Conference Applications of Mathematics 2013
in honor of the 70th birthday of Karel Segeth.
Institute of Mathematics AS CR, Prague 2013

A METHOD TO RIGOROUSLY ENCLOSE EIGENPAIRS OF COMPLEX INTERVAL MATRICES

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Abstract

In this paper, a rigorous computational method to enclose eigenpairs of complex interval matrices is proposed. Each eigenpair $x = (\lambda, \mathrm{v})$ is found by solving a nonlinear equation of the form $f(x) = 0$ via a contraction argument. The set-up of the method relies on the notion of *radii polynomials*, which provide an efficient mean of determining a domain on which the contraction mapping theorem is applicable.

1. Introduction

Computing eigenvalues and eigenvectors of matrices is a central problem in many fields of applied sciences involving mathematical modelling. When applied to real-life phenomena, models need to consider the occurrence of diverse errors in the data, due for instance to inaccuracy of measurements or noise effects. Such uncertainty in the data can be represented by intervals. In the context of studying a matrix with uncertain entries, interval matrices can be considered. Our goal here is to develop a rigorous computational method to enclose eigenpairs of complex interval matrices.

Before proceeding further, note that bounds for eigendecompositions of standard (non interval) matrices are abundant, ranging from classical perturbation theory like Bauer-Fike residual and condition number based theorems [4], via Kato-Temple bounds [5], to Rayleigh-Ritz bounds [6, 7, 8, 9], to bounds coming from Newton-Kantorovich type arguments [1, 2], to pseudospectral bounds, and so on. Many such results can, for instance, lead to bounds on the nearest eigenpair to a given approximation. Also, while the problem of computing rigorous bounds for the eigenvalue set of interval matrices is well studied, see [10, 11] and the references therein, a not so large literature has been produced regarding the simultaneous enclosure of the eigenvalues and eigenvectors of interval matrices. In this direction we refer to [1, 12], where different techniques have been developed to enclose simple eigenvalues and corresponding eigenvectors, while for double or nearly double eigenvalues a method has been introduced in [13]. For the rigorous enclosure of multiple or nearly multiple eigenvalues of complex matrices, a contribution has been made by S. Rump in [3, 14].

In this paper, we propose the new idea of enclosing rigorously the eigenpairs of complex interval matrices by using the notion of the *radii polynomials*, which provide a computationally efficient way of determining a domain on which the contraction mapping theorem is applicable. The radii polynomials approach, which is similar to the approaches of Newton-Kantorovich and the Krawczyk operator, aims at demonstrating existence and local uniqueness of solutions of nonlinear equations. The Newton-Kantorovich approach fixes *a priori* the radius r of a ball $B(r)$ around a numerically computed eigenpair and attempt to demonstrate the existence of a contraction on $B(r)$. Similarly, the Krawczyk operator approach consists of applying directly the operator to interval vectors (in the form of small neighbourhoods around a numerical approximation) and then attempt to verify *a posteriori* the hypotheses of a contraction mapping argument [15, 16]. On the other hand, the radii polynomials are *a priori* conditions that are derived analytically, and once they are theoretically constructed, they are used to *solve* for the sets (also in the form of small neighbourhoods of a numerical solution) on which a Newton-like operator is a contraction. The radii polynomials were originally introduced in [17] to compute equilibria of PDEs with the goal of minimizing the extra computational cost required to prove existence of solutions of infinite dimensional PDEs [18].

The paper is organized as follows. In Section 2, the method is introduced to enclose rigorously eigenpairs of non interval matrices and in Section 3 it is generalized to the case of interval matrices. In Section 4, we present applications and compare our method to the method of [14] and to a method based on the Krawczyk operator.

2. The computational method

We fix some notation. We denote by $\mathbb{IC}^{n\times n}$ the set of complex matrices with interval entries, $A \in \mathbb{C}^{n\times n}$ an $n\times n$ complex matrix and $\boldsymbol{A} \in \mathbb{IC}^{n\times n}$ an $n\times n$ interval complex matrix, meaning that any entry of $\boldsymbol{A}$ is a complex interval of the form

$$\boldsymbol{A}_{i,j} = [Re(\hat{A}_{i,j}) \pm rad^{(1)}_{i,j}] + \imath[Im(\hat{A}_{i,j}) \pm rad^{(2)}_{i,j}], \quad rad^{(1)}_{i,j}, rad^{(2)}_{i,j} \in \mathbb{R}_+ \ .$$

