaced by

$$\tilde{\mathcal{A}}(w) = \int_0^S \tilde{\mathcal{L}}(w, w')ds$$

under the constraint

$$T = \int_0^S |w|^2 ds$$

□

Before going on we want to recall some notations. The action functional we are taking into account is

$$\mathcal{A}(u) = \int_0^T \left\{ \frac{1}{2}|\dot{u}|^2 + \frac{1}{|u|} - F(u) \right\} dt$$

where $F \in \mathcal{C}^2$ is any function symmetric with respect the x-axis. Denote the x-axis with π and the open components disconnected by the origin with π_+,π_-. We used the notation $\mathcal{A}_{\Gamma_*}$ in order to indicate the restriction of the action on the sets of path

$$\Gamma_{P,Q} = \left\{ u(t),\ \dot{u} \in L^2([0,T]) :\ u(0) = P \in \pi_+, u(T) = Q \in \pi_- \right\}$$

$$\Gamma_P = \left\{ u(t),\ \dot{u} \in L^2([0,T]) :\ u(0) = P \in \pi_+, u(T) \in \overline{\pi_-} \right\}$$

$$\Gamma = \left\{ u(t),\ \dot{u} \in L^2([0,T]) :\ u(0) \in \overline{\pi_+}, u(T) \in \overline{\pi_-} \right\}$$

Moreover recall the Euler-Lagrange equation associated to the action functional

$$\ddot{u} = -\frac{u}{|u|^3} + f(u) \tag{5.10}$$

Theorem 22. *Let $u_{min}^{P,Q}$, u_{min}^{P}, u_{min} be a minimizer of the action $\mathcal{A}_{\Gamma_{P,Q}}$, $\mathcal{A}_{\Gamma_P}$, $\mathcal{A}_{\Gamma}$ respectively. Then*

- *$u_{min}^{P,Q}$ is collisionless,*
- *either u_{min}^{P} is collisionless or it lies on the P side of the x-axis with respect to the origin for every $t \in [0,T]$*
- *either u_{min} is collisionless or it lies on the x-axis on one side with respect to the origin for every $t \in [0,T]$*

In other words for every minimizer $u_{min}(t)$ only two alternatives are possible: either $u_{min}(t)$ is collisionless , $|u_{min}(t)| \neq 0$ for every $t \in [0,T]$, or $u_{min}(t)$ lies on one side the x-axis for all time.

Moreover, every non-collision minimal action path is free of self intersections, while a collision path can not have a reflection on the singular point.

Proof

Since the function $F(x)$ is symmetric with respect to the x-axis, on points of the x-axis the force $f(x) = \nabla F(x)$ is directed as the axis itself. Then a solution $\bar{u}(t)$ of equation (5.10) with initial data $\bar{u}(0) = (x_0, 0)$ and $\dot{\bar{u}}(0) = (v_0, 0)$ lies on the axis for all the time of existence.

Moreover from the symmetry of the Lagrangian, without losing any generality, we can restrict the sets $\Gamma_\star$ to those paths $u(t) = (u_1(t), u_2(t))$ with $u_2(t) \geq 0$ for every $t \in [0,T]$. Indeed we can replace $u(t) = (u_1(t), u_2(t))$ with $(u_1(t), |u_2(t)|)$ without changing the action. Thus we replace the previous definition of $\Gamma_\star^\varepsilon$ with the following

$$\Gamma_\star^\varepsilon = \left\{ u \in \Gamma_\star, u_2(t) \geq 0, \min_{t\in[0,T]} |u(t)| \geq \varepsilon \right\}$$

Denote

$$c_\star(\varepsilon) = \min_{u\in\Gamma_\star^\varepsilon} \mathcal{A}_{\Gamma_\star}(u)$$

and let $u_{min}^\star$ be the minimal path, $\mathcal{A}_{\Gamma_\star}(u_{min}^\star) = c_\star(\varepsilon)$.

Moreover define the sets

$$\mathcal{M}_\star(\varepsilon) = \left\{ u \in \Gamma_\star^\varepsilon\ :\ \mathcal{A}_{\Gamma_\star}(u) = c_\star(\varepsilon)\ ,\ \min_{t\in[0,T]} |u(t)| = \varepsilon \right\}$$

Fix T, if there exists ε_0 such that for every $\varepsilon < \varepsilon_0$ the sets $\mathcal{M}_\star(\varepsilon)$ are empty then the minimizers $u^\star_{min}$ are collisionless and the theorem is proved.

Otherwise suppose there exists a sequence $\{\varepsilon_n\}_{n=1}^\infty$ converging to zero and a sequence of paths $\{u_n(t)\}_{n=1}^\infty \in \Gamma_\star^{\varepsilon_n}$ such that $\mathcal{A}_{\Gamma_\star}(u_n) = c_\star(\varepsilon_n)$ and

$$\min_{t\in[0,T]} u_n(t) = \varepsilon_n$$

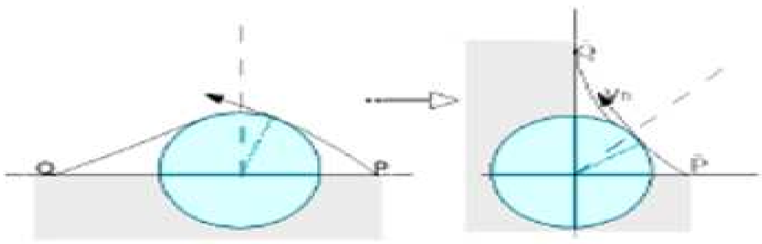


Figure 5.1: Levi-Civita transform

We now apply the Levi-Civita transform: consider $u_n(t) = (u_1(t), u_2(t))$ in complex form $u_n(t) = u_1(t) + iu_2(t))$ and set

$$w_n \in \Psi(u_n)$$
$$w_n : [0, S] \to \mathbb{C}$$

Remark 5.1. *As observed the function Ψ is multivalued. However if a path $u(t)$ is collisionless, then $\Psi(u)$ consists in two elements $w(s)$ and $-w(s)$. In our case since every $u_n(t)$ lives in the y-positive halfspace, the images $w_n^1(s)$ and $w_n^2(s) = -w_n^1(s)$ live one in the first and one in the third quadrant of the complex plane. We choose one of them and we will consider paths $w_n(s)$ in the first quadrant, joining the point $\tilde{P}$ and $\tilde{Q}$ one on the real positive semiaxis and the other on the imaginary positive semiaxis. (see figure 5.1)*

The constraint $\mathcal{B}_0(\varepsilon_n)$ corresponds by the transformation to $\tilde{\mathcal{B}}_n = \mathcal{B}_0(\sqrt{\varepsilon_n})$ and, for a fixed value of n, keeping in mind the dependence on n, we define

$$s_i = s(t_i)$$

$$\Delta S_n = \bigcup_j (t_{2j}, t_{2j+1})$$

where $\{t_i\}$ is the set of defined in (5.5). For lemma (5.2) the set of instants $\{t_i\}$ is finite in number for every n, then ΔS_n is the union of a finite number of open intervals and $|w_n(s)| > \sqrt{\varepsilon_n}$ for every $s \in \Delta S_n$.

In the following proposition we collect some properties of $w_n(s)$ needed to conclude the proof of the theorem .

Proposition 5.2. *For every n the path $w_n(s)$ has the following properties:*

i) *$w_n(s)$ is a $\mathcal{C}^1$ path in $[0,S]$*

ii) *$w_n(s)\Big|_{\Delta S_n}$ is $\mathcal{C}^2$ and there exists a constant λ_n such that it is a solution of equation*

$$4w'' + \frac{d}{dw}\left(F(w^2)|w|^2\right) - 2\lambda_n w = 0 \tag{5.11}$$

Precisely, for every n, it holds

$$\lambda_n = h_n$$

where $h_n = \frac{1}{2}|\dot{u}_n|^2 - \frac{1}{|u_n|} + F(u_n)$ is the energy constant associated to the path u_n.

iii) *the vectors $w_n'(s_j)$ are tangent to the boundary of $\tilde{\mathcal{B}}_n$ and $2|w_n'(s_j)|^2 \to 1$ as $n \to \infty$.*

Pass to the limit in Levi-Civita space.