$\hat{A} \in \mathbb{C}^{n\times n}$ is called the center of $\boldsymbol{A}$ while $rad^{(1)}_{i,j}, rad^{(2)}_{i,j}$ are called the radii of the real and imaginary part of $\boldsymbol{A}_{i,j}$, resp. We denote $A \in \boldsymbol{A}$, if $A_{i,j} \in \boldsymbol{A}_{i,j}$ for any $1 \le i,j \le n$. Bold face letters will always denote interval quantities. Moreover,

- $|\cdot|$ is the complex absolute value and, in case of matrices $M \in \mathbb{C}^{n\times m}$, it acts component-wise, that is $|M|_{i,j} = |M_{i,j}|$;
- given two real matrices M, N, we write $M \preceq N$ if and only if $M_{i,j} \le N_{i,j}$ for all i,j. The same notation holds for $\prec$, $\succ$ and $\succeq$;
- I_n denotes the $n\times n$ dimensional identity matrix, $\mathbb{1}_n$ is the column vector of length n with all the entries equal to 1;
- given any matrix $M \in \mathbb{C}^{n\times m}$, the object $(M)_{\hat{k}}$ stands for the $n\times(m-1)$ matrix obtained by deleting the k-th column of M.

Given $\boldsymbol{A} \in \mathbb{IC}^{n\times n}$, we aim to enclose eigenpairs of any $A \in \boldsymbol{A}$. To simplify the exposition, we first present the method in the context of non interval matrices $A \in \mathbb{C}^{n\times n}$. Minor modifications are needed for the extension to the interval case.

Suppose that an approximate eigenpair of A has been computed, that is $(\bar{\lambda}, \bar{\mathrm{v}})$ such that $A\bar{\mathrm{v}} \approx \bar{\lambda}\bar{\mathrm{v}}$ and let $f(x)$ be the function $f : \mathbb{C}^n \to \mathbb{C}^n$ that maps a point $x = (\lambda, \mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_{k-1}, \mathrm{v}_{k+1}, \ldots, \mathrm{v}_n)$ to

$$f(x) = A \begin{bmatrix} \mathrm{v}_1 \\ \vdots \\ \bar{\mathrm{v}}_k \\ \vdots \\ \mathrm{v}_n \end{bmatrix} - \lambda \begin{bmatrix} \mathrm{v}_1 \\ \vdots \\ \bar{\mathrm{v}}_k \\ \vdots \\ \mathrm{v}_n \end{bmatrix} \tag{1}$$

where $\bar{\mathrm{v}}_k$ is the largest component in absolute value of $\bar{\mathrm{v}}$. Fixing $\mathrm{v}_k = \bar{\mathrm{v}}_k$ ensures that the solution is isolated. Note that the more standard approach of fixing $\|\mathrm{v}\| = 1$ will fail to provide isolation if v is complex. Indeed, given an eigenpair $(\lambda, \mathrm{v}) \in \mathbb{C}^{n+1}$ where v is complex, then for any θ, $(\lambda, e^{i\theta}\mathrm{v})$ is also an eigenpair and $\|e^{i\theta}\mathrm{v}\| = 1$.

By definition, a solution x of $f(x) = 0$ corresponds to an eigenpair (λ, v) of A with the eigenvalue λ given by the first component of x and the eigenvector $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_{k-1}, \bar{\mathrm{v}}_k, \mathrm{v}_{k+1}, \ldots, \mathrm{v}_n)$. We then aim at proving existence of zeros of $f(x)$ together with rigorous bounds. Denoting $\bar{x} = (\bar{\lambda}, \bar{\mathrm{v}}_1, \bar{\mathrm{v}}_2, \ldots, \bar{\mathrm{v}}_{k-1}, \bar{\mathrm{v}}_{k+1}, \ldots, \bar{\mathrm{v}}_n)$ and $Df(\bar{x})$ the Jacobian matrix of f at $\bar{x}$, one has that

$$Df(\bar{x}) = \left(- \left[\begin{array}{c} \bar{\mathrm{v}}_1 \\ \vdots \\ \bar{\mathrm{v}}_k \\ \vdots \\ \bar{\mathrm{v}}_n \end{array} \right] \middle| (A - \bar{\lambda} I_n)_{\hat{k}} \right). \tag{2}$$

We find zeros of f by introducing a fixed point operator T. Endow $\mathbb{C}^n$ with the norm $\|x\|_\infty = \max_{i=1,\ldots,n}\{|x_i|\}$. Consider $R \approx Df(\bar{x})^{-1}$ an invertible matrix. Define

$$T : \mathbb{C}^n \to \mathbb{C}^n : x \mapsto T(x) = x - Rf(x). \tag{3}$$

so that fixed points of T are in bijection with zeros of f. In practice, the matrix R is computed numerically in MATLAB. Note that getting a good approximate inverse is fundamental for our method to provide sharp bounds. Indeed, as one shall see shortly, the better the approximate inverse R is, the smaller the bound $Z^{(1)}$ in (4) will be. Since fixed points of T correspond to zeros of $f(x)$, the idea is to construct a small set $B \subset \mathbb{C}^n$ such that $T : B \to B$ is a contraction, and then to apply the contraction mapping theorem to conclude about the existence of a unique fixed point of T in B. Note that $\bar{x}$ is an approximate zero of f and the operator T has been defined as a Newton-like operator around the point $\bar{x}$, thus it is advantageous to test the contractibility of T on neighbourhoods of $\bar{x}$ in $\mathbb{C}^n$. More precisely, denote by $B(r) = \{x \in \mathbb{C}^n, \|x\|_\infty \leq r\}$ the closed ball of radius r around the origin and let $B_{\bar{x}}(r) = \bar{x} + B(r)$ be the ball with the same radius and centered at $\bar{x}$. Treating r as a variable, we choose the balls $B_{\bar{x}}(r)$ as the candidate sets where to check if T is a contraction. The next result provides a recipe to determine the radius r.

Theorem 2.1. *Consider $\bar{x} \in \mathbb{C}^n$ and R a real $n \times n$ invertible matrix. Consider the nonlinear problem* (1) *and bounds $Y, Z^{(1)}, Z^{(2)} \in \mathbb{R}^n$ such that*

$$|Rf(\bar{x})| \preceq Y, \quad |I_n - R \cdot Df(\bar{x})|\mathbb{1}_n \preceq Z^{(1)}, \quad 2|R|(\mathbb{1}_n)_{\hat{k}} \preceq Z^{(2)}. \tag{4}$$

Define the radii polynomials $p_1(r), p_2(r), \ldots, p_n(r)$ by

$$p_i(r) = Z_i^{(2)} r^2 + (Z_i^{(1)} - 1)r + Y_i, \tag{5}$$

and define $\mathcal{I} = \cap_{i=1}^n \{r > 0 : p_i(r) < 0\}$. If $\mathcal{I} \neq \emptyset$, then for any $r \in \mathcal{I}$, there exists a unique $\hat{x} \in B_{\bar{x}}(r)$ such that $f(\hat{x}) = 0$.

Proof. Consider $r \in \mathcal{I} \neq \emptyset$. Recalling (3), consider $T(x) = x - Rf(x)$. Then

$$\begin{aligned} \sup_{b,c \in B(r)} |DT(\bar{x}+b)c| &= \sup_{b,c \in B(r)} |(I_n - R \cdot Df(\bar{x}))c + R(Df(\bar{x}) - Df(\bar{x}+b))c| \\ &\preceq \sup_{b,c \in B(r)} |(I_n - R \cdot Df(\bar{x}))c| + |R(Df(\bar{x}) - Df(\bar{x}+b))c| \\ &\preceq Z^{(1)} r + Z^{(2)} r^2. \end{aligned}$$

In the last inequality, we used that for any $b = (b_\lambda, b_1, \ldots, b_{k-1}, b_k, \ldots, b_n) \in B(r)$

$$(Df(\bar{x}) - Df(\bar{x}+b)) = \left(\begin{bmatrix} b_1 \\ \vdots \\ b_{k-1} \\ 0 \\ b_{k+1} \\ \vdots \\ b_n \end{bmatrix} \middle| (b_\lambda I_n)_{\hat{k}} \right).$$

Note that the k-th row of the above matrix is null. Since $|b_i| \leq r$, we have that $|(Df(\bar{x}) - Df(\bar{x}+b))c| \preceq 2r^2(\mathbb{1}_n)_{\hat{k}}$ and therefore $\sup_{b,c \in B(r)} |R[(Df(\bar{x}) - Df(\bar{x}+b))c]| \preceq 2r^2|R|(\mathbb{1}_n)_{\hat{k}} = Z^{(2)} r^2$.