Proof

i) Obvious

ii) For lemma (5.2) point ii) and lemma (5.4), the path $w_n(s)$ is a stationary point of the functional

$$\tilde{\mathcal{A}} = \int_0^S 2|w'|^2 + 1 - F(w^2)|w|^2 ds$$

with the condition $T = \displaystyle\int_0^S |w|^2 ds$ with respect to variations of w_n whose support is contained in $[0, s_n^-]$. Then $w_n(s)$ is solution of the Euler-Lagrange equation (5.11) where λ is the Lagrange multiplier.

For lemma (5.2) the quantity h_n is constant for every $t \in [0,T]$ and, performing the Levi Civita transform, we obtain

$$|w_n(s)|^2 h_n = 2|w'(s)|^2 - 1 + F(w(s)^2)|w(s)|^2$$

Differentiating both the terms with respect to s, since

$$\frac{dh_n}{ds} = |w(s)|^2 \frac{dh_n}{dt} = 0$$

it holds

$$2h_n w_n w_n' = 4w_n'' w_n' + \frac{d}{dw}\left(F(w_n^2)|w_n|^2\right) w_n'$$

for every $s \in [0,S]$, then $2h_n w_n = 4w_n'' + \frac{d}{dw}\left(F(w_n^2)|w_n|^2\right)$. Substituting in (5.11) it follows $\lambda_n = h_n$.

iii) Since the map Ψ is conformal and for lemma (5.2)point iv) the vector $\dot{u}_n(t_1)$ is tangent to the constraint, then the vector v_n is tangent to $\tilde{\mathcal{B}}_n$. Moreover $\dot{u}_n = 2w\dot{w} = 2w\dfrac{w'}{|w|^2}$ then $|\dot{u}_n(t_1)|^2 = 4\dfrac{|v_n|^2}{|w(s_n^-)|^2} = 4\dfrac{|v_n|^2}{\varepsilon}$ and again for lemma 5.2point iv) follows $\lim\limits_{n\to\infty} 2|v_n|^2 = 1$

□

The most important consequence that comes from the Levi -Civita transform is that the differential equation (5.11) is not singular anymore in the origin, then whenever a solution $w(s)$ reaches the origin, it passes through in the same direction.

Since the function $F(w)$ is of class $\mathcal{C}^2$ the initial value problem, associated to equation (5.11),

$$\begin{cases} 4w'' = -\dfrac{d}{dw}\left(F(w^2)|w|^2\right) + 2\lambda w \\ \\ (w(0), w'(0)) = (w_0, w_0') \end{cases} \tag{5.12}$$

admits an unique solution for every initial data (w_0, w_0') and the solutions are continuous with respect to the initial data (see [21]). In particular setting as initial position w_0 a point on an axis of coordinates and as initial velocity w_0' a vector parallel to the same axis, the symmetry of the function F implies that the solution $\bar{w}(s)$ of the system belongs to the axis for all the time.

Moreover the solutions are invariant with respect to time reversal: if $w(s)$ is a solution of the initial value problem (5.12) in the interval $[0, S]$ and $W(s)$ is the solution with initial data $(W(0), W'(0)) = (w(S), -w'(S))$, then, since the differential equation is autonomous, the path $w(s)$ and $W(s)$ have the same support and $w(s) = W(S - s)$ for any $s \in [0, S]$.

We can now conclude the proof of the theorem.

From proposition (5.1) we can extract from $\{u_n\}_{n=1}^{\infty}$ an uniform converging subsequence. Denote u_0 the uniform limit, and let w_0 be the limit of the corresponding sequence $w_n \in \Psi(u_n)$ where, for every n, the choice of w_n is done according with remark 5.1. Since the map Ψ is conformal, the path w_0 is the uniform limit of the sequence $\{w_n\}$ then $w_0(s)$ lies in the first quadrant for every $s \in [0, S]$.

Moreover $w_0(s) \in \Psi(u_0)$ then, for lemma (5.3), it has a collision at most in a finite number of instants $\{\bar{s}_k\}$. Consider the first collision time $\bar{s}_1$ and denote $v = w_0'(\bar{s}_1)$.

Since $v = \lim_{n\to\infty} w_n'(s_1)$, for point iii) of proposition (5.2), $|v|^2 = \frac{1}{2}$. For every n the vectors $w_n'(s_1)$ are tangent to the boundary of $\tilde{\mathcal{B}}_n$ in points of the first quadrant, thus passing to the limit, v is a vector directed either as one of the coordinate axis or in the second or in the fourth quadrant.

The last alternative is impossible to achieve. Indeed suppose, for instance, that v is a vector directed toward the second quadrant and let $\tilde{w}(s)$ be the solution of system (5.12) with initial data $(\tilde{w}(0), \tilde{w}'(0)) = (0, -v)$. Then there exists a time $\tilde{s}$ such that the path $\tilde{w}(s)$ belongs to the fourth quadrant for $s \in [0, \tilde{s}]$. For the uniqueness of solution and for the invariant with respect to time reversal of the system it follows that $w_0(\bar{s}_1 - s) = \tilde{w}(s)$ then also $w_0(s)$ lies in the fourth quadrant for s in a left neighbourhood of $\bar{s}_1$. This contradicts the fact that $w_0(s)$ lives in the first quadrant for all the time.

Thus only the first alternative is possible: the vector v is directed as one of the coordinate axis. In this case, for the uniqueness of the solution of system (5.12), the limit path w_0 belongs to the same axis for all time.

We can conclude that if the end-points $\tilde{P}$ and $\tilde{Q}$ are fixed, different from the origin, a collision path $w(s)$, solution of equation (5.11), joining $\tilde{P}$ and $\tilde{Q}$ does not exists, thus the minimal path $u_{min}^{P,Q}$ has to be collisionless.

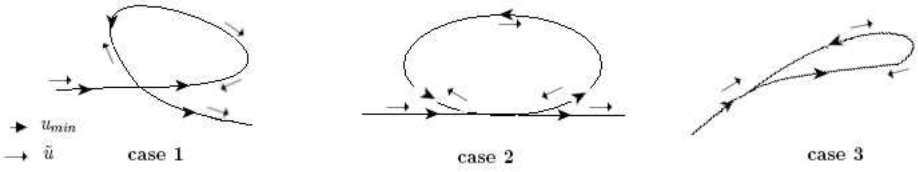
Otherwise, if we allow $\tilde{P}$ or $\tilde{Q}$ or both to vary respectively in the closure of the real and imaginary positive semiaxis, every minimal path joining $\tilde{P}$ and $\tilde{Q}$ is either collisionless or lies on one of the semiaxis and one of the end-points is the origin.

In order to prove that non-collision minimal action paths can not have self intersections suppose, for the sake of absurdity, that there exist $t_0 \neq t_1 \in (0, T)$ such that $u_{min}(t_0) = u_{min}(t_1)$. The conservation of energy yields $|\dot{u}_{min}(t_0)| = |\dot{u}_{min}(t_1)| \neq 0$ and one of the following alternatives, depicted in figure 5.2(a), may occurs: $\dot{u}_{min}(t_0)$ is transversal to $\dot{u}_{min}(t_1)$, case 1, $\dot{u}_{min}(t_0)$ is parallel to $\dot{u}_{min}(t_1)$ with same or opposite direction, case 2 and 3.

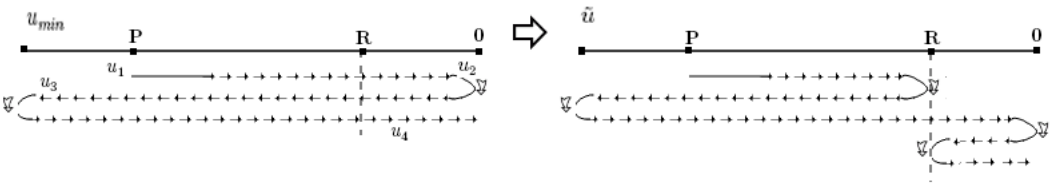
In any case the path $\tilde{u}(t)$ defined as

$$\tilde{u}(t) = \begin{cases} u_{min}(t) & t \in [0, t_0] \cup [t_1, T] \\ u_{min}(t_0 + t_1 - t) & t \in (t_0, t_1) \end{cases}$$

has the same action of $u_{min}(t)$ and therefore, being a minimal action path, it should be a solution of equation (5.10). In case 1 and 2 this is impossible to achieve since $\tilde{u}(t)$ is not a $\mathcal{C}^1$ path, while the



(a)



(b)

Figure 5.2:

theorem of local existence and uniqueness of solution of a system of regular differential equation [21] rule out the third alternative.

An other case we have to exclude is that u_{min} does not draw a loop but retraces the same path or a piece of it. This is the case of a collision path that, after a reflection in the singular point at time $t_0 < T$, comes back on the path until the velocity vanishes and then returns to the origin in time T. Also for non collision paths, it may happen that u_{min} reaches the point Q in time $t_0 < T$ with zero velocity and, like in the previous case, comes back on the path for a piece and finally reverts the motion towards Q. Consider, for instance, the collision case and, as drawn in figure 5.2(b), let $\tilde{u}(t)$ be the path obtained by a rearrangement of the path u_{min}. More precisely let R be a point on the x-axis between P and the origin and let $t_1 < t_2 < t_3$ be the instants such that $u_{min}(t) = R$.

Define

$$u_1 = u_{min}\big|_{[0,t_1)} \qquad u_2 = u_{min}\big|_{[t_1,t_2)} \qquad u_3 = u_{min}\big|_{[t_2,t_3)} \qquad u_4 = u_{min}\big|_{[t_3,T]}$$

and

$$\tilde{u}(t) = \begin{cases} u_1 & t \in [0, t_1) \\ u_3 & t \in [t_1, t_1 - t_2 + t_3) \\ u_2 & t \in [t_1 - t_2 + t_3, t_3) \\ u_4 & t \in [t_3, T] \end{cases}$$

By construction the path $\tilde{u}$ is minimal for the action then it has to solve equation (5.10) in every point different than the origin. This contradicts the fact that $\tilde{u}$ is not $\mathcal{C}^1$ out of the origin, for instance in t_1.

□

Consider now the system of one perturbed newtonian center

$$-\ddot{u} = \frac{u}{|u|^3} + f(u)$$

where f is any regular function. Let P, Q two points of the plane different from the origin and consider $\Gamma_{P,Q}$ the set of paths joining P and Q in the time interval $[0,T]$. Denoting with $\mathcal{A}$ the action functional associated to the equation of motion, our aim is to study the variational property of the minimal paths $\mathcal{A}$ in the set $\Gamma_{P,Q}$. To this purpose an asymptotic study when a solution approaches the singularity is necessary.

5.3 Asymptotic estimates at collisions

We now consider the differential equation

$$\ddot{u} = \nabla U(u) \tag{5.13}$$

and the associated action function

$$\mathcal{A}(u) = \int \left\{ \frac{1}{2}|\dot{u}|^2 + U(u) \right\} dt \tag{5.14}$$

where the potential function $U(x)$ satisfies the condition

$$U(x) = \frac{1}{|x-S|} - F(x) \tag{5.15}$$

for a function $F(x)$ of class $\mathcal{C}^2$ at least on an open neighbourhood of the point S. A function that satisfies condition (5.15) can be regarded as a regular perturbation, at least near the singularity, of the Kepler potential. We point out that, since the perturbating function has not to satisfy any kind of symmetry, a solution of the dynamical system (5.13) may not preserve the angular momentum.

The aim of this section is to prove that any minimal path for the action functional associated to the equation (5.13) can not have an interior collision. For this purpose we need to study the asymptotic behaviour of solutions as they approach a collision.

Up to a space-time translation we may suppose that the collision instant is $t = 0$ and the singular point is $S = (0,0)$. Moreover, without loss of generality, we perform the analysis in the left neighbourhood of the collision instant.

Theorem 23. *Let $u(t)$ be a solution for the dynamical system* (5.13) *in the time interval $[a, 0)$ with a collision for $t = 0$, $\lim_{t\to 0^-} u(t) = 0$. Then there exists a vector v_0 such that, as $t \to 0^-$*

$$\begin{aligned} u(t) &\sim t^{\frac{2}{3}} v_0 \\ t^{\frac{1}{3}} \dot{u}(t) &\sim v_0 \end{aligned}$$

Proof
We apply the Levi-Civita transform Ψ to the path $u(t)$ and define $w(s) \in \Psi(u)$. For proposition 5.2, $w(s)$ is a solution of a non singular second order differential equation then $w(s)$ is a $\mathcal{C}^2$ path up to $s = 0$. Since $w(0) = 0$ we can write $w(s) = s\varphi(s)$ where $\varphi(s)$ is a $\mathcal{C}^1$ complex valued function and $\varphi(0) \neq 0$. By means of a Taylor expansion $w(s) = s(\varphi(0) + s\tilde{\varphi}(s))$ for a suitable function $\tilde{\varphi}(s) \in \mathcal{C}^0$.

In order to obtain the asymptotic behaviour of the path $u(t)$ we have to invert the transformation Ψ. From the relation $dt = |w(s)|^2 ds$, by integration, it follows $t \sim s^3$ then $s(t) \sim t^{\frac{1}{3}}$, as $t \to 0^-$. Since, in complex notation, $u(t) = w^2(s(t))$, keeping in mind that $s = s(t)$, we obtain

$$\begin{aligned} u(t) &= s^2(\varphi(0) + s\tilde{\varphi}(s))^2 = s^2\varphi(0)^2 + 2s^3\varphi(0)\tilde{\varphi}(s) + s^4\tilde{\varphi}(s)^2 \\ &= t^{\frac{2}{3}}\varphi(0)^2 + 2t\varphi(0)\tilde{\varphi}(t^{\frac{1}{3}}) + t^{\frac{4}{3}}\tilde{\varphi}(t^{\frac{1}{3}})^2 \end{aligned}$$

then as t tends to 0^-

$$u(t) \sim t^{\frac{2}{3}} v_0$$

where the vector $v_0 = \big(\Re e(\varphi(0)^2), \Im m(\varphi(0)^2)\big)$.

Similarly we obtain an estimate for the time derivative of $x(t)$. Since $w(s)$ is $\mathcal{C}^2$ till $s = 0$ and noting that $\varphi(0) = w'(0)$, we have $w'(s) = \varphi(0) + s\tilde{v}(s)$ for a opportune continuous function $v(s)$. Since

$$\begin{aligned}\dot{u}(t) &= 2w(s(t))w'(s(t))\frac{ds}{dt} = 2s\varphi(s)\big(\varphi(0) + s\tilde{v}(s)\big)\frac{1}{|w(s)|^2}\\ &= 2\frac{\varphi(s)}{s|\varphi(s)^2|}(\varphi(0) + s\tilde{v}(s))\end{aligned}$$

substituting $s = s(t)$ and passing to the limit for $t \to 0^-$, it follows

$$t^{\frac{1}{3}}\dot{u}(t) \sim 2\frac{\varphi(0)^2}{|\varphi(0)^2|} \sim v_0$$

□

Roughly speaking, theorem 23 states that, whenever a solution $u(t)$ of the equation (5.13) has a collision, the path $u(t)$ approaches the singularity, asymptotically, along a direction. We remind that in case of non perturbed Kepler potential or if the perturbation is radially symmetric, a collision solution not only has to be asymptotic to a direction, but it has to lie on a straight line for all the time due to the conservation of angular momentum.

Now we deal in the non minimality of collision paths for the action functional associated to equation (5.13). In theorem 22 we proved that, for a special class of perturbations, collision paths could be minimal for the action but the collision have to be one of the end-points of the trajectory. If we avoid this possibility and we enjoin the paths to have the extremal points different from the singular point, then the minimal paths are collisionless.