Letting $Z(r) := Z^{(1)} r + Z^{(2)} r^2$, we get that $\sup_{b,c \in B(r)} |DT(\bar{x}+b)c| \preceq Z(r)$. The Mean Value Theorem applied component-wise to T implies that for any $x, y \in B_{\bar{x}}(r)$ and for any $i = 1, \ldots, n$, $T_i(x) - T_i(y) = DT_i(z)(x-y)$, for some $z \in \{tx + (1-t)y : t \in [0,1]\} \subset B_{\bar{x}}(r)$. Then,

$$|T_i(x) - T_i(y)| = \left| DT_i(z) \frac{r(x-y)}{\|x-y\|_\infty} \right| \frac{1}{r} \|x-y\|_\infty \leq \frac{Z_i(r)}{r} \|x-y\|_\infty \leq Z_i(r). \tag{6}$$

Let $x \in B_{\bar{x}}(r)$ and $y = \bar{x}$ in (6), and using that $T(\bar{x}) - \bar{x} = -Rf(\bar{x})$, one has that

$$|T_i(x) - \bar{x}_i| \leq |T_i(x) - T_i(\bar{x})| + |T_i(\bar{x}) - \bar{x}_i| \leq Z_i(r) + Y_i = Z_i^{(2)} r^2 + Z_i^{(1)} r + Y_i < r$$

by the hypothesis that $p_i(r) < 0$, which follows from the fact that $r \in \mathcal{I}$. That shows that $T(B_{\bar{x}}(r)) \subseteq B_{\bar{x}}(r)$. From (6), it follows that

$$\|T(x) - T(y)\|_\infty = \max_i \{|T_i(x) - T_i(y)|\} \leq \frac{\|Z(r)\|_\infty}{r} \|x-y\|_\infty.$$

Since $Z_i(r) \leq Z_i(r) + Y_i < r$ for any $i = 1, \ldots, n$, it follows that $\|Z(r)\|_\infty < r$. Hence T is a contraction with contraction constant $\frac{\|Z(r)\|_\infty}{r} < 1$. From the contraction mapping theorem, there exists a unique $\hat{x} \in B_{\bar{x}}(r)$ such that $T(\hat{x}) = \hat{x}$. By invertibility of R, there exists a unique $\hat{x} \in B_{\bar{x}}(r)$ such that $f(\hat{x}) = 0$. $\square$

In summary, given an approximate eigenpair $(\bar{\lambda}, \bar{\mathrm{v}})$, the method consists of computing rigorously the bounds $Y, Z^{(1)}, Z^{(2)}$ given in (4), and then to check whether there exists an interval $\mathcal{I}$ where all the polynomials $p_i(r)$ are negative. If $\mathcal{I} \neq \emptyset$ we select $r = \inf \mathcal{I}$ and we conclude that $f = 0$ has a unique solution within the ball $B_{\bar{x}}(r)$. In practice, we get the existence of an eigenpair (λ, v) of A, with $|\lambda - \bar{\lambda}| \leq r$, $|\mathrm{v}_j - \bar{\mathrm{v}}_j| \leq r$, for $j \neq k$ and $\mathrm{v}_k = \bar{\mathrm{v}}_k$. To prove the existence of another eigenpair of A, it is necessary to provide a different numerical approximate solution $(\bar{\lambda}, \bar{\mathrm{v}})$, different from the previous one, and to repeat the computation.

3. Extension to the interval case

Besides few modifications necessary to deal with interval quantities, the procedure to compute rigorously bounds for the eigenpairs of an interval matrix $\boldsymbol{A} \in \mathbb{IC}^{n \times n}$ is basically the same as for the scalar case. However, a fundamental difference is that all the computations are done using interval arithmetic [10], in which any of the basic operations $\circ \in \{+, -, \cdot, /\}$ is extended to the interval case in order to satisfy the general assumption

$$\forall P \in \boldsymbol{P} \quad \forall Q \in \boldsymbol{Q}, \quad P \circ Q \in \boldsymbol{P} \circ \boldsymbol{Q} \, . \tag{7}$$

Given an interval complex valued matrix $\boldsymbol{A}$, we now address the problem wether or not we can rigorously enclose the eigenpairs of any $A \in \boldsymbol{A}$. Recall that $\hat{A}$ is the center of the interval matrix $\boldsymbol{A}$. We first compute $(\bar{\lambda}, \bar{\mathrm{v}})$ an approximate eigenpair of $\hat{A}$ and, as before, define $\bar{x} = (\bar{\lambda}, \bar{\mathrm{v}}_1, \bar{\mathrm{v}}_2, \ldots, \bar{\mathrm{v}}_{k-1}, \bar{\mathrm{v}}_{k+1}, \ldots, \bar{\mathrm{v}}_n)$ where the missing component $\bar{\mathrm{v}}_k$ is chosen so that $|\bar{\mathrm{v}}_k| = \max_j\{|\bar{\mathrm{v}}_j|\}$. Then, replacing the scalar matrix A in (1) by the interval matrix $\boldsymbol{A}$, the function $f(x)$ and the Jacobian matrix $Df(\bar{x})$ defined in (1) and (2) are replaced respectively by $\boldsymbol{f} : \mathbb{C}^n \to \mathbb{IC}^n$ and by an interval matrix $\boldsymbol{Df}(\bar{x})$ that represents a linear operator from $\mathbb{C}^n$ to $\mathbb{IC}^n$. We choose R to be a numerical inverse of $\widehat{Df}(\bar{x})$, the center of $\boldsymbol{Df}(\bar{x})$, and we proceed to the definition of the operator $\boldsymbol{T}(x) = x - R\boldsymbol{f}(x)$ and to the bounds $Y, Z^{(1)}, Z^{(2)}$, as done before with the boldface quantities in place of the previous one. Clearly some quantities on the left hand side of relations (4) are now intervals, thus we define component-wise $Y, Z^{(1)}, Z^{(2)}$ as the supremum over the intervals involved, yielding uniform bounds

$$|R\boldsymbol{f}(\bar{x})| \preceq Y, \quad |I_n - R \cdot \boldsymbol{Df}(\bar{x})|\mathbb{1}_n \preceq Z^{(1)}, \quad 2|R|(\mathbb{1}_n)_{\hat{k}} \preceq Z^{(2)}.$$

As in Theorem 2.1, define the radii polynomials by $p_i(r) = Z_i^{(2)} r^2 + (Z_i^{(1)} - 1)r + Y_i$, for $i = 1, \ldots, n$. If $r \in \mathcal{I} = \cap_{i=1}^n \{r > 0 : p_i(r) < 0\}$, then for all $A \in \boldsymbol{A}$, there exists a unique $(\lambda, \mathrm{v}) \in B_{\bar{x}}(r)$ such that $|\lambda - \bar{\lambda}| \leq r$, $|\mathrm{v}_j - \bar{\mathrm{v}}_j| \leq r$, $\mathrm{v}_k = \bar{\mathrm{v}}_k$, and $A\mathrm{v} = \lambda \mathrm{v}$.

In other words, r is a uniform bound in $\boldsymbol{A}$ for the existence of an eigenpair of any $A \in \boldsymbol{A}$. Indeed, having fixed $(\bar{\lambda}, \bar{\mathrm{v}})$ and $R \approx \left(\widehat{Df}(\bar{x})\right)^{-1}$, for any $A \in \boldsymbol{A}$ define $f_A(x)$ and $Df_A(\bar{x})$ as in (1) and (2), and the fixed point operator $T_A(x) = x - Rf_A(x)$. The fundamental inclusion (7) implies that $f_A(x) \in \boldsymbol{f}(x)$, $Df_A(\bar{x}) \in \boldsymbol{Df}(\bar{x})$ and $T_A(x) \in \boldsymbol{T}(x)$, for any $A \in \boldsymbol{A}$ and $x \in \mathbb{C}^n$. Thus, as A varies in $\boldsymbol{A}$, the bounds (4), with T_A in place of T, are satisfied for the same $Y, Z^{(1)}, Z^{(2)}, r$ proving the existence of a fixed point in $B_{\bar{x}}(r)$ for any T_A and consequently an eigenpair for any $A \in \boldsymbol{A}$.

4. Results

In this section we report some computational results. All the computations have been done in MATLAB supported by the package INTLAB [19] where the interval arithmetic routines have been implemented. The approximate eigenpairs $(\bar{\lambda}, \bar{\mathrm{v}})$ of $\hat{A}$ have been computed running the standard `eig.m` function in MATLAB. In order to avoid rounding error and to obtain rigorous results, we emphasize that the computational algorithm treats any matrix as an interval matrix. Thus, even if one wishes to deal with a scalar matrix A, the method first constructs a (narrow) interval matrix around A and perform all the computation with interval arithmetics.

Example 1. Consider the interval matrix $\boldsymbol{A}$ centered at

$$\hat{A} = \begin{bmatrix} -10.55360193 & 5.33379647 & -5.24740415 \\ 0.31403414 & 2.33062549 & -3.32865541 \\ -7.49045333 & 5.01386821 & -5.44369022 \end{bmatrix}$$

with radius $rad = 9.66146973 \cdot 10^{-7}$, meaning that each entry $\boldsymbol{A}(i,j)$ consists of the interval $[\hat{A}(i,j) - rad, \hat{A}(i,j) + rad]$. Using the method of Section 2, it results that any $A \in \boldsymbol{A}$ admits three eigenpairs $(\lambda_i, \mathrm{v}_i)$, $i = 1, 2, 3$ each one lying in the ball of radius r_i around the approximate values $(\bar{\lambda}_i, \bar{\mathrm{v}}_i)$ given in Table 1.