Theorem 24. *Suppose $U(x)$ satisfies assumption* (5.15). *Then, for every choice of points P, Q, different from the collision point, any (local) minimal path for the action functional associated to the system* (5.13) *restricted on the set of paths joining P and Q, is free of collision.*

Proof

As usual, let S be the origin. The action functional we are taking into account is

$$\mathcal{A}(u) = \int \frac{1}{2}|\dot{u}|^2 dt + \int \left\{\frac{1}{|u|} - F(u)\right\} dt = K(u) + W(u)$$

defined on the set of paths $\Gamma_{P,Q}$ joining the points P and Q.

Suppose that $u(t)$ is a minimal path for $\mathcal{A}$ with a collision at time $t = t_0$; in order to prove the theorem we exhibit a local variation v^δ that lowers the value of the action.

Up to time translation we may assume that the collision instants is $t_0 = 0$ and by the asymptotic estimates proved in the previous theorem, there exist two vectors v_0^- and v_0^+, $|v_0^\pm| = 1$, such that in a neighbourhood of $t = 0$

$$u(t) = \begin{cases} |t|^{\frac{2}{3}}v^-(t) & t \in [-T, 0]\\ t^{\frac{2}{3}}v^+(t) & t \in [0, T]\end{cases}$$

where the function $v^\pm(t)$ are of class $\mathcal{C}^1$ and $\lim_{t\to 0^\pm} v^\pm(t) = v_0^\pm$.

Moreover

$$\dot{u}(t) = \begin{cases} -|t|^{-\frac{1}{3}}v_0^- + v_1(t) & t \in [-T, 0]\\ t^{-\frac{1}{3}}v_0^+ + v_2(t) & t \in [0, T]\end{cases}$$

where $v_1(t)$ and $v_2(t)$ are continuous function.

Let δ be a vector with the property

$$v_0^+ \cdot \delta \geq 0 \qquad v_0^- \cdot \delta \geq 0$$

We note that given two vectors it is always possible to choose a third vector with the property above; moreover the scalar products both vanish if and only if the vector v_0^+ and v_0^- are collinear.

Consider $\tilde{x}(t) = x(t) + v_\delta(t)$ where the variation is defined as

$$v_\delta(t) = \eta(t)\delta = \begin{cases} \frac{T+t}{T}\delta & t \in [-T, 0] \\ \\ \frac{T-t}{T}\delta & t \in [0, T] \end{cases}$$

We have to show that $\Delta\mathcal{A} = \mathcal{A}(u) - \mathcal{A}(\tilde{u}) > 0$ for every $|\delta|$ small enough.

We evaluate separately the estimates for the variation of the kinetic and potential parts.

$$\begin{aligned} \Delta K = K(u) - K(\tilde{u}) &= \int_{-T}^{T} \left\{ \frac{1}{2}|\dot{x}|^2 - \frac{1}{2}|\dot{x} + \dot{\eta}\delta|^2 \right\} dt \\ &= -|\delta|^2 \int_{-T}^{T} \frac{1}{2}\dot{\eta}(t)^2 dt - \int_{-T}^{T} \dot{\eta}(t)\dot{x}(t) \cdot \delta \, dt \end{aligned}$$

For $t \in [0, T]$, $\dot{\eta}(t)\dot{x}(t) \cdot \delta = -\frac{1}{T}(t^{-\frac{1}{3}} v_0^+ \cdot \delta + v_2(t) \cdot \delta) < \frac{1}{T}|v_2(t)||\delta|$, then it follows

$$-\int_0^T \dot{\eta}(t)\dot{x}(t) \cdot \delta dt > -\frac{|\delta|}{T}\int_0^T |v_2(t)| dt = -\mathcal{O}(|\delta|)$$

Reasoning in the same way in the interval $[-T, 0]$ we conclude that

$$\Delta K > -\mathcal{O}(|\delta|^2) - \mathcal{O}(|\delta|) = -\mathcal{O}(|\delta|)$$

We now perform an estimate for the variation of the potential part ΔW. For simplicity of notation we compute ΔW only for $t \in [0, T]$, indeed for negative values of t the calculation is similar.

$$\Delta W = \int_0^T \left\{ \frac{1}{|x|} - \frac{1}{|x + \eta\delta|} \right\} dt - \int_0^T \{F(x) - F(x + \eta\delta)\} dt = I + II$$

The second integral can be easily estimated by $\int_0^T \nabla F \cdot \eta\delta \; dt = \mathcal{O}(|\delta|)$ while we rewrite the first integral in the form

$$I = \int_0^T \left\{ \frac{1}{t^{\frac{2}{3}}|v^+(t)|} - \frac{1}{\sqrt{t^{\frac{4}{3}}|v^+|^2 + 2\eta t^{\frac{2}{3}} v^+ \cdot \delta + \eta^2|\delta|^2}} \right\} dt$$

With the variable changing $s = \frac{t}{|\delta|^{\frac{3}{2}}}$ we obtain

$$I = |\delta|^{\frac{1}{2}} \int_0^{\frac{T}{|\delta|^{\frac{3}{2}}}} \left\{ \frac{1}{s^{\frac{2}{3}}|v_\delta|} - \frac{1}{\sqrt{s^{\frac{4}{3}}|v_\delta|^2 + 2\eta s^{\frac{2}{3}} v_\delta \cdot \delta + \eta^2}} \right\} ds$$

where $v_\delta(s) = v^+(s|\delta|^{\frac{3}{2}})$. Since $v_\delta(s) \to v_0^+$ as $|\delta|$ goes to zero and $|v_0^+| = 1$, the second term of the integrand is equal to $(s^{\frac{4}{3}} + 1 + o(1))^{-\frac{1}{2}}$ for $|\delta|$ small. Moreover the integrand is strictly positive and, for every s, it is bounded by the function

$$\psi(s) = 2\left(\frac{1}{s^{\frac{2}{3}}} - \frac{1}{\sqrt{s^{\frac{4}{3}} + 1}} \right)$$

provided that $|\delta|$ and T small enough are chosen. The function $\psi(s)$ is integrable on $(0, \infty)$, indeed it is obviously integrable in the origin and $\psi(s) \sim s^{-2}$ as s goes to infinity, then we can apply the Lebesgue theorem and conclude that I is positive and $I = \mathcal{O}(|\delta|^{\frac{1}{2}})$. The variation of the potential part is then estimated by $\Delta W = I + II = \mathcal{O}(|\delta|^{\frac{1}{2}})$ and it is positive.

Thereby for every $|\delta|$ sufficiently small $\Delta\mathcal{A} = \Delta K + \Delta W > -\mathcal{O}(|\delta|) + \mathcal{O}(|\delta|^{\frac{1}{2}}) = \mathcal{O}(|\delta|^{\frac{1}{2}})$ and positive. This proves that the collision path $u(t)$ is not a minimizer of the action. $\square$

5.3.1 Minimization on connected components

It's important to remark that in the previous theorem we proved that a collision path can not be a minimazer for the action among the set of all the paths joining P to Q without any other topological constraint. Furthermore, reasoning as in the proof of theorem 22, a minimal path can not have self intersections.

We want now to investigate deeper on this argument performing the minimizing procedure on open connected components of paths with a certain topological characterisation.

Fix S, the collision point, and let P and Q be two points different from S; denote with $\Gamma_{P,Q}$ the set of paths joining P to Q,

$$\Gamma_{P,Q} = \{u(t) \in H^1([0,T], \mathbb{R}^2),\, u(0) = P,\, u(T) = Q\} \tag{5.16}$$

and with $\tilde{\Gamma}_{P,Q}$ the set of collision-free elements of $\Gamma_{P,Q}$.

Suppose for a moment that P,Q,S are not collinear and let Σ be the segment joining P and Q then Σ does not intersect S. For every path $u(t) \in \tilde{\Gamma}_{P,Q}$ let $\bar{u}$ be the closed path obtained from $u(t)$ attaching to it the path from Q to P on Σ and define $I_S(u)$ the winding number of $\bar{u}$ around S. Of course, if $P = Q$ the path $\bar{u}$ is equal to u. On the other hand, if P, Q, S belong to the same line l, we choose Σ any non collision path connecting Q to P that lies in one of the halfplane delimited by l.