	$i = 1$	$i = 2$	$i = 3$
r_i	$2.7747640834393 \cdot 10^{-6}$	$3.5677963538014 \cdot 10^{-5}$	$3.6494066386385 \cdot 10^{-5}$
$\bar{\lambda}_i$	-13.9620493680589	$-9.3632556453596 \cdot 10^{-14}$	0.2953827013923
$\bar{\mathrm{v}}_i$	$\begin{bmatrix} -0.77788012985175 \\ -0.11136179959087 \\ -0.61846669528252 \end{bmatrix}$	$\begin{bmatrix} 0.12133203779007 \\ 0.80769996168880 \\ 0.57697427021802 \end{bmatrix}$	$\begin{bmatrix} 0.15662675418092 \\ 0.83598562894630 \\ 0.52592403260356 \end{bmatrix}$

Table 1: Rigorous enclosure of the eigenpairs of $\boldsymbol{A}$.

We remark that in the general situation the genuine solution (λ, v) of the eigenproblem is proved to exist in a complex neighborhood of the approximate solution

$(\bar{\lambda}, \bar{\mathrm{v}})$. Therefore, even if one or both $\bar{\lambda}$ and $\bar{\mathrm{v}}$ are real vectors, the same cannot be concluded for λ or v. However, if the matrix A and the approximate solution $\bar{\lambda}$ and $\bar{\mathrm{v}}$ are real and the computation is successful, then the genuine solution so obtained by solving the radii polynomials is also real. Indeed, suppose the contrary, that is the exact solution λ and v are complex. Since A is real, the complex conjugate couple $(\mathcal{C}(\lambda), \mathcal{C}(\mathrm{v})))$ is also a solution of the eigenproblem, $A\mathcal{C}(\mathrm{v}) = \mathcal{C}(\lambda)\mathcal{C}(\mathrm{v})$. But both the solutions (λ, v) and $(\mathcal{C}(\lambda), \mathcal{C}(\mathrm{v})))$ belong to the same ball in $\mathbb{C}^n$ around $\bar{x}$ and this violates the uniqueness result stated in Theorem 2.1. The same argument extends in the case of interval matrices.

Example 2: matrices with interval entries of large radius. In this example, we rigorously enclose all eigenpairs of an interval matrix $\boldsymbol{A}$ constructed as follows: consider the complex number $\lambda_0 = 0$ and $\lambda_j = e^{\imath \frac{2\pi}{5} j}$, $j = 1, \ldots, 5$ and define D as the diagonal matrix with entries λ_i, $i = 0, \ldots, 5$. Let $\hat{A} = XDX^{-1}$, for a random matrix X with values in the complex square $[-1, 1] + \imath[-1, 1]$ and finally let $\boldsymbol{A}$ be the interval complex matrix centered at $\hat{A}$ with component-wise radius rad both in the real and imaginary part. For different values of rad we compute the enclosure of the eigenvalues of $\boldsymbol{A}$. Consider $\bar{\lambda}_i$ the approximate eigenvalues of $\hat{A}$ given by $\bar{\lambda}_0 = 0$, $\bar{\lambda}_1 = 0.30901 + 0.95105\imath$, $\bar{\lambda}_2 = -0.80901 + 0.58778\imath$, $\bar{\lambda}_3 = -0.80901 - 0.58778\imath$, $\bar{\lambda}_4 = 0.30901 - 0.95105\imath$ and $\bar{\lambda}_5 = 1$. Denote by r_i, $i = 0, \ldots 5$ the radius of the ball in the complex plane centered at $\bar{\lambda}_i$ inside which, for any $A \in \boldsymbol{A}$, a unique eigenvalues of A has been proved to exist. The results are presented in Table 2. See also Figure 1 for the enclosure of the six eigenvalues of any $A \in \boldsymbol{A}$ for $rad = 1.3 \cdot 10^{-3}$.

rad	r_0	r_1	r_2	r_3	r_4	r_5
$1 \cdot 10^{-5}$	$1.2 \cdot 10^{-4}$	$1.0 \cdot 10^{-4}$	$9.2 \cdot 10^{-5}$	$1.4 \cdot 10^{-4}$	$1.3 \cdot 10^{-4}$	$1.0 \cdot 10^{-4}$
$1 \cdot 10^{-4}$	0.0013	0.0011	0.0001	0.0015	0.0014	0.0012
$1 \cdot 10^{-3}$	0.0149	0.0115	0.0103	0.0184	0.0168	0.0122
$1.5 \cdot 10^{-3}$	0.0254	0.0186	0.0163	–	0.0307	0.0197
$2.0 \cdot 10^{-3}$	–	0.0272	0.0234	–	–	0.0287
$2.5 \cdot 10^{-3}$	–	0.0390	0.0322	–	–	0.0407
$3.0 \cdot 10^{-3}$	–	–	0.0450	–	–	–
$3.5 \cdot 10^{-3}$	–	–	–	–	–	–

Table 2: Enclosures of the eigenpairs of the complex interval matrix $\boldsymbol{A}$, as rad grows.