In any case, the function $I_S : \tilde{\Gamma}_{P,Q} \to \mathbb{Z}$ is well defined and

Definition 5.2. *Given a point S and two points P, Q different from S, for every integer n, we define*

$$\tilde{\Gamma}^n_{P,Q} = I_S^{-1}(n)$$

the open components of paths in $\tilde{\Gamma}_{P,Q}$ that wind n-times around the collision point S.

It holds that $\tilde{\Gamma}_{P,Q} = \bigcup_{n\in\mathbb{Z}} \tilde{\Gamma}^n_{P,Q}$, $\tilde{\Gamma}^n_{P,Q} \cap \tilde{\Gamma}^m_{P,Q} = \emptyset$ for every $n \neq m \in \mathbb{Z}$ and, moreover, denoting $\Gamma^n_{P,Q}$ the union of $\tilde{\Gamma}^n_{P,Q}$ with the collision elements of $\Gamma_{P,Q}$, we have that $\Gamma^n_{P,Q}$ is weakly closed with respect to the H^1 norm.

Lemma 5.5. *Suppose $U(x)$ satisfies condition (5.15) and let P,Q two points different from the singularity S. Let $\{u_\varepsilon(t)\}$ be a family of paths in $\tilde{\Gamma}_{P,Q}$ such that for every $\varepsilon > 0$*

a) u_ε belongs to the same connected components $\tilde{\Gamma}^n_{P,Q}$

b) $u_\varepsilon(t)$ is a minimizer for the action (5.14) among the set $\{u \in \tilde{\Gamma}^n_{P,Q},\ \min|u - S| \geq \varepsilon\}$

c) there exists R and ε_0 such that for every $\varepsilon < \varepsilon_0$ the path u_ε has not self intersections in $B_S(R)$

Then one of the following occurs:

- *there exists a non collision path u_0 minimizer for the action restricted on $\Gamma^n_{P,Q}$, uniform limit, up to a subsequence, of the family u_ε*
- *the family u_ε uniformy converge to a collision-reflection path u_0 passing through P and Q, solution of the dynamical system (5.13) out of the singular point.*

Remark 5.2. *With the term 'collision-reflection path', used in the lemma, we refer to a path that, starting from P, solves the equation (5.13) till the singularity and, after a reflection, retraces the same path. This collision-reflection phenomena could be repeated many times before the path ends in Q.*

Proof
Since the set $\Gamma^n_{P,Q}$ is weakly closed in H^1, the lower semicontinuity and the coercivity of the action functional $\mathcal{A}$ implies the existence of a minimizer in $\Gamma^n_{P,Q}$. If the minimum is achieved in $\tilde{\Gamma}^n_{P,Q}$ then the first case holds and the theorem is proved. Otherwise, suppose that the action attains its minimum on the set of collision paths. As usual, without loss of generality, suppose $S = (0, 0)$. We apply the Levi Civita transform to the paths $u_\varepsilon(t)$ and denote $w_\varepsilon(s) = \Psi(u_\varepsilon)$, where, for every ε, the same choice of $w_\varepsilon \in \Psi(u_\varepsilon)$ is done, according to remark 5.1. Moreover denote $\tilde{B}_\varepsilon = B_0(\sqrt{\varepsilon})$ and $\tilde{B}_R = B_0(\sqrt{R})$ the $\sqrt{\varepsilon}$-ball and $\sqrt{R}$-ball around the origin in the Levi Civita space.

For every ε denote with I_ε and O_ε the points on $\tilde{B}_\varepsilon$ where the path w_ε reaches and leaves the constraint and with α_ε the angle between I_ε and O_ε. Condition c) implies that for every $\varepsilon < \varepsilon_0$ the paths w_ε restricted on the ball $\tilde{B}_R$ lies in one halfspace.

We claim the $\lim_{\varepsilon\to 0} \alpha_\varepsilon = 0$. For the sake of absurdity, suppose $\alpha_\varepsilon \not\to 0$. For proposition 5.2 the paths w_ε are of class $\mathcal{C}^1$ and the modulus of the velocities on I_ε and O_ε tend to positive values. It follows that for ε small enough the path w_ε are not more contained in one halfspace and we have a contradiction.

Thus I_ε tends to O_ε and $w'(I_\varepsilon)$ tends to $w'(O_\varepsilon)$ then, for the uniqueness of solution of equation (5.11), $w_0 = \lim_{\varepsilon\to 0} w_\varepsilon$ is a global solution of equation (5.11). The symmetry of (5.11) implies that the solution w_0 is odd in time with respect to the singularities; that is, setting $w_0(s_0) = 0$, we have $w_0(s_0 - s) = -w_0(s_0 + s)$. Applying the inverse of transformation Ψ, we conclude that the limit path u_0 is of collision reflection type. □

The previous lemma has important implications in what we are going to show about the existence of eight shaped solutions for the N centres problem. It may be considered as a technique to prove, under some assumptions, that a non collision minimizer has to exist. Suppose for instance we are performing a minimization procedure on a connected component of paths connecting P to Q and that, for any reason, a collision-reflection path starting from P and ending in Q can not exist in the closure of the component, then, if the hypothesis of the lemma are satisfied, the minimizer path has to be free of collisions.

In the following lemma we illustrate a situation in which the hypotesis of theorem are satisfied.

Lemma 5.6. *Let P and Q be different from the singular point S and P, Q, S non collinear. Denote with l the line passing through P and Q and with $\mathscr{H}^S$ the halfplane delimited by l where S is placed. Consider the open component of paths joining P to Q that wind 0 or 1 time around S and that lies in $\mathscr{H}^S$*

$$\Lambda_i = \tilde{\Gamma}^i_{P,Q} \cap \{u, u(t) \in \mathscr{H}^S, \forall t\} \quad i = 0, 1$$

Then for every u_ε that satisfies hypothesis a) and b) of lemma (5.5) on the set Λ_i, also hypothesis c) holds.

Proof
Consider the case $i = 1$, the other one is similar.

Let u_ε be a path in Λ_1, minimizer for the action on the set $\{u \in \Lambda_1,\ \min|u - S| \geq \varepsilon\}$ and, for the sake of absurdity, suppose that the path u_ε reaches the boundary of $B_S(\varepsilon)$ at instant t_A and, before leave it at instant t_B, covers an angle Θ bigger then 2π. Suppose $2\pi < \Theta < 4\pi$, and denote $A = u_\varepsilon(t_A)$, $B = u_\varepsilon(t_B)$. Moreover denote the restrictions $\gamma_1 = u|_{[0,t_*]}$, $\gamma_2 = u|_{[t_*,t_B]}$, $\gamma_3 = u|_{[t_B,T]}$ in such a way that $u_\varepsilon = \gamma_1 \sqcup \gamma_2 \sqcup \gamma_3$, where t_* is the first instant such that $u_\varepsilon(t_*) = B$.

We note that if a curve in Λ_1, by a self intersection, has a loop that encloses the point S, then the curve must have at least another self intersection.

A complete circle drawn by the path u_ε on the boundary of $B_S(\varepsilon)$ may be considered as a loop enclosing the point S, then the path has to self intersect at least one more time. It comes out that γ_3 has to intersect γ_1 out of the constraint or it has to return on the constraint and turn around in the opposite direction. We consider the first case(the second one follows with similar argumentation): suppose that γ_3 intersects at least one time γ_1 out of the constraint and denote C the first intersection point. The minimality of u_ε implies that the every intersection of γ_3 with

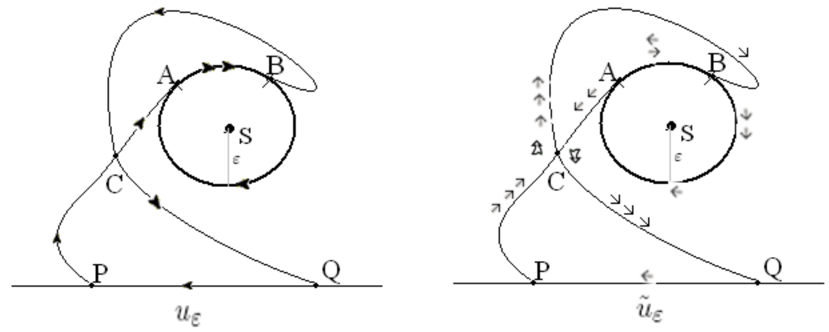


Figure 5.3:

γ_1 out of the constraint must be transversal, for reasons of uniqueness of solution of a regular differential equation.

As shown in figure consider the path $\tilde{u}_\varepsilon = \sqcup \tilde{\gamma}_i$ where $\tilde{\gamma}_1$ is the restriction of γ_1 connecting P to C, $\tilde{\gamma}_2$ connects C to B travelling on γ_3, $\tilde{\gamma}_3$ is the equal to γ_2, $\tilde{\gamma}_4$ connects B to C along γ_1 and finally $\tilde{\gamma}_4$ is the restriction of γ_3 between C and Q. It follows that $\tilde{u}_\varepsilon$ belongs to the same open component Λ_1 and has the same action of u_ε. Then $\tilde{u}_\varepsilon$ is a minimizer of the action functional then it must be of class $\mathcal{C}^1$. This leads to a contradiction since $\tilde{u}_\varepsilon$ is not differentiable in the point C. $\square$

5.4 8-shaped solutions for the N-centres problem

The N-centres problem concerns the motion of a particle in a force field generated by N fixed newtonian centres of attraction. We restrict our analysis to the planar case and we are interested in finding periodic solution in which the particle winds around at least two centres drawing an orbits that looks like an eight, in case of two centres, or something like a flower if the number of centres is larger. We will use the notation '8-shaped solutions' in order to indicate all this kind of orbits.

Since the existence and the shape of these periodic orbits strongly depends on many variables, for instance the number of centres, their mutual position in the plane or the strength of each of them, we can't give a general statement but we have to study, step by step, every case separately.

Among all the possible positions that N-centres could place in a plane, we are mostly interested on configurations in which the centres lie on the same line or they have the same attractive power and are placed on the vertices of a N-polygon with certain suitable symmetry. All these problems have the property that in the neighbourhood of each centre they can be regarded as a perturbed one-centre problem. The equation of motion has the same form of equation (5.4) where the perturbation $f(u)$ is a function symmetric with respect to an axis where the axis is either the line where all the centres are placed, or the segment joining the vertex to the center of the polygon.

For this reason the results given in the previous section together with the Levi Civita transform will be very useful in the following.

5.4.1 The symmetric 2-centre problem

In this section we consider the symmetric 2-centres problem and, without loss of generality, suppose that the centres are placed on the x-axis on opposite side and symmetric with respect to the origin, say $S_\pm = (\pm C, 0)$, $C > 0$. Then the equation of motion is

$$\ddot{u} = -\frac{u - S_+}{|u - S_+|^3} - \frac{u - S_-}{|u - S_-|^3} \tag{5.17}$$

We dealt with this problem for a long in the previous chapter, mainly about the integrability of the system of motion and about a qualitative description of some class of solution it may have. As shown the integrability implies the existence of infinitely many periodic orbits, included the 8-shaped one. In particular, following the classification introduced by Pauli and Charlier, the 8-shaped orbits are periodic orbits on the class of the lemniscate orbits. (see [25, 26, 9, 8, 27])

Now our intention is to give a variational characterization to the 8-shaped orbits, in order to extend the knowledge gained for the two centres problem to systems with more centres, that are not integrable and more difficult to study. To this aim, besides some results obtained in the last section, especially in theorem 22, we need the following one, valid in a more general framework, about the non minimality of bounded paths in presence of vanishing potentials at infinity.

For $r > 0$ define the set of r-bounded paths

$$Ball_r = \{|u(t)| \le r \ \forall t \in [0, T]\}$$

Theorem 25. *Let $U(x)$ be a positive function such that*

$$\liminf_{|x| \to \infty} U(x) = 0 \tag{5.18}$$

For fixed $r > 0$ and for $T > 0$ let $u^T_{min}(t)$ the minimal path for the action functional

$$\mathcal{A}_T : \Gamma^r \to \mathbb{R}$$

$$\mathcal{A}_T(u) = \int_0^T \left\{ \frac{1}{2}|\dot{u}|^2 + U(u) \right\} dt$$

where

$$\Gamma^r = \{u(t) : |\dot{u}| \in L^2, |u(0)| < r, |u(T)| < r\}$$

Then if

$$\inf_{|x| \le r} U(x) > 0$$

there exists $\overline{T} > 0$ such that for every $T > \overline{T}$, $u^T_{min} \notin Ball_r$.

Proof
For every $u(t) \in Ball_r$

$$\mathcal{A}_T(u) = \int_0^T \left\{ \frac{1}{2}|\dot{u}|^2 + U(u) \right\} dt \ge \int_0^T U(u) dt > mT$$

where $m = \min_{|x| \le r} U(x) > 0$.

The property (5.18) implies the existence of a point A such that $U(A) < \frac{m}{2}$.

Define the path

$$\bar{u}(t) = \begin{cases} u_1(t) & t \in [0, 1] \\ A & t \in [1, T-1] \\ u_2(t) & t \in [T-1, T] \end{cases}$$

where $u_1(t)$ and $u_2(t)$ are the minimal paths for the action among those that connect, respectively, P to A and A to Q, in a time interval of length 1. Since

$$\mathcal{A}_T(\bar{u}) = \mathcal{A}_T(u_1) + \mathcal{A}_T(u_2) + (T-2)\frac{m}{2} = cost + T\frac{m}{2}$$

we conclude that for T sufficently large $\mathcal{A}_T(\bar{u}) < \mathcal{A}_T(u)$ for every $u \in Ball_r$. □

We can now state the theorem concerning the existence of 8-shaped solutions for the symmetric two body problem.

Theorem 26. *For T large enough the system* (5.17) *admits T-periodic 8-shaped solutions.*

Proof
Denote with $S_+ = (C, 0)$ and $S_- = (-C, 0)$, $C > 0$, the points on the x-axis where the equation (5.17) is singular and with $\pi_+^C = \{(x, 0), x > C\}$, the right open components of the x-axis disconnected by S_+. Moreover, for every fixed T, define the set of path

$$\Gamma^+ = \{u(t), \dot{u} \in L^2([0, T]) : \ u(0) = 0, u(T) \in \overline{\pi_+^C}\}$$

and the action functional $\mathcal{A} : \Gamma^+ \to \mathbb{R}$

$$\mathcal{A} = \int_0^T \left\{ \frac{1}{2}|\dot{u}|^2 + \frac{1}{|u - S_+|} + \frac{1}{|u - S_-|} \right\} dt$$

We approach to the problem by a variational method and we look for minimal path for the action in the set Γ^+.

Since the potential function

$$U = \frac{1}{|u - S_+|} + \frac{1}{|u - S_-|}$$

is symmetric with respect to both the coordinated axes, in order to find minimal path for the action functional, we can restrict our minimization procedure to those paths $u(t) = (u_1(t), u_2(t))$ in Γ^+ with $u_i(t) \geq 0$. As a consequence, the potential function $U(x)$ can be considered as a perturbation of the potential function for the one center problem

$$U = \frac{1}{|u - S_+|} + F(u)$$

where the perturbation $F(u)$ is a regular function symmetric with respect to the x-axis, hence theorem 22 is applicable. Consequently, the minimal action path $u_{min}(t)$ is either collisionless or it lies for all time on the x-axis between the origin and S_+. For theorem 25 we choose a time interval sufficiently large in order to exclude the collision alternative, then we regard $u_{min}(t)$ as a non-collision $\mathcal{C}^2$ path, solution of equation (5.17) in $[0, T]$.