We see in Table 2 that for values of $rad \approx 10^{-3}$ the method starts to fail. A natural question is whether it is possible to predict up for which values of rad the method will be successful. We underline that the technique we propose is a verification method, therefore among other conditions, the success or the failure is strictly related to the accuracy of the approximate solution that could change from one computation to the other. Hence it is not possible to determine a priori the maximum

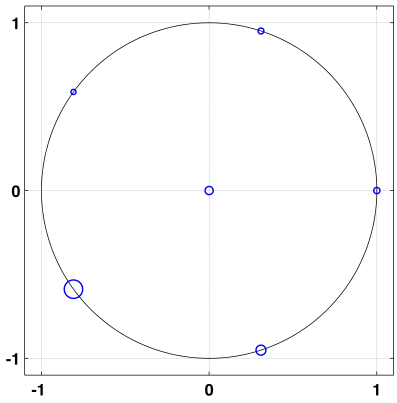

Figure 1: Balls in $\mathbb{C}$ enclosing the six eigenvalues of any $A \in \boldsymbol{A}$ for $rad = 1.3 \cdot 10^{-3}$.

value for rad. However we can to get an idea of what happens when rad increases and, based on that, we can guess which is the maximal admissible value of rad. A necessary condition for the method to be successful is that all the radii polynomials defined in (5) cross the r-axis. That occurs if

$$(Z_i^{(1)} - 1)^2 - 4Y_i Z_i^{(2)} > 0 \tag{8}$$

for all $i = 1, \ldots, n$. Roughly speaking, if the value of rad increases, while the numerical solution is kept fixed, the norms of the components of the vectors Y and $Z^{(1)}$ increase. Thus there is a value of rad large enough such that some of the inequalities (8) are not satisfied anymore. To be more precise, assuming that $\bar{x}$ is a good numerical approximate solution, we can estimate $|f(\bar{x})|_\infty \approx |\bar{x}|_1 rad$, where $|x|_1 = \sum_i |x_i|$. Then we can write $|Y|_\infty \approx \|R\|_\infty |\bar{x}|_1 rad$. Concerning $Z^{(1)}$, we see from (2) that the radius of $\boldsymbol{Df}$ is the same as the radius of $\boldsymbol{A}$. Then, assuming that the matrix R has been properly computed so that $I - R \cdot \widehat{Df}(\bar{x}) \approx 0$, we estimate $|Z^{(1)}|_\infty \approx n\|R\|_\infty rad$. Finally $|Z^{(2)}|_\infty \approx 2\|R\|_\infty$. By substituting into (8), we obtain

$$rad_{max} = \frac{n + 4\|R\|_\infty |\bar{x}|_1 - 2\sqrt{4\|R\|_\infty^2 |\bar{x}|_1^2 + 2n\|R\|_\infty |\bar{x}|_1}}{n^2 \|R\|_\infty}.$$

For $\boldsymbol{A}$ considered above, rad_{max} are computed and presented in Table 3. Note there that the prediction of rad_{max} is more precise when the dimension n is larger.

#	0	1	2	3	4	5
rad_{max}	0.0016	0.0024	0.0028	0.0012	0.0014	0.0022

Table 3: rad_{max} as a function of $\bar{\lambda}_i$, for $i = 0, 1, \ldots, 5$.

Table 4 displays the results for the enclosure of two eigenpairs of $\boldsymbol{A}$ around $\hat{A} = XDX^{-1}$, where X is a random matrix and $D = diag(1, 2, \ldots, n)$ both for $n = 50$ and $n = 100$ and the value of rad_{max}.