Rewrite $u_{min}(t) = (u_1(t), u_2(t))$ and define the path $u(t) : [0, 4T] \to \mathbb{R}^2$ as

$$u(t) = \begin{cases} u_{min}(t) & t \in [0, T] \\ (u_1, -u_2)(2T - t) & t \in (T, 2T] \\ (-u_1, u_2)(t - 2T) & t \in (2T, 3T] \\ (-u_1, -u_2)(4T - t) & t \in (3T, 4T) \end{cases}$$

For lemma .7 the path u_{min} reaches the end-point $u(T)$ perpendicularly with respect to the x-axis, hence $u(t)$ is $\mathcal{C}^1$ and, for the symmetric property of potential $U(x)$, $u(t)$ solves equation (5.17) except for a finite number of instants.

The theorem of uniqueness of solution for ordinary differential equations implies that $u(t)$ solves equation (5.17) for every $t \in [0, 4T]$ and $u(t)$ is $\mathcal{C}^2$. □

5.4.2 N-centre: a particular placement of centres

Suppose $\{S_i\}_{i=1}^n$ be n points placed on an axis, say the axis π, and $\{(Z_j, \tilde{Z}_j)\}_{j=1}^k$ be k couples of points such that, for every j, the points Z_j and $\tilde{Z}_j$ are symmetric with respect to π.

We want to study the N-centres problem

$$\ddot{u} = \nabla U(u)$$
$$U(x) = \sum_{i=1}^n \frac{\alpha_i}{|x - S_i|} + \sum_{j=1}^k \beta_j \left(\frac{1}{|x - Z_j|} + \frac{1}{|x - \tilde{Z}_j|} \right) \quad \alpha_i, \beta_j > 0 \tag{NCP}$$

We note that the particular choice of the positions of the singularities we done implies that the potential is a function symmetric with respect the axis π. We want to find non collision solutions for the problem by a variational approach. In particular, we will perform a minimizating procedure on suitable sets of paths where we will be able to exclude the collisions on the minimal paths.

In order to define the components of paths where the minimization works, we have to fix some notations. The axis π is disconnected by the points S_i into $n+1$ open components and, ordering in a natural way the points S_i on the axis, denote with π_i the open components bounded by S_i and S_{i+1}, $i = 1 \dots n-1$ and with π_0, π_n the unbounded open component whose estremal point is S_1 and S_n respectively. Denote with $\mathcal{H}$ and $\tilde{\mathcal{H}}$ the halfspaces delimited by the axis π, suppose that $Z_j \in \mathcal{H}$, $\tilde{Z}_j \in \tilde{\mathcal{H}}$ and let $\Delta = \{S_i, Z_j, \tilde{Z}_j\}$ be used to indicate the set of all the singularities of the potential.

For every $P \in \pi_0$, $Q \in \pi_n$, let $\Gamma_{P,Q}$ and $\tilde{\Gamma}_{P,Q}$ be defined as in (5.16). Suppose γ be a path in $\tilde{\Gamma}_{P,Q}$: to any intersection of γ with the axis π we associate the couple $(\mathcal{P}, I)$ where $\mathcal{P}$ is the point of intersection and $I \in \{\pm 1\}$ indicates the direction of the intersection, say $+1$ if γ passes from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ and -1 otherwise. Moreover let $Int(\gamma) = \{(P, I_0), (\mathcal{P}_1, I_1), \dots, (\mathcal{P}_{K-1}, I_{K-1}), (Q, I_K)\}$ be the sequence of all consecutive π-crossing of γ; given two consecutive intersections $(\mathcal{P}_k, I_k)$,$(\mathcal{P}_{k+1}, I_{k+1})$ with opposite directions, denote with $\gamma|_k$ the restiction of γ between $\mathcal{P}_k$ and $\mathcal{P}_{k+1}$ and $\sharp(\mathcal{P}_1, \mathcal{P}_2)$ the number of centres that are enclosed by $\gamma|_k$ together with its symmetric with respect to π. Finally we use the notation $\tilde{\Gamma}(\gamma)$ to indicate the open components of paths in $\tilde{\Gamma}_{P,Q}$ homotopic to γ where P and Q are free to vary in the components π_0, π_n respectively.

Theorem 27. *Consider the N-center problem (NCP) with $n > 2$. For any choice of sequence of π-crossing $\{(\mathcal{P}_k, I_k)\}_{k=0}^{K}$, where $\mathcal{P}_0 \in \pi_0$, $\mathcal{P}_K \in \pi_n$, with the conditions that $\mathcal{P}_k$ and $\mathcal{P}_{k+1}$ belongs to different components π_i and $I_k I_{k+1} = -1$, let $\gamma \in \tilde{\Gamma}_{P,Q}$ such that:*

i) $Int(\gamma) = \{(\mathcal{P}_k, I_k)\}_{k=0}^{K}$

ii) γ has not self intersection between two consecutive crossing points with π

If

iii) $\sharp(\mathcal{P}_k, \mathcal{P}_{k+1}) \geq 2 \quad \forall k = 0, \dots, K-1$

then there exists a non collision minimal path for the action on the open components $\tilde{\Gamma}(\gamma)$.

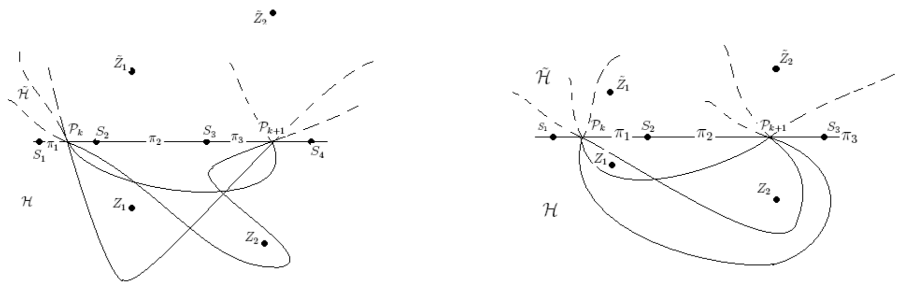
Before proving the theorem we want to explain the meaning of condition iii).

Let $\mathcal{P}_k$, $\mathcal{P}_{k+1}$ be two consecutive π-crossing of γ and π_k, π_{k+1} be the open components of π where the points $\mathcal{P}_k$, $\mathcal{P}_{k+1}$ are placed. Then, if condition iii) holds, an alternative occurs for the behaviour of the path γ between $\mathcal{P}_k$, $\mathcal{P}_{k+1}$:

type 1: the components π_k and π_{k+1} are not contiguous then $\gamma|_k$ is free to wind around or not a centres out of the axis, see figure 5.4(a),

type 2: π_k and π_{k+1} are adjacent and $\gamma|_k$ has to wind around at least one of the centres $\{Z, \tilde{Z}\}$ as show in figure 5.4(b)

Proof
To the sequence $Int(\gamma) = \{(\mathcal{P}_k, I_k)\}_{k=0}^{K}$ we can associate the sequence $Com(\gamma) = \{(\Pi_k, I_k)\}_{k=0}^{K}$ where $\Pi_k \in \{\pi_j\}$ indicates the component of the axis π where $\mathcal{P}_k$ is placed. A path γ_1 homotopic to γ could have a bigger number of crossing points in each components π_i but, since we are looking for minimal paths for the action and the potential function is symmetric with respect to π, by opportune reflections with respect to π, we can restrict the components $\tilde{\Gamma}(\gamma)$ to those paths that, in each component π_i, have the same number of crossing points, with the same direction, as γ. We still denote such restriction with $\tilde{\Gamma}(\gamma)$. Thus, although the crossing points $\mathcal{P}_k$ may change, the sequence $Com(\gamma)$ is invariant on $\tilde{\Gamma}(\gamma)$. It follows that the choice of the sequence $\{(\mathcal{P}_k, I_k)\}_{k=0}^{K}$



(a) non contiguous case (b) contiguous case

Figure 5.4:

prescribes in which components of γ and with which direction the minimal path has to intersect the axis of symmetry.

For every $\varepsilon > 0$ let u_ε be the minimal path for the action on the set

$$\tilde{\Gamma}_\varepsilon(\gamma) = \{u \in \tilde{\Gamma}(\gamma),\ \min_{C \in \Delta} |u - C| \geq \varepsilon\}$$

and, up to subsequence,

$$u_{min} = \lim_{\varepsilon \to 0} u_\varepsilon$$

Suppose *iii*) holds, we have to exclude the occurrence of collisions for the minimal path u_{min}.

Since u_ε has the same number as γ of intersections in each components π_i with same direction, relations *iii*) hold for every u_ε too, once we substitute to the points $\mathcal{P}_k$ the points of intersection of u_ε with π.

The minimality of u_ε implies that they are at least of class $\mathcal{C}^1$ then u_ε can not reach the axis π out of the crossing points. Indeed if u_ε reaches the axis π but does not cross it, the path has to be tangent to the axis; since π is an axis of symmetry for the potential the path u_ε has to collide in one of the S_i.

Condition ii) implies that the path γ, and by omotopy every path u_ε, between two consecutive π-crossing, winds 0 or 1 times around any center Z_j, $\tilde{Z}_j$ with respect the axis π. Thus the minimality of u_ε and lemma (5.6) imply that any restriction $u_\varepsilon|_k$ are free of self intersections.

In any case what is important and crucial for the theorem is that any u_ε can not make a loop around only a center, but, whenever it has a self intersection at least two centres are enclosed by the path.

It follows from lemma (5.5) that if a collision occurs in any centre, the path u_{min} is a collision reflection path. Suppose for instance that between two consecutive crossing points $\mathcal{P}_k$, $\mathcal{P}_{k+1}$ the sequence u_ε uniformly converge to u_{min} with a collision in Z_i. Then, since u_{min} is of reflection type, the point $\mathcal{P}_k$ have to converge to $\mathcal{P}_{k+1}$ and since they can not belong to the same component π_j, the only possibility is that both the crossing points converge to a center S_k. As a consequence, the limit path u_{min} is a collision reflection path between Z_i and S_k.

On the other hand suppose γ be of type 1 and that does not wind around any other center out of the axis between two consecutive crossing points $\mathcal{P}_k$, $\mathcal{P}_{k+1}$. Without lose of generality let, for instance, $\Pi_k = \pi_q$, $\Pi_{k+1} = \pi_r$, $r > q$, $r \neq q + 1$ and suppose u_ε converges to u_{min} with a collision in S_r. Since u_{min} must be of reflection type and since u_ε can not intersect π_{r-1}, the limit path belongs to the axis π and it is a collision reflection solution between two consecutive centres.

In any case, since the estremal points P and Q are free to vary in π_0 and π_n and $n > 2$, if the limit path $u_{\min}$ has a collision, it does not belongs to the closure of $\tilde{\Gamma}(\gamma)$, then all the minimizer of the action on $\tilde{\Gamma}(\gamma)$ are free of collisions. □

.1 Appendix 1

We deal with the so called *Principle of Least Action* that connects solutions of a dynamical system to the stationary points of the associated action functional.

Given a Lagrangian function

$$\mathcal{L}(\dot{u}, u) = \frac{1}{2}|\dot{u}(t)|^2 + U(u(t))$$

the *Action* is defined as

$$\mathcal{A}(u) = \int_0^T \mathcal{L}(\dot{u}, u)dt$$

The Principle of Least Action states that a path $u(t)$ is a stationary point for the Action in the interior of an appropriate class of paths, $\mathrm{d}\mathcal{A}(u) = 0$, if and only if $u(t)$ is solution of equation

$$-\ddot{u}(t) + \nabla U(t) = 0 \tag{19}$$

From a variational point of view the property of a path to be a stationary point of a functional is weaker than to be a minimal; on the other hand in order to find stationary points of a functional it could be easier look for a minimal point since many tools from the functional analysis may be useful to this end.

In the sequel the terms *singular* or *collision* are used to indicate points where the potential function is not differentiable.

Principle of Least Action *Let $u(t)$ be any collisionless local minimal path for the action $\mathcal{A} = \int_0^T \mathcal{L}(\dot{u}, u)dt$ with respect to variations $\varphi(t) \in \mathcal{C}_c^\infty(0, T)$. Then for every $t \in (0, T)$ the path $u(t)$ is a solution of the Euler-Lagrange equation*

$$-\ddot{u}(t) + \nabla U(t) = 0$$

Proof

Since $u(t)$ is a local minimal path for the action $\mathcal{A}$, for every variation $\varphi(t) \in \mathcal{C}_c^\infty(0, T)$ and for $\varepsilon > 0$ small enough it holds $\mathcal{A}(u + \varepsilon\varphi) - \mathcal{A}(u) \geq 0$. Then

$$\begin{aligned}
0 &\leq \mathcal{A}(u + \varepsilon\varphi) - \mathcal{A}(u) \\
&= \int_0^T \left\{ \frac{1}{2}\left(|\dot{u}(t) + \varepsilon\dot{\varphi}(t)|^2 - |\dot{x}(t)|^2\right) + U\big(u(t) + \varepsilon\varphi(t)\big) - U\big(u(t)\big) \right\} dt \\
&= \varepsilon \int_0^T \Big\{ \dot{u}(t)\dot{\varphi}(t) + \nabla U\big(u(t)\big)\varphi(t) \Big\} dt + \mathscr{O}(\varepsilon^2)
\end{aligned}$$

Dividing by ε and noting that the same inequality holds for the variation $-\varphi(t)$, we glean

$$\int_0^T \Big\{ \dot{u}(t)\dot{\varphi}(t) + \nabla U\big(u(t)\big)\varphi(t) \Big\} dt = 0$$

Integrating by parts the first term, reminding that the function $\varphi(t)$ has compact support, we obtain

$$\int_0^T \Big\{ -\ddot{u}(t) + \nabla U\big(u(t)\big) \Big\} \varphi(t) dt = 0$$

and the arbitrarity of the variation $\varphi(t)$ implies that

$$-\ddot{u}(t) + \nabla U\big(u(t)\big) = 0$$

for every $t \in (0, T)$. □

In the following lemma we state that if $u(t)$ is any (local) minimal path of the action functional, joining the points P and Q, in the case that one of the end-points, for instance Q, is free to vary on a subspace or, in general, on a variety V, then the path $u(t)$ is normal to V in Q. This property holds for every kind of Lagrangian and it could be considered as a Neumann condition for the minimal path.

Lemma .7. *Let $u(t)$ be a minimizer for the action functional*

$$\mathcal{A} = \int_0^T \left\{ \frac{1}{2}|\dot{u}(t)|^2 + U(u(t)) \right\} dt$$

with respect to variations $\varphi \in \mathcal{C}^\infty([0,T])$ with the constrain $\varphi(T) \in V$ where V indicates a vector subspace or a variety. If $u(T)$ is not a singular point for the potential $U(x)$, then $\dot{u}(T) \in V^\perp$ or, in case V is a variety, $\dot{u}(T) \in \big(T_{u(T)}V\big)^\perp$.

Proof
We repeat the same argument done in the first part of the previous proof in order to say that for every $\varphi \in \mathcal{C}^\infty([0,T])$ it holds

$$\int_0^T \Big\{ \dot{u}(t)\dot{\varphi}(t) + \nabla U\big(u(t)\big)\varphi(t) \Big\} dt = 0$$

By integration by parts it follows

$$\dot{u}(t)\varphi(t)\Big|_0^T + \int_0^T \big(-\ddot{u}(t) + \nabla U\big(u(t)\big)\big)\,\varphi(t)dt = 0$$

For the Principle of Least Action the path $u(t)$ solve the equation (19) for every $t \in (0,T)$ then the integral in the previuos equation is zero. It follows that $\dot{u}(T)\varphi(T) = 0$ for every φ such that $\varphi(T) \in V$. This means that $\dot{u}(T) \in V^\perp$. $\square$

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