$n = 50$

$\bar{\lambda}_{20} = 20$, $rad_{max} = 5.698 \cdot 10^{-6}$		$\bar{\lambda}_{47} = 47$, $rad_{max} = 1.701 \cdot 10^{-5}$	
rad	r	rad	r
$1 \cdot 10^{-7}$	$3.806 \cdot 10^{-5}$	$1 \cdot 10^{-7}$	$2.163 \cdot 10^{-5}$
$2 \cdot 10^{-6}$	$8.399 \cdot 10^{-4}$	$1 \cdot 10^{-5}$	$2.621 \cdot 10^{-3}$
$5 \cdot 10^{-6}$	$1.956 \cdot 10^{-3}$	$1.7 \cdot 10^{-5}$	$6.134 \cdot 10^{-3}$
$6 \cdot 10^{-6}$	-	$1.8 \cdot 10^{-5}$	-

$n = 100$

$\bar{\lambda}_{5} = 5$, $rad_{max} = 2.753 \cdot 10^{-6}$		$\bar{\lambda}_{95} = 95$, $rad_{max} = 7.9715 \cdot 10^{-7}$	
rad	r	rad	r
$1 \cdot 10^{-7}$	$6.474 \cdot 10^{-5}$	$1 \cdot 10^{-7}$	$1.239 \cdot 10^{-4}$
$1 \cdot 10^{-6}$	$7.154 \cdot 10^{-4}$	$6 \cdot 10^{-7}$	$9.630 \cdot 10^{-4}$
$2 \cdot 10^{-6}$	$1.693 \cdot 10^{-3}$	$8 \cdot 10^{-7}$	$1.857 \cdot 10^{-3}$
$3 \cdot 10^{-6}$	-	$9 \cdot 10^{-7}$	-

Table 4: Test the theoretically derived rad_{max} to some rad used in computations.

Example 3: comparison. We now compare our method, denoted by radiipol, with two different algorithms developed by S. Rump. The first one, denoted by verifyeig, has been introduced in [14] with the primary goal of computing enclosures of multiple of nearly multiple eigenvalues (and related eigenvectors) of interval matrices. The second one, denoted by verifynlss, is based on a Krawczyk operator [15, 16] and is a general routine to rigorously compute well separated zeros of nonlinear functions. In fact, in the code `verifyeig.m` (available in the library INTLAB [19]), where the method verifyeig has been implemented, the author suggests to use verifynlss to compute simple and well separated eigenpairs. This method is implemented in the code `verifynlss.m` in the library INTLAB [19]. Table 5 provides the average of the radius of the balls enclosing the exact eigenpairs for each method.

For both experiments the test matrices $\boldsymbol{A}$ have been constructed as in the previous section: given N we define $D \in \mathbb{C}^{N+1,N+1}$ as a diagonal matrix with entries given by N equispaced values on the unit circle in the complex plane and 0, i.e. $diag(D) = [0, e^{i\frac{2\pi}{N}j}], j = 1, \ldots, N$. Then let $\hat{A} = XDX^{-1}$, where X is a complex random matrix with entries in the complex square $[-1, 1] + i[-1, 1]$ and finally define $\boldsymbol{A}$ as the interval complex matrix centered in $\hat{A}$ and of radius rad.

The results presented in Table 5 confirm that the new approach radiipol is satisfactory from the point of view of the accuracy of the results. Indeed, while the algorithm verifynlss fails quite soon as N and rad increase (it fails for $rad = 0$ and for all $N \geq 15$), the new algorithm is successful also for large entries of $\boldsymbol{A}$.

	N=5			N=10			
rad	10^{-20}	10^{-10}	10^{-4}	10^{-10}	10^{-5}	10^{-4}	10^{-3}
radiipol	$9.14{\cdot}10^{-15}$	$2.76{\cdot}10^{-9}$	0.0019	$3.25{\cdot}10^{-9}$	$4.61{\cdot}10^{-4}$	–	–
verifyeig	$4.69\cdot10^{-15}$	$2.07{\cdot}10^{-9}$	0.0016	$2.08{\cdot}10^{-9}$	$3.02{\cdot}10^{-4}$	0.0049	–
verifynlss	$6.26\cdot10^{-9}$	–	–	–	–	–	–

	N=50			N=100			N=150
rad	10^{-10}	10^{-8}	10^{-5}	10^{-10}	10^{-8}	10^{-7}	10^{-10}
radiipol	$2.69{\cdot}10^{-7}$	$4.94{\cdot}10^{-5}$	–	$9.02{\cdot}10^{-7}$	–	–	$1.31{\cdot}10^{-6}$
verifyeig	$5.59{\cdot}10^{-8}$	$9.45{\cdot}10^{-6}$	–	$1.31{\cdot}10^{-7}$	$2.07{\cdot}10^{-5}$	–	$1.64{\cdot}10^{-7}$
verifynlss	–	–	–	–	–	–	–

Table 5: Each number is the average of the radius of the disks enclosing the eigenvalues for each method. Comparison of the accuracy of the three methods as the dimension N and the radius rad of the test matrix $\boldsymbol{A}$ change. The entry – means that the method fails in the enclosure of at least one of the eigenpair.

Acknowledgements

This work was supported by grant No. MTM2011–24766 of the MICINN, Spain.

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